10

HIGHER ORDER MODAL LOGIC

Reinhard Muskens

1 INTRODUCTION

A logic is called *higher order* if it allows for quantification (and possibly abstraction) over higher order objects, such as functions of individuals, relations between individuals, functions of functions, relations between functions, etc. Higher order logic (often also called *type theory* or the *Theory of Types*) began with Frege, was formalized in Russell [46] and Whitehead and Russell [52] early in the previous century, and received its canonical formulation in Church [14].[1] While classical type theory has since long been overshadowed by set theory as a foundation of mathematics, recent decades have shown remarkable comebacks in the fields of *mechanized reasoning* (see, e.g., Benzmüller et

[1]For a good survey of (non-modal) higher order logic, see van Benthem and Doets [8]; for a textbook development, Andrews [3].

al. [9] and references therein) and *linguistics*. Since the late 1960's philosophers and logicians, for various reasons which we will dwell upon, have started to combine higher order logic with modal operators (Montague [35, 37, 38], Bressan [11], Gallin [22], Fitting [19]). This combination results in *higher order modal logic*, the subject of this chapter.

The chapter will be set up as follows. In the next section we will look at possible motivations behind the idea of combining modality and higher order logic. Then, in Section 3, Richard Montague's system of 'Intensional Logic', by far the most influential of higher order modal logics to date, will be discussed. This logic will be shown to have some limitations. One of these is that, despite its name, the logic is not fully intensional, as it validates the axiom of Extensionality. This leads to a series of well-known problems centering around 'logical omniscience'. Another limitation is that the logic is not Church-Rosser (it matters in which order λ-conversions are carried out). These limitations can be overcome and the remaining sections of the chapter will contain an exposition of a modal type theory that is intensional in two ways: in the sense of being a modal logic and in the sense that Extensionality does not hold. The logic in itself is not strong enough to make the usual rules of λ-conversion derivable, but these rules can consistently be added as an axiomatic extension and in that case the Church-Rosser property will hold (as an alternative, the rules can be hard-wired into the theory, in which case the theory is also Church-Rosser). Section 4 will introduce the basic syntax and semantics of this logic, Section 5 will give a tableau calculus, and Section 6 provides some elementary model theory in the form of a model existence theorem and its usual corollaries, such as generalized completeness. We conclude with a conclusion.

2 MOTIVATION

Why should one want to combine modality with quantification or abstraction over objects of higher type? Possible reasons come from areas as diverse as rational theology, the axiomatization of classical mechanics, the semantics of natural language, and modal logic itself. Let us look at each of these in turn.

2.1 The Ontological Argument

Anselm (1033–1109) proved the existence of God by defining him as "a being than which none greater can be thought" and by arguing that, since that definition can be understood, such a being must "exist in the understanding". But if this being exists in the understanding, one can also think of it as existing in reality and, since real existence is "greater" than mere conceptual existence, the "being than which none greater can be thought" must truly exist. Otherwise one could think of an even greater being that did truly exist. Moreover, by an analogous argument, Anselm comes to the conclusion that it is even *impossible* to think of God as nonexistent. For something that cannot be thought of as nonexisting is greater than something that can be so thought of. It follows that a "being than which none greater can be thought" cannot merely exist contingently, otherwise one could think of an even greater being with necessary existence.

Anselm's original argument was phrased in ordinary Latin and its lack of precision may be deemed a weakness by some, but increasingly more precise variants of the argument have been put forward by Descartes, Leibniz and, more recently, Gödel [24]. Gödel's argument centers around "positive" properties and being a god can be defined as having