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COMBINING MODAL LOGICS

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1 INTRODUCTION

When can we say that a logic is a *combination* of others? In general, *any* logical system having more than one connective can be considered as a combination of logical systems having fewer connectives. In particular, *any* multimodal logic can be considered as a combination of, say, unimodal logics. So, in this general sense, *any* result on multimodal logics can be considered as a result on combining modal logics. What makes this chapter special among other ones studying multimodal logics is that here we investigate the following kind of problems:

> *Given a family* L *of modal logics and a combination method* C*, do certain properties of the 'component logics'* $L \in \mathsf{L}$ *transfer to their 'combination'* $\mathsf{C}(\mathsf{L})$*?*

Most of the *combination methods* considered in this chapter satisfy the following three criteria:

(C1) They are *finitary*, that is, C is defined only on *finite* families L of modal logics.

(C2) The combination $\mathsf{C}(\mathsf{L})$ of (multi)modal logics from L is a (multi)modal logic itself.

(C3) The combined logic $\mathsf{C}(\mathsf{L})$ is an extension of each component logic $L \in \mathsf{L}$.

For each considered combination method, we discuss in detail the possible transfer of the following two kinds of properties:

- **Axiomatisation/completeness.**
  There are two versions, depending on whether the combination method results in a syntactically or semantically defined logic. In the former case, the question is whether the combination of recursively (finitely) axiomatisable components remains recursively (finitely) axiomatisable, and in the latter, whether the Kripke completeness of the components transfers to their combination.

- **Decidability/complexity of the validity/satisfiability problem.**
  We study whether decidability of the validity problem transfers from the components to their combination and if so, what is the change in complexity. We also discuss the possible transfer of the finite model property.

For transfer results about several other properties (like versions of interpolation, decidability of various consequence relations, etc.) see [23] and the references therein. Combinations of deductive calculi (such as combined tableaux) are not considered either, see Chapter 2 of this handbook for some examples.

Combination methods not satisfying **(C1)**–**(C3)** are in general out of our scope, though see Section 5 for a discussion.

Notation and terminology. We will mainly consider possible world (or Kripke) semantics. *Kripke models* are pairs $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ that are based on relational structures $\mathfrak{F} = \langle W, R_1, \dots, R_n \rangle$, where $n > 0$ is a natural number, W is a non-empty set and the R_i are binary relations on it. Such structures are called *n-frames* (or *frames*, for short). We say that an n-frame $\mathfrak{G} = \langle U, S_1, \dots, S_n \rangle$ is a *subframe* of an n-frame $\mathfrak{F}$ ($\mathfrak{G} \subseteq \mathfrak{F}$, in symbols) if $U \subseteq W$ and $S_i = R_i \cap (U \times U)$, for $i = 1, \dots, n$. A *path of length k* from point x to point y in an n-frame $\mathfrak{F}$ is a sequence $\langle x_0, \dots, x_k \rangle$ of points such that $x_0 = x$, $x_k = y$ and $x_i R_j x_{i+1}$, for each $i < k$ and some j, $1 \le j \le n$. We call an n-frame $\mathfrak{F}$ *rooted* if there exists some $x \in W$ such that for every $y \in W$, $y \neq x$ there is a path from x to y. Such an x is called a *root of* $\mathfrak{F}$. We say that $\mathfrak{F}$ is of *depth k* if k is the length of the longest path in $\mathfrak{F}$. If such a longest path does not exist, then we say that $\mathfrak{F}$ is of *infinite depth*. An n-frame $\mathfrak{F}$ is called *tree-like* if it is rooted and $R = \bigcup_{i=1}^{n} R_i$ is weakly connected on the set $\{y \in W \mid yRx\}$ for every $x \in W$. If a tree-like frame is well-founded (i.e., there are no infinite descending R-chains $\dots Rx_2Rx_1Rx_0$ of points) then we call $\mathfrak{F}$ a *tree*. The *depth* $d^{\mathfrak{F}}(x)$ of a point x in a tree $\mathfrak{F}$ is defined to be the length of the unique path from the root to x. If for no $n < \omega$ the point x is of depth n, then we say that x is of *infinite depth*. By the *co-depth* of a point x in a tree $\mathfrak{F}$ we understand the depth of the subtree of $\mathfrak{F}$ with root x.

Given a natural number n, the *n-modal language* $\mathcal{ML}_n$ has propositional variables $p, q, s, \dots$, Boolean connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\top$, $\bot$, and (unary) modal operators $\Box_1, \dots, \Box_n$ and $\Diamond_1, \dots, \Diamond_n$. $\mathcal{ML}_n$-formulas are formed inductively in the usual way. Given an $\mathcal{ML}_n$-formula φ, we let $sub\,\varphi$ denote the set of all subformulas of φ, and $md(\varphi)$ denote the *modal depth* of φ. We will also use the following abbreviations. For every formula φ, let

$$\Box^0\varphi = \varphi \qquad \text{and, for } n < \omega, \quad \Box^{n+1}\varphi = \Box\Box^n\varphi, \quad \Box^{\le n}\varphi = \bigwedge_{k \le n} \Box^k\varphi.$$

The *truth-relation* '$(\mathfrak{M}, w) \models \varphi$' connecting syntax and semantics is defined by induction on the construction of φ as usual. We say that φ is *true in* $\mathfrak{M}$ ($\mathfrak{M} \models \varphi$, in symbols),