Complexity and online algorithms for a coloring problem on a line

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1 Introduction

We consider a coloring problem that arises in the context of wavelength assignment on an optical line network [1, 6]. Related coloring problems on a line network with different objective functions include [2, 3, 7, 8].

We are given a set of n intervals $\mathcal{I} = \{I_1, I_2, \cdots, I_n\}$. Each I_j is an interval $[s_j, e_j)$, where s_j and e_j denote the start and end time of the interval I_j, respectively. Two intervals I_i and I_j are said to be *overlapping* if $I_i \cap I_j \neq \emptyset$. We say that interval I_j contains time t if $t \in I_j$. The load at time t, denoted by $load(t)$ is the number of intervals containing t. The *length* of I_j, denoted by $\ell(I_j)$, is defined as $e_j - s_j$. The maximum and minimum length over all intervals in $\mathcal{I}$ is denoted by $\ell_{\max}$ and $\ell_{\min}$, respectively. The length of any set of intervals $\mathcal{S}$ is defined as the sum of the length of all intervals in $\mathcal{S}$, i.e., $\ell(\mathcal{S}) = \sum_{I \in \mathcal{S}} \ell(I)$. Without loss of generality, we can assume that the intervals in $\mathcal{I}$ form a contiguous interval, otherwise, the scheduling of each contiguous interval is independent of the others, so we can focus on just one such contiguous interval.

We are also given an infinite set of colors $\Lambda = \{1, 2, 3, \cdots\}$, each color i is associated with a cost $\lambda(i) \geq 1$ which is an increasing function such that $\lambda(i) < \lambda(i')$ if $i < i'$. Unless specified otherwise we assume $\lambda(i) = i$. A coloring $\omega : \mathcal{I} \to \Lambda$ is *valid* if for any pair of overlapping intervals $I_i \neq I_j$, $\omega(I_i) \neq \omega(I_j)$. The cost of ω at any time t, denoted by $cost(\omega, t)$, is the maximum cost of the color over all intervals containing t, i.e., $cost(\omega, t) = \max_{I: t \in I} \lambda(\omega(I))$. The total cost of ω, denoted as $cost(\omega)$, is defined as the sum of the cost over all times t, i.e., $cost(\omega) = \int_t cost(\omega, t)$. The objective of the MinSumMax problem is to find a valid coloring ω such that $cost(\omega)$ is minimized.

For any algorithm $\mathcal{A}$ we also denote its coloring by $\mathcal{A}$, and its cost is $cost(\mathcal{A})$. We denote by $\mathcal{O}$ the optimal offline algorithm.

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2 Our contribution

2.1 The MinSumMax problem

A more general problem has been studied in [1] which was proved to be NP-complete. On the other hand, their 2-approximation algorithm applies to our problem.

We first show by a reduction from circular arc coloring [5] that, our problem MinSumMax which is simpler remains NP-complete.

Theorem 1. MinSumMax *is NP-complete.*

We then propose online algorithms which give $O(1)$-competitiveness when the ratio of maximum and minimum length of intervals is bounded by a constant and $O(\log \frac{\ell_{\max}}{\ell_{\min}})$-competitiveness for arbitrary length intervals where $\ell_{\max}$ and $\ell_{\min}$ denote the maximum and minimum length of intervals.

Bounded length intervals. Consider bounded length intervals for which we assume there is a constant k such that for any interval I in the input, we have $\ell(I) \in [\ell_{\min}, k \cdot \ell_{\min})$. When an interval $I_j \in \mathcal{I}$ arrives, $\mathcal{G}$ assigns the minimum color that is valid for it, i.e., $\mathcal{G}$ assigns the minimum color i such that for all $j' < j$ and $I_{j'} \cap I_j \neq \emptyset$, we have $\mathcal{G}(I_{j'}) \neq i$.

Our analysis identifies intervals that "contribute" to the cost of a coloring and uses their length to bound the total cost. The *skyline* of a coloring ω is the function of the maximum color used at any time t. We denote the set of intervals on the skyline by $\mathcal{I}_{\mathrm{S}}$ which contains intervals whose color is on the skyline for some time t, i.e., an interval I is in $\mathcal{I}_{\mathrm{S}}$ if there exists $t \in I$ such that $cost(\omega, t) = \lambda(\omega(I))$. We further select a subset $\mathcal{I}^*_S$ of intervals on the skyline of $\mathcal{G}$ from $\mathcal{I}_{\mathrm{S}}$, partition the time horizon into segments based on $\mathcal{I}^*_S$, and show that we can "charge" the cost of $\mathcal{G}$ and $\mathcal{O}$ to this subset, thus allowing us to bound the two costs. The partition of time is based on a notion of extended interval. For any interval I_j, we define its *hat* interval as $I^{\mathrm{h}}_j = [s_j - k\ell_{\min}, e_j + k\ell_{\min})$ and *extended hat* interval as $I^{\mathrm{e}}_j = [s_j - 3k\ell_{\min}, e_j + 3k\ell_{\min})$.

We show that we can select $\mathcal{I}^*_S$ satisfying the following properties. We denote by $q(j)$ the indexes of intervals chosen, i.e., $\mathcal{I}^*_S = \{I_{q(1)}, I_{q(2)}, \cdots\}$ where $q(j) < q(j+1)$.

Lemma 2. *We can select $\mathcal{I}^*_S$ such that for any $j \geq 1$, we have* (i) $I^{\mathrm{h}}_{q(j)} \cap I^{\mathrm{h}}_{q(j+1)} = \emptyset$; (ii) $I^{\mathrm{e}}_{q(j)} \cup I^{\mathrm{e}}_{q(j+1)}$ *forms a contiguous interval;* (iii) $s_{q(j+1)} - s_{q(j)} \leq 7k\ell_{\min}$ *and* $e_{q(j+1)} - e_{q(j)} \leq 7k\ell_{\min}$. (iv) *For any interval $I \in \mathcal{I}_S$, there exists $j \geq 1$ such that $I \subseteq I^{\mathrm{e}}_{q(j)}$ and $\mathcal{G}(I) \leq \mathcal{G}(I_{q(j)})$.*

The above properties then lead to the following theorem.

Theorem 3. *The greedy algorithm $\mathcal{G}$ is $7k$-competitive when $k = \ell_{\max}/\ell_{\min}$ is a constant.*

Arbitrary length intervals. One can show that the above competitive ratio of the greedy algorithm is tight up to a constant factor, meaning that for arbitrary length intervals, the competitive ratio can be very large. To have a better competitive ratio, we use a standard technique of classification to partition the input intervals $\mathcal{I}$ into $O(\log \frac{\ell_{\max}}{\ell_{\min}})$ classes where in each class the length of the intervals is bounded to be within a ratio of 2. This means that if we apply the greedy algorithm for individual class, then we obtain a competitive ratio of $14 = 2 \times 7$. Let $L = 1 + \lceil \log \frac{\ell_{\max}}{\ell_{\min}} \rceil$. We denote the classes by $C_1, C_2, \cdots, C_L$. We assume that the classified-greedy algorithm, denoted by $\mathcal{C}$, knows about the ratio of the maximum to minimum length of intervals.

$\mathcal{C}$ divides the colors into bands of size L, the i-th color in each band is reserved for intervals in class C_i. Precisely, the color set $\Lambda_i = \{i, i+L, i+2L, i+3L, \cdots\}$ is reserved for intervals in C_i. Intervals in each class C_i are colored independently using the greedy algorithm $\mathcal{G}$ over the color set Λ_i.

At first glance, the competitive ratio of $\mathcal{C}$ is $O(L^2)$. A more detailed analysis reveals that the competitive ratio is indeed $O(L)$.

Theorem 4. *For arbitrary length intervals, $\mathcal{C}$ is $O(\log \frac{\ell_{\max}}{\ell_{\min}})$-competitive.*

Lower bound for arbitrary length intervals. There is a lower bound for any deterministic online algorithm:

Theorem 5. *There is an adversary such that for any deterministic online algorithm $\mathcal{A}$, $cost(\mathcal{A})/cost(\mathcal{O}) \geq \frac{1}{2} \log \frac{\ell_{\max}}{\ell_{\min}}$.*

2.2 Variant — permutation problem

In the MinSumMax problem the algorithm has to assign a color to each interval. We consider a variant of the problem where the input is sets of intervals each of which contains intervals that are pairwise non-overlapping and all such intervals in a set are given the same color. The problem is to find a permutation of the colors such that the cost of the coloring (as defined above) is minimized. This problem may sound easier since we do not need to assign color but only need to find a permutation of colors. It turns out that this variant is still NP-complete via a reduction from the directed optimal linear arrangement problem [4].

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