Overview

- Last time
  - Logic as a knowledge representation scheme
  - Reminder of Propositional logic
  - Introduction to first-order predicate logic
- Today
  - Introduction to first-order predicate logic
  - Terminology
  - Equivalences
  - Proof
  - Decidability

Quantifiers

- Quantifiers allow us to express properties about collections of objects
- The quantifiers are
  - $\forall$ universal quantifier ‘For all . . . ’
  - $\exists$ existential quantifier ‘There exists . . . ’
- If $P(x)$ is a predicate then we can write
  - $\forall x \cdot P(x)$; and
  - $\exists x \cdot P(x)$;
  - where $x$ is a variable which can stand for any object in the domain.

Universal Quantification

- Note that universal quantification is similar to conjunction
- Suppose the domain is the numbers $\{2, 4, 6\}$. Then
  - $\forall n \cdot \text{Even}(n)$
    - is the same as
    - $\text{Even}(2) \land \text{Even}(4) \land \text{Even}(6)$

Universal Quantification and $\Rightarrow$

- Typically, $\Rightarrow$ is the main connective with $\forall$
- If we have domain $\{\text{Kitty}, \text{Horace}\}$ where Kitty is a cat (and a mammal) and Horace is a lizard (not a mammal)
  - $\forall x \cdot \text{cat}(x) \Rightarrow \text{mammal}(x)$ really means
    - $\text{cat(Kitty)} \Rightarrow \text{mammal(Kitty)}$ \land
    - $\text{cat(Horace)} \Rightarrow \text{mammal(Horace)}$
- This evaluates to true as both $\text{cat(Kitty)}$ and $\text{mammal(Kitty)}$ are true, also $\text{cat(Horace)}$ is false so the second implication is true
A Common Mistake to Avoid

• Common mistake: using ∧ as the main connective with ∀:

$\forall x \cdot \text{At}(x, \text{Berkeley}) \land \text{Smart}(x)$

means “Everyone is at Berkeley and everyone is smart”

Existential Quantification

• Existential quantification is the same as disjunction

Thus with the same domain

$\exists n \cdot \text{Even}(n)$

is the same as

$\text{Even}(2) \lor \text{Even}(4) \lor \text{Even}(6)$

Existential Quantification

• Existential quantification allows us to make a statement about some object without naming it

• There exists an x such that x is a man and x is a father

  $(\text{some men are fathers})$

  $\exists x \cdot \text{man}(x) \land \text{father}(x)$

• Some cats are white and have three legs

  $(\text{some cats are white and have three legs})$

  $\exists y \cdot \text{cat}(y) \land \text{white}(y) \land \text{three_legs}(y)$

Another Common Mistake to Avoid

• Common mistake: using ⇒ as the main connective with ∃:

  $\exists x \cdot \text{At}(x, \text{Stanford}) \Rightarrow \text{Smart}(x)$

  is true if there is anyone who is not at Stanford!

Examples

• $\forall x \cdot \text{Man}(x) \Rightarrow \text{Mortal}(x)$

  ‘For all x, if x is a man, then x is mortal.’

  (i.e. all men are mortal)

• $\forall x \cdot \text{Man}(x) \Rightarrow (\exists y \cdot \text{Woman}(y) \land \text{MotherOf}(y, x))$

  ‘For all x, if x is a man, then there exists a y such that y is a woman and y is the mother of x.’

  (i.e., every man has a mother)

• $\exists m \cdot \text{Monitor}(m) \land \text{MonitorState}(m, \text{ready})$

  ‘There exists a monitor that is in a ready state.’

• $\forall r \cdot \text{Reactor}(r) \Rightarrow (\exists t \cdot (100 \leq t \leq 1000) \land \text{temp}(r) = t)$

  ‘Every reactor will have a temperature in the range 100 to 1000.’
More Than One Quantifier

- For all $x$ and $y$, if $x$ is the parent of $y$ then $y$ is the child of $x$

$$\forall x \forall y \cdot \text{parentOf}(x, y) \Rightarrow \text{childOf}(y, x)$$

The Order of Quantifiers is Important

- Everyone ate something
  $$\forall x \exists y \cdot \text{ate}(x, y)$$
- There is something that was eaten by everyone
  $$\exists y \forall x \cdot \text{ate}(x, y)$$
- Everything was eaten by someone
  $$\forall y \exists x \cdot \text{ate}(x, y)$$
- Someone ate everything
  $$\exists x \forall y \cdot \text{ate}(x, y)$$

Syntax of Predicate Logic

- The formulae of predicate logic are constructed from the following symbols
  - a set $\text{PRED}$ of predicate symbols with arity
  - a set $\text{FUNC}$ of function symbols with arity
  - a set $\text{CONS}$ of constant symbols
  - a set $\text{VAR}$ of variable symbols
  - the quantifiers $\forall$ and $\exists$
  - $\text{true}$, $\text{false}$ and the connectives $\land$, $\lor$, $\Rightarrow$, $\neg$, $\iff$

Terms

- The set of terms, $\text{TERM}$, is constructed by the following rules
  - any constant is in $\text{TERM}$
  - any variable is in $\text{TERM}$
  - if $t_1, \ldots, t_n$ are in $\text{TERM}$ and $f$ is a function symbol of arity $n$ then $f(t_1, \ldots, t_n)$ is a term
  - $f(x, y)$
  - $\text{add}(2, 4)$
  - $\text{motherOf}(\text{Annabel})$

Well-Formed Formulae

- The set of sentences or well-formed formulae of predicate logic are:
  - $\text{true}$, $\text{false}$ and propositional formulae are in WFF
  - if $t_1, \ldots, t_n$ are in $\text{TERM}$ and $p$ is a predicate symbol of arity $n$ then $p(t_1, \ldots, t_n)$ is in WFF
  - If $A$ and $B$ are in WFF then so is $\neg A$, $A \lor B$, $A \land B$, $A \Rightarrow B$ and $A \Leftrightarrow B$
  - If $A$ is in WFF and $x$ is a variable then $\forall x \cdot A$ and $\exists x \cdot A$ are in WFF
  - If $A$ is in WFF then so is $(A)$

Binding

- A variable in the scope of a quantifier is said to be $\text{bound}$
- A variable not in the scope of a quantifier is said to be $\text{free}$
Equivalences Between Quantifiers (I)

• The universal and existential quantifiers are in fact duals of each other:

• Saying that everything has some property is the same as saying that there is nothing that does not have the property

\[ \forall x \cdot P(x) \equiv \neg \exists x \cdot \neg P(x) \]

• everyone doesn’t like sprouts

\[ \forall x \cdot \neg \text{likes}_{\text{sprouts}}(x) \]

is the same as saying it’s not the case that someone likes sprouts

\[ \neg \exists x \cdot \text{likes}_{\text{sprouts}}(x) \]

Domains and Interpretation

• Suppose we have a formula \( \forall x \cdot P(x) \). What does \( x \) range over? Physical objects, numbers, people, times, . . . ?

• Depends on the domain that we intend. Often, we name a domain to make our intended interpretation clear

  – Suppose our intended interpretation is the positive integers

  – Suppose \( +, \times, \ldots \) have the usual mathematical interpretation

  – Is this formula satisfiable under this interpretation?

\[ \exists n \cdot n = (n \times n) \]

  – Now suppose that our domain is negative integers (where \( \times \) has the usual mathematical interpretation)

  – Is the formula satisfiable under this interpretation?

Equivalences Between Quantifiers (II)

• Saying that there is something that has the property is the same as saying that its not the case that everything doesn’t have the property

\[ \exists x \cdot P(x) \equiv \neg \forall x \cdot \neg P(x) \]

• Also

\[ \forall x \cdot \neg P(x) \equiv \neg \exists x \cdot P(x) \]

\[ \exists x \cdot \neg P(x) \equiv \neg \forall x \cdot P(x) \]

Semantics of Predicate Logic

• We haven’t given the formal semantics of predicate logic

  – See good logic and AI books

  – Informally we’ve seen we need a domain of interest

  – Constants, predicates, functions have mappings into this domain

  – To evaluate quantifiers we must check whether all objects in the domain satisfy the formula (\( \forall \)) or some object does (\( \exists \))

Validity

• A formula of FOL that is true under all interpretations is said to be valid

• So we could try to check for validity by writing down all the possible interpretations and looking to see whether the formula is true or not

First-Order Example

• Unfortunately in general we can’t use this method

• Consider the formula:

\[ \forall n \cdot \text{Even}(n) \Rightarrow \neg \text{Odd}(n) \]

and the domain Natural Numbers, i.e. \( \{1, 2, 3, 4, \ldots \} \)

• There are an infinite number of interpretations

• Is there any other procedure that we can use, that will be guaranteed to tell us, in a finite amount of time, whether a FOL formula is, or is not, valid?
Proof in FOL Decidable?

- The answer is no
- FOL is for this reason said to be undecidable. FOL is often called semi-decidable, as given a formula that is not valid, the procedure may not terminate

Proof in FOL

- Proof in FOL is similar to that in propositional logic; we just need an extra set of rules, to deal with the quantifiers
- FOL inherits all the rules of propositional logic
- The most obvious rule, ∀-elimination
  - Tells us that if everything in the domain has some property, then we can infer that any particular individual has the property,
  \[
  \forall x \cdot \varphi(x) \\
  \varphi(a)
  \]
  for any a in the domain
- Going from general to specific

Example

- Let's use ∀-Elimination to consider the cat/mammal example
  \[
  \text{cat(Kitty), } \forall x \cdot \text{cat}(x) \Rightarrow \text{mammal}(x) \\
  \]
  \[
  \vdash \text{mammal(Kitty)}
  \]
  1. \text{cat(Kitty)} [Given]
  2. \forall x \cdot \text{cat}(x) \Rightarrow \text{mammal}(x) [Given]
  3. \text{cat(Kitty)} \Rightarrow \text{mammal(Kitty)} [2, ∀-Elimination]
  4. \text{mammal(Kitty)} [1, 3, MP]

Summary

- We’ve seen the formal syntax of first-order predicate logic and informally considered the semantics
- We’ve considered decidability for predicate logic
- We’ve seen an example of a proof rule for predicate logic
  - There are other proof rules related to quantifiers
    - See good logic or AI books
- Next time: Propositional resolution