Foundations of Computer Science (COMP109)

Exercise 10

1. Explain intuitively the definition of *sentences*.

2. Suppose $S$ and $\prec$ are binary predicate symbols and $\top$ is an individual constant. Consider the structure

$$\mathcal{N} = \langle \mathbb{N}, S^\mathcal{N}, \prec^\mathcal{N}, \top^\mathcal{N} \rangle$$

with

- $\mathbb{N} = \{1, 2, 3, \ldots \}$,
- $S^\mathcal{N} = \{(n, n+1) \mid n \in \mathbb{N}\}$,
- $\prec^\mathcal{N} = <$,
- $\top^\mathcal{N} = 1$,

Which of the following are true?

- $\mathcal{N} \models \exists x \prec (\top, x)$?
- $\mathcal{N} \models \exists x \prec (x, \top)$?
- $\mathcal{N} \models \exists x \forall y \prec (x, y)$?
- $\mathcal{N} \models \forall x \forall y \prec (x, y)$?
- $\mathcal{N} \models \forall x \exists y \prec (x, y)$?
- $\mathcal{N} \models \forall x \exists y \prec (y, x)$?

Let $a$ be an assignment into $\mathbb{N}$ such that $a(x) = 2$ and $a(y) = 4$. Which of the following are true?

- $(\mathcal{N}, a) \models \prec (x, y)$?
- $(\mathcal{N}, a) \models S(x, y)$?
3. Which of the following are tautologies? If \( \varphi \) is not a tautology, then define a structure refuting \( \varphi \).

- \( \forall x(P(x) \lor \neg P(x)) \)?
- \( \forall x\exists yQ(x, y) \)?
- \( \forall x((P_1(x) \land P_2(x)) \leftrightarrow (P_2(x) \land P_1(x))) \)?
- \( P(c) \to \exists xP(x) \) (\( c \) is a constant.)
- \( P(c) \to \forall xP(x) \) (\( c \) is a constant.)

4. Let \( \Gamma \) be a set of formulas and \( \varphi \) a formula. Define ‘\( \Gamma \models \varphi \)’.

5. Show that

\[ \{P(c)\} \not\models \forall xP(x) \]

6. Show that

\[ \{\exists x(P(x) \land \neg P(x))\} \models \varphi, \]

for every formula \( \varphi \).