Worst-case equilibria for selfish restricted job scheduling

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Model

Previous Results

New Results

Model

1

2

A

B

C

n = 5, m = 3

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Worst-case equilibria for selfish restricted job scheduling
Restricted Job Scheduling

$n = 5, m = 3$
Restricted Unrelated Job Scheduling

$w_{4A} = 5$

$w_{2A} = 31$

$w_{3B} = 55$

$w_{4C} = 24$

$w_{5C} = 7$

$n = 5, m = 3$

$w_{min} = 1.5$

$w_{max} = 55$

$s = \frac{w_{max}}{w_{min}} = \frac{55}{1.5}$
Restricted Unrelated Job Scheduling

\[ j_1 = A \]
\[ j_2 = B \]
\[ j_3 = C \]
\[ j_4 = C \]
\[ j_5 = C \]

\[ w_{2A} = 31 \]
\[ w_{3B} = 55 \]
\[ w_{4C} = 24 \]
\[ w_{5C} = 7 \]

\[ A \]
\[ B \]
\[ C \]

\[ n_A = 1 \]
\[ n_B = 1 \]
\[ n_C = 3 \]

\[ c_A = 5 \]
\[ c_B = 23 \]
\[ c_C = 32.5 \]

\[ j_1 = A \]
\[ j_2 = B \]
\[ j_3 = C \]
\[ j_4 = C \]
\[ j_5 = C \]

\[ w_{1A} = 5 \]
\[ w_{2B} = 23 \]
\[ w_{3C} = 1.5 \]

\[ n = 5, m = 3 \]
\[ w_{\text{min}} = 1.5 \]
\[ w_{\text{max}} = 55 \]

\[ s = \frac{w_{\text{max}}}{w_{\text{min}}} = \frac{55}{1.5} \]
Definition

An **Optimal Solution** \((OPT)\) is an assignment that minimize the objective function selected by the designer. (Ex: \(\sum_i c_j\))

\[
\begin{align*}
  j_1^* &= A & w_{1A} &= 5 & n_A^* &= 1 & c_A^* &= 5 \\
  j_2^* &= B & w_{2B} &= 23 & n_B^* &= 1 & c_B^* &= 23 \\
  j_3^* &= C & w_{3C} &= 1.5 & n_C^* &= 3 & c_C^* &= 32.5 \\
  j_4^* &= C & w_{4C} &= 24 \\
  j_5^* &= C & w_{5C} &= 7
\end{align*}
\]
A Nash Equilibrium Solution ($NASH$) is an assignment where no job can unilaterally improve its latency moving to another machine.

\[ \forall \text{ job } i, \forall \text{ machine } j, \quad c_{ji} \leq c_j + w_{ij}. \]
Price of Anarchy

**Definition**

The **Price of Anarchy** (PoA) [KP99] is the ratio between the cost of the worst Nash Equilibrium Solution and the cost of the Optimal Solution.

\[
PoA = \frac{C(NASH)}{C(OPT)}
\]

It measures how much the lack of central coordination can affect the system.
In this setting...

A Nash Equilibrium Solution always exists. [EdKM03]

For the Maximum Latency objective function (min max\_j c\_j):

\[
PoA = \Theta \left( s + \frac{\log m}{\log(1 + \frac{\log m}{s})} \right). \quad [AART03]
\]
Definition

This objective function minimizes the total latency suffered by jobs.

$$\min \sum_i c_i = \min \sum_j n_j c_j$$
With Total Latency objective function...

<table>
<thead>
<tr>
<th>Unrestricted Weighted Related setting</th>
</tr>
</thead>
<tbody>
<tr>
<td>$PoA \leq 4s$. [BGGM06]</td>
</tr>
<tr>
<td>$\frac{n}{2w} \leq PoA \leq \frac{n}{w} + \frac{m^2 + m}{w^2}$. [HS07]</td>
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</tbody>
</table>

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<tr>
<td>$2.5 - \varepsilon \leq PoA \leq 2.5$. [STZ04][CFK+06]</td>
</tr>
</tbody>
</table>
Overview

- $\max_{j: c_j^* \neq 0} \frac{c_j}{c_j^*} = O \left( \max \left( s, \frac{\log m}{\log \left(1 + \frac{\log m}{s}\right)}\right) \right)$

- $PoA = O \left( \max \left( s, \frac{\log m}{\log \left(1 + \frac{\log m}{s}\right)}\right) \right)$

- $PoA = \Omega(s)$
<table>
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<tbody>
<tr>
<td>Total Latency</td>
<td>Restricted Identical Setting</td>
<td>Weighted Total Latency</td>
</tr>
</tbody>
</table>

### New Results - 2

**Upper Bound**

\[
\forall \text{ job } i, \quad c_{ji} \leq c_{j^*} + w_{ij^*}
\]

\[
\sum_i c_{ji} \leq \sum_i (c_{j^*} + w_{ij^*}) \leq \sum_i (c_{j^*} + c_{j^*}^*)
\]

\[
C(NASH) = \sum_j n_j c_j \leq \sum_j n_j^* c_j + \sum_j n_j^* c_j^* = \sum_{j : c_j^* \neq 0} n_j^* c_j + C(OPT)
\]

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Worst-case equilibria for selfish restricted job scheduling
With an analysis similar to [AART03] we obtain:

\[
\max_{j : c_j^* \neq 0} \frac{c_j}{c_j^*} = O \left( \max \left( s, \frac{\log m}{\log (1 + \frac{\log m}{s})} \right) \right)
\]

So

\[
\sum_{j : c_j^* \neq 0} n_j^* c_j = O \left( \max \left( s, \frac{\log m}{\log (1 + \frac{\log m}{s})} \right) \right) \sum_j n_j^* c_j
\]

And so

\[
PoA = O \left( \max \left( s, \frac{\log m}{\log (1 + \frac{\log m}{s})} \right) \right)
\]
New Results - 4

Lower Bound

\[
\begin{align*}
1 & \quad w_{1A} = 1 \\
2 & \quad w_{1B} = s, \quad w_{3A} = s \\
3 & \quad w_{2B} = 1 \\
& \quad w_{2C} = s \\
& \quad w_{3C} = 1
\end{align*}
\]
New Results - 4

Lower Bound

\[ w_{1A} = 1 \]
\[ w_{1B} = s \]
\[ w_{2B} = 1 \]
\[ w_{2C} = s \]
\[ w_{3C} = 1 \]

\[ C(OPT) = 3 \]
New Results - 4

Lower Bound

\[ C(\text{NASH}) = 3s \]
Restricted Identical Setting [HS08]

- $w_1 = 0.4$
- $w_2 = 0.06$
- $w_3 = 0.9$
- $w_4 = 0$
- $w_2 = 1$

$n = 5, m = 3$

$w_i \in [0, 1]$
Restricted Identical Setting - 2

Previous Result

\[ PoA = O\left(\frac{n\sqrt{m}}{w}\right) \quad (w = \sum_i w_i) \quad [HS08] \]

New Result

With the same analysis than general setting, we obtain

\[ \max_{j: c_j^* \neq 0} \frac{c_j}{c_j^*} = O\left(\frac{\log m}{\log \log m}\right) \]

and so

\[ PoA = O\left(\frac{\log m}{\log \log m}\right) \]
Set \( k = \frac{\log m}{\log \log m} \) and \( n = m \left( \lceil k + \frac{2k}{m} \rceil \right) \).

\[ |a| = m \left( \lceil k + \frac{2k}{m} \rceil - 1 \right), \quad w_a = 0 \]

\[ w_b = 1 \]
Set $k = \frac{\log m}{\log \log m}$ and $n = m \left( \lceil k + \frac{2k}{m} \rceil \right)$.

$|a| = m \left( \lceil k + \frac{2k}{m} \rceil - 1 \right)$, $w_a = 0$

$w_b = 1$

$C(OPT) = m + 2$
Set $k = \frac{\log m}{\log \log m}$ and $n = m \left( \lceil k + \frac{2k}{m} \rceil \right)$.

$|a| = m \left( \lceil k + \frac{2k}{m} \rceil - 1 \right)$, $w_a = 0$

$w_b = 1$

$C(NASH) \geq k(m + 2)$
Weighted Total Latency

**Definition**

This objective function minimizes the latency suffered by any unit of jobs’ weight.

\[
\min \sum_i w_{ij} c_i = \min \sum_j c_j^2
\]
Weighted Total Latency - 2

Previous Result - Restricted Weighted Related Setting

\[
2.5 \leq PoA \leq \frac{3 + \sqrt{5}}{2} \approx 2.618 [AAE05] [CFK^+06]
\]

New Result - General Setting

\[
PoA = \Theta(s^2)
\]
Upper Bound

\[ \forall \text{ job } i, \quad c_{ji} \leq c_{j_i}^* + w_{ij_i}^* \implies w_{ij_i} c_{ji} \leq w_{ij_i} (c_{j_i}^* + w_{ij_i}^*) \]

\[ \sum_i w_{ij_i} c_{ji} \leq \sum_i w_{ij_i} (c_{j_i}^* + w_{ij_i}^*) \leq \]

\[ \leq \sum_i w_{ij_i} (c_{j_i}^* + c_{j_i}^*) \leq s \sum_i w_{ij_i}^* (c_{j_i}^* + c_{j_i}^*) \]

\[ C(NASH) = \sum_j c_j^2 \leq s \left( \sum_j c_j^* c_j + \sum_j (c_j^*)^2 \right) = s \left( \sum_j c_j^* c_j + C(OPT) \right) \]
Using the “Cauchy-Schwartz inequality”:

\[ \sum_{j} c_j c_j^* \leq \sqrt{C(NASH)} \cdot \sqrt{C(OPT)} \]

Merging this result in previous statement

\[ C(NASH) \leq s \left[ \left( \sqrt{C(NASH)} \sqrt{C(OPT)} \right) + C(OPT) \right] \Rightarrow \]

\[ \Rightarrow PoA - s \sqrt{PoA} - s \leq 0 \]

Solving this disequation

\[ PoA = O(s^2). \]
Weighted Total Latency - 5

Lower Bound

1. \( w_1A = 1 \)
2. \( w_1B = s \)
3. \( w_3A = s \)
4. \( w_2B = 1 \)
5. \( w_2C = s \)
6. \( w_3C = 1 \)
Lower Bound

\[ w_{1A} = 1 \]
\[ w_{1B} = s \]
\[ w_{2B} = 1 \]
\[ w_{2C} = s \]
\[ w_{3A} = s \]
\[ w_{3C} = 1 \]

\[ C(\text{OPT}) = 3 \]
Weighted Total Latency - 5

Lower Bound

\[ w_1A = 1 \]
\[ w_1B = s \]
\[ w_3A = s \]
\[ w_2B = 1 \]
\[ w_2C = s \]
\[ w_3C = 1 \]

\[ C(NASH) = 3s^2 \]
Conclusions

General setting - Total Latency
- $PoA = O \left( \max \left( s, \frac{\log m}{\log (1 + \frac{\log m}{s})} \right) \right)$
- $PoA = \Omega(s)$
- Open problem: Close the gap.

Restricted Weighted Identical setting - Total Latency
- $PoA = \Theta \left( \frac{\log m}{\log \log m} \right)$
- Our bound is independent from the instance of game.

General setting - Weighted Total Latency
- $PoA = \Theta(s^2)$
Baruch Awerbuch, Yossi Azar, and Amir Epstein.
The price of routing unsplitable flow.

Baruch Awerbuch, Yossi Azar, Yossi Richter, and Dekel Tsur.
Tradeoffs in worst-case equilibria.

Petra Berenbrink, Leslie Ann Goldberg, Paul W. Goldberg, and Russell Martin.
Utilitarian resource assignment.

Ioannis Caragiannis, Michele Flammini, Christos Kaklamanis, Panagiotis Kanellopoulos, and Luca Moscardelli.
Tight bounds for selfish and greedy load balancing.

