How to buy a subgraph: 
game theory meets algorithm design

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Need to motivate the selfish agents to reveal their inputs truthfully.
The Cost of Truth?

• How to elicit truthful inputs?
• Techniques from game theory:
  – auction design, cooperative games, etc.
• Algorithmic mechanism design: constructing mechanisms (outcome selection rule + payment rule) that are:
  – good (outcome close to optimal)
  – computationally efficient
  – truthful
The goal: find a path with the smallest cost.

A complication: edges are owned by agents…
Mechanism Design

• A mechanism is a pair \((Q, M)\):
  – \(Q\) (allocation rule): vector of bids \(\rightarrow S - T\) path
    • does not have to be the shortest path.
  – \(M\) (payment rule): vector of bids \(\rightarrow\) payments
    • does not have to be equal to the bid, but must cover the costs (individual rationality).

Participants know \(Q\) and \(M\) and select their bidding strategies \(\beta_i: v_i \rightarrow b_i\) accordingly.

• Benchmark: payment comparable to the actual cost of the shortest path? … 2\textsuperscript{nd} shortest path?
Candidate Solution I: First-Price Auction

- Pick the **cheapest** path, pay each agent **his** bid
  - deterministic draw resolution rule (e.g., lexicographic)
- Not truthful: edges will **overbid**

Nash equilibrium:
noone wants to deviate given others’ bids
First-Price Auction: Issues (1/2)

• Nash equilibrium may be really bad:

• Neither SY nor YT can bid low enough to win…
First-Price Auction: Issues (2/2)

- Nash equilibrium may not exist:
  - If $b(\text{SYT}) \neq b(\text{SXT})$, winning path wants to bid up
  - If $b(\text{SYT}) = b(\text{SXT}) > 4$, SY and YT lose and would be happy to bid less
Better Solution Concept?

• Even when **NE** exists, players need to know each others’ costs to find it.
• **Dominant strategy**: optimal irrespective of other’s choices.
• Vector of **DS** forms a **NE**, converse is not true
Candidate Solution II: VCG Mechanism

- Allocation rule: pick the cheapest path (in terms of bids).
- Payment rule: a losing edge gets 0, a winning edge gets $t$, where $t$ is the highest bid at which it still wins (threshold bid).

- a can raise its bid to $6 and still win, so a gets $6.
VCG: Good and Bad

- **Good:** each agent’s best strategy is to bid his true cost no matter what everyone else is doing, that is, truth-telling is a dominant strategy (DS).
- **Bad:** huge payments:
  on the graph below, each edge on the upper path can raise its bid by $\delta$, so the total payment is $L+n\delta$.

Can we do better?
Frugality

• How much are we willing to pay?
  – true cost of the shortest path $P$?
  – cost of the 2nd shortest path?
  ✔ – cost of the shortest path in $G \setminus P$? [AT’02]
  ✔ – best Nash in 1st price auction? [KKT’05, EGG’06]

Frugality ratio

\[
\text{Frugality ratio} = \max_{G, c} \frac{c \text{ (shortest path in } G \setminus P \text{) } - c \text{ (shortest path)}}{c} \geq n.
\]

[Archer, Tardos 02]:
for all “nice” DS mechanisms frugality ratio $\geq n$.

[E., Sahai, Steiglitz 04] (this talk):
for all DS mechanisms frugality ratio $\geq n/2$. 
Revelation Principle [M81]

In looking for better DS mechanisms, we can consider truthful mechanisms only.

**Revelation principle**: given a mechanism $T = (Q, M)$ that has dominant strategies, we can design a truthful mechanism $T'$ that for every cost vector $v$ in $R^{|E|}$ chooses the same path and pays the same amounts to all parties.
Truthful Mechanisms: Payment Rules [GHW01]

• Further simplification: have to choose the allocation rule only.

• **Claim:** in a **truthful** mechanism, in which each edge can bid high enough to lose, and **losing** edges are paid **0**, each **winning** agent gets his **threshold bid** (the highest amount he can bid and still win).
Truthful Mechanisms Are Not Frugal

Theorem: for any truthful mechanism on a graph that has two edge-disjoint paths of length n, there is a bid vector s.t. the cheapest path costs $L$, the 2nd cheapest path costs $L + \delta$, and the mechanism pays at least $L + \delta n/2$. 
Each vertex of the bipartite graph corresponds to a vector of bids along $P$ or $Q$: $P_i$ corresponds to bids $(L/n, \ldots, L/n+\delta, \ldots, L/n)$ on $P$.

An arrow from $P_i$ to $Q_j$ means that when edges on $P$ bid according to $P_i$ and edges on $Q$ bid according to $Q_j$, path $P$ wins.
Proof (continued)

- **$P_1$**: vertex with the highest indegree (loses to at least $n/2$ other perturbed paths).
- **Upper** path bids $P_1$, every edge on the lower path bids $L/n$.
- **Total payment**:
  - **blue** edges ($\geq n/2$): threshold bid $\geq L/n + \delta$
  - **other** edges on lower path: at least $L/n$

\[ L + \delta n/2 \]
Remarks

• Upper and lower path can be of different length: \( n \rightarrow \min\{n_1, n_2\} \)

• This costly example can be embedded into any graph with two edge-disjoint paths of suitable length.

• The argument generalizes to randomized mechanisms.
Optimal Bayes-Nash Mechanism

• **New goal:** minimize expected total payments (average-case analysis).

• **New assumption:** distributions of edge costs are public knowledge

• **Setup and Notation:**
  - \( x_i \): cost of the \( i \)th edge;
  - \( \Pr[x_i < t] = F_i(t) \): cumulative density function (CDF).
  - \( f_i(t) = dF_i/dt(t) \): probability density function (PDF).

• **Virtual cost** of the \( i \)th agent: \( c_i(x_i) = x_i + F_i(x_i)/f_i(x_i) \)
  - intuition: “affirmative action”
Optimal Mechanism: Results

- **Theorem:** [ESS 04]: the optimal mechanism picks a path with the smallest *virtual* cost and pays each agent his *threshold* bid.

- **Example:** exponential distribution (\( f(x) = e^{-x} \)).
  - Virtual cost: \( c(x) = e^x + x - 1 \)
  - VCG: \( O(n^{1.5}) \).
  - Optimal mechanism: \( O(n \ln n) \)

\[
E[|c(P)-c(Q)|] = O(n^{0.5})
\]
Simple(?) Way to Reduce Payments

• Running the optimal auction requires computing virtual valuations;
  – can be too slow in practice.
• Is there a simpler way to lower payments?
• Idea [E. 05]: delete some edges and run VCG on the remaining graph.

Edge deletion reduces payments by a factor of $\Omega(n^{1/2})$!
Edge Deletion: Bad News

- Which edges should we delete?
- Finding the best set of edges to delete is NP-hard (and hard to approximate, too)
  - even if edge costs are small constants or
  - the graph is very simple (e.g., series-parallel)
  - but not both
- Also: selecting the best set of edges to delete based on expected edge costs does not work very well...
Graph-Specific Mechanisms?

Recall:

Frugality ratio: \[ \frac{\text{overpayment}}{\max_{G, c} \left( c \right. \text{(shortest path in } G \backslash P \left.) - c \text{ (shortest path)} \right) } \]

Idea [KKT 05]: what if we fix G and let c vary?

Frugality ratio for G: \[ \frac{\text{overpayment}}{\max_{c} \left( c \right. \text{(shortest path in } G \backslash P \left.) - c \text{ (shortest path)} \right) } \]

For many graphs, can design mechanisms that are much better than VCG.
How to Buy a Subgraph

• Frugality ratio for min cost bipartite matching?

Similar reductions for some other problems…
When Is VCG Frugal?

• **Theorem** [Talwar 03, KKT 05]: for Minimum Spanning Tree frugality ratio of VCG is 1
  – i.e., total VCG payment $\leq$ cost of MST in $G \setminus T$.

Idea: charge the payment to each edge in $T$ to the cost of a different edge in $T'$

True for all matroids.
Vertex Cover Auctions (EGG’06)

- Vertices have costs, want to buy a cheap vertex cover
- NP-hard problem → need approximation algorithms
- Approximation does not mix well with threshold payments
  - a winning vertex may lose by bidding lower
  - need monotone algorithms
Conclusions

• Many problems become more difficult if inputs are held by selfish agents.
• Need techniques from both CS and game theory.
• Lots of cool problems.