Buying Cheap is Expensive
Computational Aspects of Unit-Demand Pricing

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Joint work with Piotr in SODA’07 + some other stuff.
Multi-Product Pricing Problems

- Websites comparing available products help customers make optimal buying decisions.

- Customers reveal their preferences and budgets.

**Goal**: use available data to compute optimal pricing schemes for a company’s product range.
Unit-Demand Pricing (UDP-MIN-NPL)

Given:
- \( P \) - set of products, \( C \) - set of consumer samples
- \( b(c, e) \in \mathbb{R}_0^+ \) - budget values

For prices \( p : P \rightarrow \mathbb{R}_0^+ \) let

\[
A(p) = \{ c \in C \mid \exists e \in P : p(e) \leq b(c, e) \},
\]

maximize \( \sum_{c \in A(p)} \min \{ p(e) \mid p(e) \leq b(c, e) \} \).

Price Ladder Constraint (UDP-MIN-PL): predefined order on the prices of all products
Example:

Buying Cheap is Expensive

0,–
+ 8,—
+ 8,—
+ 15,—
+ 17,—
48,—

8,—  8,—  8,—  15,—  17,—
1. Inapproximability of $\text{UDP-MIN}$

2. $\text{UDP-MIN}$ with Uniform Budgets

3. The Rest
Inapproximability of \textsc{Udp-Min}
The **Single-Price Algorithm**:

- Try every budget value as uniform price for all the goods.
- Return best such pricing.

**Theorem [Aggarwal et al., ICALP’04]**

The Single-Price Algorithm has approximation ratio $\ln |C|$. 

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*Patrick Briest*  
*Buying Cheap is Expensive*
Independent Set Problem (Is)

Given undirected graph $G = (V, E)$, $|V| = n$, $|E| = m$, find maximum cardinality subset $V' \subseteq V$ with $\{v, w\} \not\in E$ for any $v, w \in V'$.

Many hardness results, among others hard to approximate within

- $O(n^\varepsilon)$
- $O(\Delta^\varepsilon)$ in graphs of maximum degree $\Delta$

for some $\varepsilon > 0$, unless $P=NP$. 
Independence via geometrically increasing threshold prices:

\[ G = (V, E), \quad V = \{v_1, \ldots, v_n\} \]

\[ \mathcal{P} = \{e_1, \ldots, e_n\}, \quad \mu_j = 1/n^{n-j} \]

\[ \mathcal{V}_j = \{e_j\} \cup \{e_i|\{v_i, v_j\} \in E, i < j\} \]

\[ v_j \sim n^{n-j} \text{ consumers } C_j \text{ with budgets } \mu_i \text{ for all } i \in \mathcal{V}_j, \text{ 0 else} \]
Independence via geometrically increasing threshold prices:

\( G = (V, E), \ V = \{v_1, \ldots, v_n\} \)

\( P = \{e_1, \ldots, e_n\}, \ \mu_j = 1/n^{n-j} \)

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\( v_j \sim n^{n-j} \) consumers \( C_j \) with

budgets \( \mu_i \) for all \( i \in \mathcal{V}_j \), 0 else

\[
\text{rev}(C_j) = 1 \Rightarrow \text{rev}(C_i) \leq 1/n \text{ for all } i \text{ with } e_i \in \mathcal{V}_j \text{ or } e_j \in \mathcal{V}_i
\]

Gadgets allow to encode independence. But graphs of size \( n \) result in instances of size \( \Omega(n^n) \).
Way out: Trade some hardness for sparser problem instances.

Theorem

IS in graphs on $n$ vertices with maximum degree $\Delta(n) = \mathcal{O}(\log n)$ is not approximable within $\mathcal{O}(\log^{\varepsilon} n)$ for some $\varepsilon > 0$, unless $P=NP$.

These graphs are just what we need, because...

Graphs of maximum degree $\Delta$ are $(\Delta + 1)$-colorable.

...and vertices of one color can be realized on one price level.
Theorem

The $\textsc{UDP-MIN-\{PL,NPL\}}$ problem is not approximable within $O(\log^\varepsilon |C|)$ for some $\varepsilon > 0$, unless $\text{NP} \subseteq \text{DTIME}(n^{O(\log \log n)})$. 
Inapproximability of \textsc{Udp-Min}
\textsc{Udp-Min} with Uniform Budgets
The Rest

\textbf{\textsc{Udp-Min} with Uniform Budgets}
Maybe restricting the problem helps...

The **uniform budget** case: Each consumer $c$’s budgets are defined by $(S_c, b_c)$ as:

$$b(c, e) = \begin{cases} b_c, & \text{if } e \in S_c \\ 0, & \text{else} \end{cases}$$

- Encoding independence using threshold prices not possible, since consumers do not distinguish between products in $S_c$.
- Encode some sort of independence using only different price levels?
A combinatorial formulation of *interaction between price levels*:

**Maximum Expanding Sequences (Mes)**

Given a sequence of sets $S_1, \ldots, S_m$, we call the subsequence $\phi = (\phi(1) < \cdots < \phi(\ell))$ *expanding*, if

$$S_{\phi(j)} \not\subseteq \bigcup_{i=1}^{j-1} S_{\phi(i)}$$

for $2 \leq j \leq \ell$. *Mes* asks for a subsequence of maximum length.

Connection between *Mes* and *UDP*: later.

First: Is *Mes* difficult?
Constructing random instances:

Random $\text{Mes}$ instances tend to have long expanding sequences:

- $\mathcal{E}_0 = \emptyset$, $\mathcal{U}(\mathcal{E}_0) = \emptyset$
- $\mathcal{E}_{j+1} = \mathcal{E}_j \cup \{S_{j+1}\}$ if $|\mathcal{U}(\mathcal{E}_j \cup \{S_{j+1}\})| = |\mathcal{U}(\mathcal{E}_j)| + 1$
Let $j \leq n/2$. When is $S_{j+1} \in \mathcal{E}_{j+1}$?

\[
\Pr(S_{j+1} \in \mathcal{E}_{j+1}) = \sum_{e \in U \setminus U(\mathcal{E}_j)} \Pr(e \in S_{j+1}) \prod_{e' \in U \setminus U(\mathcal{E}_j) \cup \{e\}} \Pr(e' \notin S_{j+1})
\]
\[
= \sum_{e \in U \setminus U(\mathcal{E}_j)} \frac{1}{n} \left(1 - \frac{1}{n}\right)^{|U \setminus U(\mathcal{E}_j) \cup \{e\}|}
\]
\[
\geq \frac{n}{2} \frac{1}{n} \left(1 - \frac{1}{n}\right)^{n/2-1} \geq \frac{1}{2 \sqrt{e}}
\]
Balanced Bipartite Independent Set (BBIS)

Given a bipartite graph $G = (V, W, E)$, find maximum cardinality subsets $V' \subset V$, $W' \subset W$ with $|V'| = |W'|$, such that $\{v, w\} \notin E$ for all $v \in V'$, $w \in W'$.

**Theorem**

$Mes$ is hard to approximate within $O(m^\varepsilon)$ for some $\varepsilon > 0$, unless $NP \subseteq \bigcap_{\delta > 0} BPTIME(2^{n^\delta})$. 
Balanced Bipartite Independent Set (BBIS)

Given a bipartite graph $G = (V, W, E)$, find maximum cardinality subsets $V' \subset V$, $W' \subset W$ with $|V'| = |W'|$, such that $\{v, w\} \notin E$ for all $v \in V'$, $w \in W'$.

\[ |V'| \leq b_m \varepsilon' \]

\[ |W'| \leq b_m \varepsilon' \]

\[ \mathcal{U}(\varepsilon|V'|) \]

\[ 2 \cdot |\mathcal{E}| \leq \Theta(m \varepsilon') \]

Theorem

$\text{MES}$ is hard to approximate within $\mathcal{O}(m^\varepsilon)$ for some $\varepsilon > 0$, unless $\text{NP} \subseteq \bigcap_{\delta > 0} \text{BPTIME}(2^{n^\delta})$. 
Reducing Mes to Udp, key idea:

Familiar problem: Instances of (formally) exponential size.
Theorem

**UDP-MIN** with uniform budgets is hard to approximate within \( O(\log^\varepsilon |C|) \) for some \( \varepsilon > 0 \), unless *Random-3SAT* allows (randomized) poly-time refutation algorithms with exponentially small failure probability.

Indicates hardness of envy-free pricing.
The Rest
Has been done:

- **UDP-MAX**: 1.59-approx-algo / APX-hard (PTAS / strong NP-hardness with PL-constraint)
- Interesting cases “between” **MIN** and **MAX**?

<table>
<thead>
<tr>
<th>UDP-MAX</th>
<th>UDP-RAND</th>
<th>UDP-MIN</th>
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<tbody>
<tr>
<td>(tractable)</td>
<td>?</td>
<td>(intractable)</td>
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Needs to be done:

- Hardness of envy-free pricing under standard assumption.
- Tractable realistic models in the **NPL**-scenario.