Linguistic Relevance in Modal Logic

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Abstract

The paper shows how a specific notion of (ir)relevance can elegantly be captured in modal logic. More concretely, the paper puts forth a formal definition of linguistic relevance in terms of logical semantics, and provides an axiomatization of it in modal logic.

1 Introduction

The paper presents a short formal study of the notion of linguistic relevance by means of logical semantics and modal logic. By linguistic relevance we mean the possibility of restricting logical evaluation functions and satisfaction relations to subsets of the non-logical language of the original logic. In other words, relevance—as intended here—has to do with the possibility of abstracting from given parts of the non-logical language. For instance, given a propositional logic theory expressed on the alphabet \{p, q, r\}, one might want to study the logical consequences of the theory by considering atom r irrelevant and hence abstracting from it. The paper shows how modal logic offers an elegant way to capture this phenomenon without resorting to non-standard tools such as partial evaluation functions.

First, in Section 2, the paper formalizes the notion of linguistic (ir)relevance in logical semantics. Then, in Section 3, the formalization is studied from the point of view of modal logic by first relating it to interesting but not very well-known work done on release logics [3,4], and then by providing a sound and complete axiomatization for it. Some conclusions are drawn in Section 4.

2 “In the beginning was the Word”

The present section introduces the problem of linguistic relevance by illustrating an example first introduced by [2] in the context of Deontic Logic.

2.1 Adam & Eve

Consider the propositional language \( L \) built from the alphabet \( P \) of propositional atoms: eat\_apple (“the apple has been eaten”), V (“a violation has occurred”). We have of course four possible models such that: \( w_1 \models \text{eat\_apple} \land V \), \( w_2 \models \text{eat\_apple} \land \neg V \), \( w_3 \models \neg \text{eat\_apple} \land V \) and \( w_4 \models \neg \text{eat\_apple} \land \neg V \). That is, we have the state in which the apple is eaten and there is a violation \( (w_1) \), the state in which the apple is eaten but there is no violation \( (w_2) \), the state where the apple is not eaten and there is a violation \( (w_3) \), and finally the state where no apple is eaten nor there is a violation \( (w_4) \).

Obviously, all these states can be distinguished from each other. But suppose now to compare the models ignoring atom V. Models \( w_1 \) and \( w_2 \) would not be distinguishable any more, nor would states \( w_3 \) and \( w_4 \). Which is just another way to say that, had we used a sublanguage \( L_i \) of \( L \) containing only atom eat\_apple, we would have been able to distinguish only states \( w_1 \) from \( w_3 \) and \( w_2 \) from \( w_4 \). This latter can be considered to be the language at disposal of Adam & Eve in their pre-moral stage, before hearing God commanding “you shall not eat of the fruit of the tree that is in the middle of the garden”—rather than before actually eating the apple. In fact, after hearing
God’s command they were already endowed with the possibility to discern good (~eat,apple) from evil (eat,apple), that is, their language was enriched and they got to distinguish also states \( w_1 \) from \( w_2 \) and \( w_3 \) from \( w_4 \), thanks to the newly introduced notion of violation (\( \forall \)).

2.2 Propositional sublanguage equivalence

The intuitions sketched in the previous section are here made formal. Take two propositional models \( m \) and \( m' \) for a propositional language \( L \). Models \( w \) and \( w' \) are equivalent if they satisfy the same formulae expressible in \( L \): \( w \models \phi \) iff \( w' \models \phi \). If \( w \) and \( w' \) are equivalent \( (w \sim w') \) then there is no set \( \Phi \) of formulae of \( L \) whose models contain \( w \) but not \( w' \), or vice versa. That is to say, the two models are indistinguishable for \( L \). However, two models which are not equivalent with respect to a given alphabet (a given set of atomic propositions), may become equivalent if only a sub-alphabet (a subset of the atomic propositions) is considered.

**Definition 1.** (Propositional sublanguage equivalence) Two models \( w \) and \( w' \) for a propositional language \( L \) are equivalent w.r.t. sublanguage \( L_i \), if they satisfy the same set of formulae expressible using the alphabet of \( L_i \). For any \( \phi \in L_i \): \( w \models \phi \) iff \( w' \models \phi \). If \( w \) and \( w' \) are equivalent w.r.t. \( L_i \), \( (w \sim_i w') \) then they cannot be distinguished by any set \( \Phi \) of formulae of \( L_i \).

The definition makes precise the idea of two propositional models agreeing up to what is expressible on a given alphabet. To put it another way, it formalizes the idea that two models \( w \) and \( w' \) are equivalent if they are indistinguishable by any set \( \Phi \) of formulae of \( L \) whose models contain \( w \) but not \( w' \), or vice versa. That is to say, the two models are indistinguishable for \( L \). However, two models which are not equivalent with respect to a given alphabet (a given set of atomic propositions), may become equivalent if only a sub-alphabet (a subset of the atomic propositions) is considered.

**Proposition 1.** (Properties of \( \sim_i \)) Let \( w \) and \( w' \) be two models for the propositional language \( L \). The following holds:

1. For every sublanguage \( L_i \) of \( L \), relation \( \sim_i \) is an equivalence relation on the set of all models of language \( L \).
2. For all sublanguages \( L_i \) and \( L_j \) of \( L \), if \( L_i \subseteq L_j \) then \( \sim_i \subseteq \sim_j \). It follows that for every sublanguage \( L_i \) of \( L \), \( \sim \subseteq \sim_i \), that is, standard equivalence implies sublanguage equivalence.

**Proof.** Claim (1) is straightforwardly proven. It is easy to see that: identity is a subrelation of \( \sim_i \) for any sublanguage \( L_i \); and that \( \sim_i \circ \sim_i \) and \( \sim_i^{-1} \) are subrelations of \( \sim_i \) for any sublanguage \( L_i \). Claim (2) is proven by considering that, if \( L_i \) is a sublanguage of \( L_j \) and \( m \sim_j m' \), then for all propositions \( \phi \in L_i \): \( m \models \phi \) iff \( m' \models \phi \). Hence, \( w \sim_i w' \). \( \Box \)

3 Logics for Sublanguage Equivalence Relations

The notion of sublanguage equivalence is here studied from the point of view of modal logic.

3.1 Release logic

Propositional release logics (PRL) have been first introduced and studied in [3, 4] in order to provide a modal logic characterization of a general notion of irrelevancy. Irrelevancies are, in short, those aspects which we can choose to ignore. Irrelevancy is represented via modal release operators, specifying what is relevant to the current situation and what can instead be ignored. Release operators are indexed by an abstract ‘issue’ denoting what is considered to be irrelevant for evaluating the formula in the scope of the operator: \( \Delta_I \phi \) means ‘formula \( \phi \) holds in all states where issue \( I \) is irrelevant’, or ‘\( \phi \) holds in all states modulo issue \( I \)’ or ‘\( \phi \) necessarily holds while releasing issue \( I \)’; \( \forall_I \phi \) means ‘formula \( \phi \) holds in at least one of the states where issue \( I \) is irrelevant’, or ‘\( \phi \) possibly holds while releasing issue \( I \)’.

Issues can be in principle anything, but their essential feature is that they yield equivalence relations which cluster the states in the model. An issue \( I \) is conceived as something that determines a partition of the domain in clusters of states which agree on everything but \( I \), or which are
equivalent modulo $I$. Release operators are interpreted on these equivalence relations. As such, propositional release logic can be thought of as a "logic of controlled ignorance" [3]. They represent what we would know, and what we would ignore, by choosing to disregard some issues.

3.1.1 Syntax of PRL

The syntax of PRL is the syntax of a standard multi-modal language $L_n$ [1] where $n$ is the cardinality of the set $\text{Iss}$ of releasable issues. The alphabet of $L_n$ contains: an at most countable set $P$ of propositional atoms $p$; the set of boolean connectives $\{\neg, \land, \lor, \rightarrow\}$; a finite non-empty set $\text{Iss}$ of issues. Metavariables $I, J, \ldots$ are used for denoting elements of $\text{Iss}$. The set of well formed formulae $\phi$ of $L^{\text{Prl}}_i$ is defined by the usual BNF:

$$
\phi :: \top \mid p \mid \neg \phi \mid \phi_1 \land \phi_2 \mid \phi_1 \lor \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \Delta_I \phi \mid \forall I \phi.
$$

where $I$ denotes elements in $\text{Iss}$.

One last important feature of PRL should be addressed before getting to the semantics. We have seen that modal operators are indexed by an issue denoting what is disregarded when evaluating the formula in the scope of the operator. The finite set $\text{Iss}$ of these issues is structured as a partial order, that is to say, $(\text{Iss}, \preceq)$ is a structure on the non-empty set $\text{Iss}$, where $\preceq$ ("being a sub-issue of") is a binary relation on $\text{Iss}$ which is reflexive, transitive and antisymmetric. The aim of the partial order is to induce a structure on the equivalence relations denoting the release of each issue in $\text{Iss}$: if $I \preceq J$ then the clusters of states obtained by releasing $I$ contain the clusters of states obtained by releasing $I$. Intuitively, if $I$ is a sub-issue of $J$ then by disregarding $J$, $I$ is also disregarded. This aspect is made explicit in the models which, for the rest, are just Kripke models.

3.1.2 Semantics of PRL

The semantics of PRL is given via the class $\Psi\text{rl}$ of frames $\mathcal{F} = (\mathcal{W}, \{R_I\}_{I \in \text{Iss}})$ such that $\mathcal{W}$ is a non-empty set of states and $\{R_I\}_{I \in \text{Iss}}$ is a family of equivalence relations such that: if $I \preceq J$ then $R_I \subseteq R_J$.

Models are, as usual, structures $\mathcal{M} = \langle \mathcal{F}, I \rangle$ where $I$ is an evaluation function $I : P \rightarrow \mathcal{P}(\mathcal{W})$ associating to each atom the set of states which make it true. PRL models are therefore just $S5_n$ models with the further constraint that the granularity of the equivalence relations follows the partial order defined on the set of issues: the $\preceq$-smaller is the issue released, the more granular is the partition obtained via the associated equivalence relation. The satisfaction relation is standard. Boolean clauses are omitted.

**Definition 2. (Sat\ nification for PRL models)** Let $\mathcal{M}$ be a PRL model.

$$
\mathcal{M}, w \models \Delta_I \phi \iff \forall w', w R_I w' : \mathcal{M}, w' \models \phi
$$

$$
\mathcal{M}, w \models \forall I \phi \iff \exists w', w R_I w' : \mathcal{M}, w' \models \phi.
$$

where $I \in \text{Iss}$. As usual, a formula $\phi$ is said to be valid in a model $\mathcal{M}$, in symbols $\mathcal{M} \models \phi$, iff for all $w$ in $\mathcal{W}$, $\mathcal{M}, w \models \phi$. It is said to be valid in a frame $\mathcal{F} \in \Psi\text{rl}$ ($\mathcal{F} \models \phi$) if it is valid in all models based on that frame. Finally, it is said to be valid on the class of frames $\Psi\text{rl}$ ($\Psi\text{rl} \models \phi$) if it is valid in every frame $\mathcal{F}$ in $\Psi\text{rl}$.

3.1.3 Axiomatics of PRL

Finally, the axiomatics amounts to a multi-modal $S5$ plus the P0 (partial order) axiom:

(P) all tautologies of propositional calculus

(K) $\Delta_I (\phi_1 \rightarrow \phi_2) \rightarrow (\Delta_I \phi_1 \rightarrow \Delta_I \phi_2)$

(T) $\Delta_I \phi \rightarrow \phi$

(4) $\Delta_I \phi \rightarrow \Delta_I \Delta_I \phi$

(5) $\forall I \phi \rightarrow \Delta I \forall I \phi$

(P0) $\Delta_I \phi \rightarrow \Delta_I \phi$ if $J \preceq I$

(Dual) $\forall I \phi \leftrightarrow \neg \forall I \neg \phi$

(MP) If $\vdash \phi_1$ and $\vdash \phi_1 \rightarrow \phi_2$ then $\vdash \phi_2$

(M') If $\vdash \phi$ then $\vdash \Delta_I \phi$
where \( I, J \in \text{Iss} \). A proof of the soundness and completeness of this axiomatics w.r.t. to the semantics presented in Definition 2 is exposed in [4].

### 3.1.4 PRL with Boolean Algebras

The partial order structure of a PRL logic is reflected in the axiomatics by axiom \( \text{PO} \), and in the semantics by a partial order on the accessibility relations. By adding structure to the partial order on the set of issues more validities can be derived which mirror that structure. Interesting for our purposes is the case when \( \text{Iss} \) is structured according to a Boolean Algebra. The following propositions lists some of the PRL validities holding in that case.

**Proposition 2.** *(Validities of PRL with BA)* Let \( \text{Iss} \) be ordered as \( \langle \text{Iss}, \sqcup, \cap, -, 1, 0, \leq \rangle \), where the structure \( \langle \text{Iss}, \sqcup, \cap, -, 1, 0 \rangle \) is a Boolean Algebra. The following formulae can be derived in PRL:

\[
\begin{align*}
\Delta_I \phi &\rightarrow \Delta_{I'} \phi \\
\Delta_I \phi &\rightarrow \Delta_{0} \phi \\
\Delta_{I'} \cap \Delta_{I''} \phi &\rightarrow (\Delta_I \phi \wedge \Delta_{I'} \phi) \\
\Delta_{I'} \cup \Delta_{I''} \phi &\rightarrow (\Delta_I \phi \wedge \Delta_{I'} \phi) \\
\Delta_I \phi \vee \Delta_{I'} \phi &\rightarrow \Delta_{I'} \phi \\
\Delta_I \phi &\leftrightarrow \Delta_{-I} \phi \\
(\Delta_I \phi &\rightarrow \Delta_{I'} \phi) \leftrightarrow (\Delta_{-I} \phi \rightarrow \Delta_{-I'} \phi)
\end{align*}
\]

*Proof.* The desired derivations are easily obtainable: some (Formulae 1, 2, 3) are just instances of \( \text{PO} \), some (Formulae 5, 6, 7) can be proven by application of \( \text{PO} \) and propositional logic. Formula 4 is derived by applying \( \text{PO} \) and propositional logic. \( \square \)

### 3.2 Sublanguage equivalence in PRL

Logic PRL is a viable tool for reasoning about sublanguage equivalence, and thus about the type of linguistic relevance it captures.

#### 3.2.1 Sublanguage equivalence as release

Reasoning about propositional sublanguage equivalence is an instance of reasoning in release logic.

**Proposition 3.** *(Sublanguage equivalence is a form of release)* Consider a propositional language \( \mathcal{L} \) on the set of atoms \( P \), and a non-empty set of states \( W \). Any evaluation function \( I : P \rightarrow \mathcal{P}(W) \) determines a PRL model \( m = \langle W, \{ \sim \}_{i \in \text{Sub}(\mathcal{L})}, I \rangle \).

*Proof.* It follows from the properties of \( \sim \) proven in Proposition 1. \( \square \)

Notice that the release issues \( \text{Iss} \) are the complements \(-L_i\) of the sublanguages in \( \text{Sub}(\mathcal{L}) \). In fact, what is released is just what cannot be expressed. The accessibility relations should therefore be taken to be the sublanguage-equivalence relations \( \sim \).

Notice also that the set \( \text{Iss} \) is ordered by set-theoretic inclusion \( \subseteq \) between sublanguages of \( \mathcal{L} \). In fact, sets of issues have a natural algebraic structure. Let \( \mathcal{L} \) be a propositional at most countable language, and let us denote with \( \mathcal{L}_i \) any of its sublanguages, i.e., languages defined on a set of atomic propositions \( P_i \subseteq P \), where \( P \) is the set of atomic propositions. Now let \( \text{Sub}(\mathcal{L}) \) be the set of all the sublanguages \( \mathcal{L}_i \) of \( \mathcal{L} \). If we allow \( P_i \) for any \( i \) to be possibly empty, it is immediately clear that the structure \( \langle \text{Sub}(\mathcal{L}), \cup, -, \mathcal{L}, 0 \rangle \) is a set algebra and therefore a Boolean Algebra. Leaving technicalities aside, this just means that by choosing an alphabet, a set of sublanguages is consequently chosen which is structured according to a Boolean Algebra. As a consequence, sublanguage equivalence satisfies also the schemata in Proposition 2.

\(^1\)Obviously, if \( \mathcal{L} \) is infinite then \( |\text{Sub}(\mathcal{L})| > \aleph_0 \). For our purposes, we are typically interested in finite languages.
3.2.2 Adam & Eve in PRL models

Intuitively, what Proposition 3 says is that PRL is a suitable logic to reason about scenarios like the Adam & Eve one sketched in Section 2.1. Let us get back to that example. Now it is possible to represent both the pre- and post-God’s commandment situations, within the same formalism, by making use of the release operators of PRL. Suppose Adam & Eve to be at state $w_1$ in the model with domain $W = \{w_1, w_2, w_3, w_4\}$ and evaluation $I$ as in Section 2.1. Recall that the language was built on atoms $P = \{eat, apple\}$. So let us denote with $\{V\}$ and $\{eat, apple\}$ the sublanguages containing only atom $V$ and, respectively, atom $eat, apple$. These sublanguages represent the releasable issues together with the empty language 0 and the full language 1 = $P$. Let $M = \langle W, \{\sim \}, \sim, I \rangle$ be the resulting release model. We have that:

$$M, w_1 \models eat, apple \land V$$
$$M, w_1 \models \Delta_0(eat, apple \land V)$$
$$M, w_1 \models \Delta_{\{V\}}eat, apple \land \sim \Delta_{\{V\}}V$$

So Formula 8 just states what holds in $w_1$, which is the actual state where Adam & Eve eat the apple committing a violation. Formula 9 does the same by saying that, if you evaluate $eat, apple$ and $V$ after releasing nothing, i.e., by using the full descriptive power of the language, then both $eat, apple$ and $V$ necessarily hold. In fact, in the model at issue the set of states reachable from $w_1$ via $\sim_0$ coincides with $w_1$ itself, since there are no other states in $W$ which are equivalent with $w_1$ if all available atoms are used in the comparison. Hence, in the model at issue, $\Delta_0$ refers to the current evaluation state, i.e., $w_1$. Formula 10 shows what the effects of releasing atom $V$ are. In fact, by abstracting from $V$, state $w_1$ is not distinguishable any more from state $w_2$: $w_1 \sim_V w_2$. Hence there exists a state $w_2 \in W$ such that $M, w_2 \models eat, apple \land \sim V$.

Formulae 10 and 9 represent Adam & Eve’s situation after and, respectively, before God’s commandment “you shall not eat of the fruit of the tree that is in the middle of the garden”. Such commandment introduces a further characterization of reality, exemplified here by the notion of violation, which was not available to Adam & Eve before the commandment was uttered.

3.3 Linguistic release logic: PRL

Although all sublanguage equivalence relations are PRL accessibility relations (Proposition 3), it is easy to see that the reverse does not hold. In a sense the characterization of sublanguage equivalence in terms of PRL is too liberal. This section proposes a logic specifically tailored for talking about sublanguage equivalence w.r.t. a propositional language $L$: linguistic release logic (PRL$^L$ in short).

3.3.1 Syntax of PRL

Linguistic release logic makes use of the same multi-modal language $L_0$ of PRL. The important difference is that the set of issues $Iss \subseteq P(L)$, that is, the issues are sublanguages $L_i$ of a given propositional language $L$. To avoid clutter in the notation, modal operators will be indexed with $i, j, \ldots$ instead of $L_i, L_j, \ldots$.

3.3.2 Semantics of PRL

Let us first precisely define the notion of sublanguage equivalence model, which was somehow already sketched in Proposition 3.

**Definition 3.** (se-models) A sublanguage equivalence model (se-model in short) $m = \langle W, \{\sim_i\}_{i \in \mathbb{N}(L)}, I \rangle$ is a PRL model where each $\sim_i$ is a sublanguage equivalence relation w.r.t the sublanguage $L_i$ of a given propositional language $L$, that is, it is a PRL model where the release accessibility relation $\sim_i$ obeys the constraint: $w \sim_i w'$ iff $\forall p \in L_i: w \models p$ iff $w' \models p$.

In other words, se-models are those PRL models where the accessibility relations and the evaluation function are related in such a way that two states can access one another iff they satisfy
the same propositional atoms. Therefore, the set of se-models built on a $\Psi r l$ frame is a subset of PRL models built on that frame.

The satisfaction relation for these models is defined as follows. Boolean clauses are omitted.

**Definition 4. (Satisfaction for se-models)** Let $M$ be a se-model.

\[ M, w \models \Delta_i \phi \iff \forall w', w \sim_i w' : M, w' \models \phi \]
\[ M, w \models \nabla_i \phi \iff \exists w', w \sim_i w' : M, w' \models \phi. \]

where $i \in Iss = \text{Sub}(L)$. As usual, a formula $\phi$ is said to be valid in a se-model $M$, in symbols $M \models \phi$, iff for all $w$ in $W$, $M, w \models \phi$. It is said to be valid in a frame $F \in \Psi r l (F \models \phi)$ if it is valid in all se-models based on that frame. Finally, it is said to be valid on the class of frames $\Psi r l (\Psi r l \models \phi)$ if it is valid in every frame $F$ in $\Psi r l$.

Intuitively, $\Delta_i \phi$ means that $\phi$ holds in all states that are equivalent to the evaluation state up to sublanguage $-L_i$ (or, that can be reached by releasing sublanguage $L_i$), and $\nabla_i \phi$ means that $\phi$ holds in at least one state which is equivalent to the evaluation state up to sublanguage $-L_i$ (or, that can be reached by releasing sublanguage $L_i$).

### 3.3.3 Axiomatics of PRL$^L$

The point is to find, given a language $L$ an axiomatics which is sound and complete w.r.t. the class of se-models built on the class of $\Psi r l$ frames. The axiomatics extends the axiomatics of PRL with axiom NoCross:

\[(P)\] all tautologies of propositional calculus
\[(K)\] $\Delta_i(\phi_1 \rightarrow \phi_2) \rightarrow (\Delta_i \phi_1 \rightarrow \Delta_i \phi_2)$
\[(T)\] $\Delta_i \phi \rightarrow \phi$
\[(4)\] $\Delta_i \phi \rightarrow \Delta_i \Delta_i \phi$
\[(5)\] $\nabla_i \phi \rightarrow \Delta_i \nabla_i \phi$
\[(PO)\] $\Delta_i \phi \rightarrow \Delta_i \phi$ if $j \leq i$
\[(\text{NoCross})\] $\Delta_i p \lor \Delta_i \neg p$ if $p \notin i$
\[(\text{Dual})\] $\nabla_i \phi \leftrightarrow \neg \Delta_i \neg \phi$
\[(\text{MP})\] If $\vdash \phi_1$ and $\vdash \phi_1 \rightarrow \phi_2$ then $\vdash \phi_2$
\[(\text{N}^1)\] If $\vdash \phi$ then $\vdash \Delta_i \phi$

where $i, j \in Iss$. The idea behind the new axiom is that, if an atom $p$ does not belong to a sublanguage $i \in \text{Sub}(L)$ (i.e., if $p$ belongs to the complement of $i$), then all worlds which are accessible by releasing sublanguage $i$ must either satisfy $p$ or $\neg p$. Technically, NoCross forces the release accessibility relation not to cross the bipartitions of the domain $W$ yielded by each atom $p$, when $p$ does not belong to the released language. This could be considered to be the distinctive feature of linguistic release with respect to other forms of release\(^2\). A proof of the soundness and completeness of this axiomatics w.r.t. the class of se-models built on $\Psi r l$ frames (Definition 4) is provided in Appendix A.

### 4 Conclusions and Future Work

The contribution of the paper consisted in the formalization of linguistic (ir)relevance in terms of propositional sublanguage equivalence (Section 2), its study as an instance of propositional release logic (Section 3.2), and its axiomatization in modal logic (PRL$^L$) which we called linguistic release logic (Section 3.3).

A future research line which is worth pursuing is the study of linguistic release logic in combination with other modal operators, in particular epistemic and doxastic ones. This could clarify

\(^2\)The reader is referred to [3, 4]
interesting epistemic phenomena such as the so-called “unknown-unknowns”\textsuperscript{3}. Getting back to the Adam & Eve example, and stretching it a little bit, we can see that before God’s commandment it was an “unknown-unknown” for both Adam and Eve that eating the apple counted as a sin. In other words, neither did they know that eating the apple was a sin, nor did they know that they did not know that eating the apple was a sin. We think that logic $\text{PRL}^L$ in combination with $\text{S5}$ could provide interesting insights on this issue.

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**References**


### A Linguistic Release Logic is Sound and Complete

The proof of soundness is routinary. It is well-known that inference rules $\text{MP}$ and $\text{N}$ preserve validity on any class of frames, and that axioms $\text{T}$, $\text{4}$ and $\text{5}$ are valid on models built on equivalence relations\textsuperscript{4}. Providing the soundness of $\text{PRL}^L$ w.r.t. the class of se-models built on $\text{Prl}$ frames boils down to checking the validity of axioms $\text{P0}$ and $\text{NoCross}$.

**Theorem 1.** (Soundness of $\text{PRL}^L$ w.r.t. $\text{Prl}^L$ models) Logic $\text{PRL}^L$ is sound w.r.t. the class $\text{Prl}^L$ of models, i.e., if $\vdash_{\text{PRL}^L} \phi$ then $\models_{\text{Prl}^L} \phi$.

**Proof.** Validity of $\text{P0}$. Suppose axiom $\text{P0}$ is not valid. This means that $\exists M \in \text{Prl}^L$ s.t. for at least one world $w$, it holds that $j \leq i$ and $M, w \models \Delta_i \phi \land \neg \Delta_i \phi$, that is: $\forall w'$ such that $w \sim_j w'$ $M, w' \models \phi$ but $\exists w''$ s.t. $w \sim_j w'$ and $M, w'' \not\models \phi$. This contradicts Proposition 1 since $L_{-j} \subseteq L_{-i}$. Validity of $\text{NoCross}$. If $p \in i$, then $p \in i$, hence $\forall w, w'$ s.t. $w \sim_j w'$ we have that $M, w' \models p$ if $M, w \models p$ by Definition 1. It follows that $\forall w$ either $M, w \models \Delta p$ or $M, w \models \Delta \neg p$. \hfill $\Box$

The proof of completeness makes use of the standard canonical model technique. Notice that we have to deal with completeness w.r.t. to the class of se-models which can be built on the class $\text{Prl}$ of propositional release logic frames.

**Lemma 1.** (Redefining strong completeness for $\text{PRL}^L$) Logic $\text{PRL}^L$ is strongly complete w.r.t. the class of frames $\text{Prl}$ iff every $\text{PRL}^L$-consistent set $\Phi$ of formulae is satisfiable on some se-model built on a frame in class $\text{Prl}$.

\textsuperscript{3}We thank Martin Caminada for pointing us to this notion. “Unknown unknowns” have obviously to do with the failure of negative introspection.

\textsuperscript{4}See [1].
Proof. From right to left we argue by contraposition. If $\text{PRL}^L$ is not strongly complete w.r.t. the class $\mathfrak{Prl}$ then there exists a set of formulae $\Phi \cup \{\phi\}$ s.t. $\Phi \models_{\text{Pr}l} \phi$ and $\Phi \not\models_{\text{PRL}^L} \phi$. It follows that $\Phi \cup \{\neg \phi\}$ is $\text{PRL}^L$-consistent but not satisfiable on any sublanguage equivalent model built on a frame in class $\mathfrak{Pr}l$. From left to right we argue per absurdum. Let us assume that $\Phi \cup \{\neg \phi\}$ is $\text{PRL}^L$-consistent but not satisfiable in any sublanguage equivalent model built on a frame in class $\mathfrak{Pr}l$. It follows that $\Phi \models_{\text{Pr}l} \phi$ and hence $\Phi \cup \{\neg \phi\}$ is not $\text{PRL}^L$-consistent, which is impossible. \[\square\]

Now let $M_{\text{PRL}^L}$ be the canonical model of logic $\text{PRL}^L$ in the multi-modal language $\mathcal{L}_\mathfrak{P}$. Model $M_{\text{PRL}^L}$ is the structure $\langle W_{\text{PRL}^L}, \{R_i\}_{1 \leq i \leq n}, I_{\text{PRL}^L} \rangle$ where:

1. The set $W_{\text{PRL}^L}$ is the set of all maximal $\text{PRL}^L$-consistent sets.
2. The canonical relations $\{R_i\}_{1 \leq i \leq n}$ are defined as follows: for all $w, w' \in W_{\text{PRL}^L}$, if for all formulae $\phi, \psi \in w'$ implies $\implies_{\text{Pr}l} \phi \in w$, then $w R_{\text{PRL}^L} w'$.
3. The canonical interpretation $I_{\text{PRL}^L}$ is defined by $I_{\text{PRL}^L}(p) = \{w \in W_{\text{PRL}^L} \mid p \in w\}$.\[\square\]}

Lemma 2. (Existence lemma) For all states in $W_{\text{PRL}^L}$, if $\forall \phi \in w$ then there exists a state $w' \in W_{\text{PRL}^L}$ s.t. $R_{\text{PRL}^L}(w, w')$ and $\phi \in w'$.

Proof. The claim is proven by construction. Assume $\forall \phi \in w$ and let $w_0 = \{\phi\} \cup \{\psi \mid \Delta \psi \in w\}$. Set $w_0'$ must be consistent otherwise there would exist $\psi_1, \ldots, \psi_m \in w_0'$ such that $\vdash_{\text{PRL}^L} (\psi_1 \land \ldots \land \psi_m) \rightarrow \neg \phi$, from which we obtain $\vdash_{\text{PRL}^L} (\Delta \psi_1 \land \ldots \land \Delta \psi_m) \rightarrow \Delta \neg \phi$. Since $\Delta \psi_1, \ldots, \Delta \psi_m \in w$ we have that $\neg \forall \phi \in w$, which contradicts our assumption. Therefore, $w_0'$ is consistent and can be extended to a maximal $\text{PRL}^L$-consistent set (for Lindenbaum’s Lemma\(^5\)). By construction, $w'$ contains $\phi$ and is such that for all $\psi_i$, if $\Delta \psi_i \in w$ then $w'$ contains $\psi_i$. From this it follows $R_{\text{PRL}^L}(w, w')$ since, if this was not the case, then there would exist a formula $\psi'$ s.t. $\psi' \in w'$ and $\forall \psi' \notin w$. Since $w$ is maximal $\text{PRL}^L$-consistent, $\Delta \neg \psi' \in w$ and hence $\neg \psi' \in w'$, which contradicts the $\text{PRL}^L$-consistency of $w'$. \[\square\]

Lemma 3. (Truth lemma) For any formula $\phi$: $M_{\text{PRL}^L}. w \models \phi$ iff $\phi \in w$.

Proof. The claim is proven by induction on the complexity of $\phi$. The Boolean case follows by the properties of maximal $\text{PRL}^L$-consistent sets. As to the modal case, it follows from the definition of the canonical relations $R_{\text{PRL}^L}$ and Lemma 2. \[\square\]

Everything is now put into place to prove the strong completeness of $\text{PRL}^L$.

Theorem 2. (Completeness of $\text{PRL}^L$ w.r.t $\mathfrak{Pr}l$) Logic $\text{PRL}^L$ is strongly complete w.r.t. the class $\mathfrak{Pr}l$ of frames, i.e., if $\Phi \models_{\text{Pr}l} \phi$ then $\Phi \vdash_{\text{PRL}^L} \phi$.

Proof. By Proposition 1, given a $\text{PRL}^L$-consistent set $\Phi$ of formulae, it suffices to find a model state pair $(M, w)$ such that: (a) $M, w \models \Phi$, (b) $M$ is an s-model. Let $M_{\text{PRL}^L} = \langle W_{\text{PRL}^L}, \{R_i\}_{1 \leq i \leq n}, I_{\text{PRL}^L} \rangle$ be the canonical model of $\text{PRL}^L$, and let $\Phi^*$ be any maximal consistent set in $W_{\text{PRL}^L}$ extending $\Phi$. By Lemma 3 it follows that $M_{\text{PRL}^L}, \Phi^* \models \Phi$, which proves (a). To prove (b), we show that $M_{\text{PRL}^L}$ is s.t.: (b.1) the frame $\mathcal{F}$ on which $M$ is based is a $\mathfrak{Pr}l$ frame; and (b.2) $R_{\text{PRL}^L}(w, w')$ iff $\forall p \in \mathcal{L}_{\mathfrak{P}r}$. $p \in w$ iff $p \in w'$. As to (b.1), it is well-known that axioms T, 4 and 5 force the relations $R_{\text{PRL}^L}$ to be equivalence relations. It remains to be shown that if $i \leq j$ then $R_{\text{PRL}^L} \subseteq R_{\text{PRL}^L}^{i+j}$. Assume $R_{\text{PRL}^L}^{i+j}(w, w')$ and $\Delta \phi \in w$. It follows that $\phi \in w'$ and, for axiom P0, that $\Delta \phi \in w$ and, for axiom T that $\forall_p \phi \in w$. Therefore, $R_{\text{PRL}^L}(w, w')$.

As to (b.2), form left to right. Assume $R_{\text{PRL}^L}^{i+j}(w, w')$. For any $p \in \mathcal{L}_{\mathfrak{P}r}$ if $p \in w$, then for axiom NoCross $p \notin w'$. If $p \notin w$ then, for axiom NoCross $p \notin w'$. From right to left. Assume that for all $p \in \mathcal{L}_{\mathfrak{P}r}$, $p \in w$ iff $p \in w'$. Suppose that $p \notin w'$. For the maximal $\text{PRL}^L$-consistency of $w$ it holds that $\forall \phi \in w$. Hence, if $p \in w$ then $\forall \phi \in w$, and therefore $R_{\text{PRL}^L}(w, w')$. This completes the proof. \[\square\]

\(^5\)See [1]