5

MODEL THEORY OF MODAL LOGIC

Valentin Goranko and Martin Otto

INTRODUCTION

Model theory is about semantics; it studies the interplay between a logical language (logic) and the models (structures) for that language. Key issues therefore are *expressiveness* and *definability*. At the basic level these concern the questions which structural properties are expressible and which classes of structures are definable in the logic. These basic questions immediately lead to the study of *model constructions*; to the *analysis of models and of model classes* for given formulae or theories; to notions of *equivalence between structures* with respect to the truth of formulae; and to the study of preservation phenomena.

Modal logics[1] come as members of a loosely knit family and have various links to other logics – classical first- and second-order logic as well as, for instance, temporal and process logics stemming from particular applications. Correspondingly, the key issues mentioned above may also be studied comparatively, both within the family and in relation to other relevant logics. Such a comparative view can support an understanding of the internal coherence of the rich family of modal logics. It also offers a perspective to place modal logics in the wider logical and model theoretic context.

In regard to the coherence of the family of modal logics, it is important to understand in model theoretic terms what it is that makes a logic 'modal'. For that aim we devote a major part of this chapter to the discussion of *bisimulation*. Many other features of the 'modal character' can be understood in terms of bisimulation invariance; this is true most notably of the *local* and *restricted nature of quantification*. Due to these features modal logic enjoys very specific features, and in many respects its model theory can be developed along lines that have no direct counterparts in classical model theory.

In regard to the wider logical context, there is a rich body of classical work in modal model theory that measures modal logic against the backdrop of classical first- and second-order logic into which it can be naturally embedded. But, beside this 'classical picture', there are also many links with other logics, partly designed for other purposes or studied with a different perspective from that of classical model theory.

In the classical picture, both first- and second-order logic have their role to play. This is because modal logic actually offers several distinct semantic levels, as will be reviewed in the following section which provides an introduction to the model theoretic semantics of modal logic. So, a modal formula is traditionally viewed in four different ways, subject to two orthogonal dichotomies – *Kripke structures* (also called *Kripke models*) versus *Kripke frames* and *local* versus *global*.

The fundamental semantic notion in basic modal logic is *truth of a formula at a state in a Kripke structure*; this notion is *local* and of a *first-order* nature. Semantics in Kripke frames is obtained, if instead one looks at all possible propositional valuations

[1]In this chapter we use the term *modal logic* (despite the established tradition in the literature on modal logic) in a typical model-theoretic sense, as a (propositional) modal language equipped with suitable relational (Kripke) semantics, rather than proof systems over such languages, determined by a set of axioms and inference rules, such as K, S4, etc. We refer to the latter as 'axiomatic extensions'.

over the given frame (in effect an abstraction through implicit universal second-order quantification over all valuations); this semantics, accordingly, is of essentially *second-order* nature. On the other hand, the passage from local to *global* semantics is achieved if one looks at truth in all states (an abstraction through implicit universal first-order quantification over all states).

While all these semantic levels are ultimately based on the local semantics in Kripke structures, the two independent directions of generalisation, and in particular the divide between the (first-order) Kripke structure semantics and the (second-order) frame semantics, give rise to very distinct model theoretic flavours, each with their own tradition in the model theory of modal logic. Still, these two semantics meet through the notion of a *general frame* (closely related to a *modal algebra*).

History. The origins of model theory of modal logic go back to the fundamental papers of Jónsson and Tarski [78, 79], and Kripke [86, 87] laying the foundations of the *relational (Kripke) semantics*, followed by the classical work of Lemmon and Scott [91].

Some of the most influential themes and directions of the classical development of the model theory of modal logic in the 1970/80s have been: the *completeness theory* of modal axiomatic systems with respect to the frame-based semantics of modal logic, and the closely related *correspondence theory* between that semantics and first-order logic [117, 28, 123, 124, 113, 42, 51, 125, 127, 128]; and the *duality theory* between Kripke frames and modal algebras, via general frames [42, 43, 44, 45, 114]. Also at that time, the *theory of bisimulations and bisimulation invariance* emerged in the semantic analysis of modal languages in [125, 128]. For detailed historical and bibliographical notes see [5], and the survey [49] for a recent and comprehensive historical account of the development of modal logic, and in particular its model theory.

Overview. The sections of this chapter are roughly arranged in three parts or main tracks, reflecting the semantic distinctions outlined above.

The first part provides a common basic introduction to some of the key notions, in particular the different levels of semantics in section 1, followed by the concept of bisimulation and bisimulation respecting model constructions in section 2. This more general thread is taken up again in section 6 with some more advanced model constructions, and also in the final section 9 devoted to some ideas in the finite model theory of modal logic.

A second track, comprising sections 3 to 5, is primarily devoted to modal logic as a logic of Kripke structures (first-order semantics): section 3 continues the bisimulation theme; section 4 is specifically devoted to the role of modal logic as a fragment of first-order logic; section 5 illustrates some of the richness of modal logics over Kripke structures in terms of variations and extensions.

The third track is devoted to a study of modal logic as a logic of frames (the second-order semantics). This comprises more advanced constructions such as ultrafilter extensions and ultraproducts in section 6, basic model theory of general frames in section 7, and a survey of classical results on frame definability and relations with second-order logic in section 8.

Most of the other chapters in this handbook supplement this chapter with important model-theoretic topics and results. In particular, we refer the reader to Chapters 1, 3, 6, 7 and 8.