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ALGEBRAS AND COALGEBRAS

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1 INTRODUCTION

Modal logic is not an isolated field. When studied from a mathematical perspective, it has evident connections with many other areas in logic, mathematics and theoretical computer science. Other chapters of this handbook point out some of the links between modal logic and areas like (finite) model theory or automata theory. Here we will outline the *algebraic* and *coalgebraic* environments of the theory of modal logic.

First we approach modal logic with the methodology of *algebraic logic*, a discipline which aims at studying all kinds of logics using tools and techniques from universal algebra — in fact, much of the theory of universal algebra was developed in tandem with that of algebraic logic. The idea is to associate, with any logic L, a class $\mathsf{Alg}(L)$ of algebras, in such a way that (natural) logical properties of L correspond to (natural) algebraic properties of $\mathsf{Alg}(L)$. Carrying out this program for modal logic, we find that normal modal logics have algebraic counterparts in varieties of *Boolean algebras with operators* (BAOs). In the simplest case of monomodal logics, the algebras that we are dealing with are simply *modal algebras*, that is, expansions of Boolean algebras with a single, unary operation that preserves finite joins (disjunctions). One advantage of the algebraic semantics over the relational one is that it allows a general *completeness* result, but the algebraic approach may also serve to prove many significant results concerning properties of modal logics such as completeness, canonicity, and interpolation. As we will see, a crucial observation in the algebraic theory of modal logic is that standard algebraic constructions correspond to well-known operations on Kripke frames. These correspondences can be made precise in the form of categorical *dualities*, which may serve to explain much of the interaction between modal logic and universal algebra. Our discussion of the algebraic approach towards modal logics takes up the sections 3 to 8.

The *coalgebraic* perspective on modal logic is much more recent (see section 9 for references). Coalgebras are simple but fundamental mathematical structures that capture the essence of dynamic or evolving systems. The theory of universal coalgebra seeks to provide a general framework for the study of notions related to (possibly infinite) behavior such as invariance, and observational indistinguishability. When it comes to modal logic, an important difference with the algebraic perspective is that coalgebras *generalize* rather than *dualize* the model theory of modal logic. Many familiar notions and constructions, such as bisimulations and bounded morphisms, have analogues in other fields, and find their natural place at the level of coalgebra. Perhaps even more important is the realization that one may generalize the concept of modal logic from Kripke frames to arbitrary coalgebras. In fact, the link between (these generalizations of) modal logic and coalgebra is so tight, that one may even claim that modal logic is the natural logic for coalgebras — just like equational logic is that for algebra. The second and last part of this chapter, starting from section 9, is devoted to coalgebra.

What is the point of taking such an abstract perspective on modal logic, be it algebraic or coalgebraic? Obviously, making the above kind of mathematical generalizations, one should not aim at solving all concrete problems for specific modal logics. Rather, the approach may serve to isolate those aspects of a problem that are easy in the sense of being solvable by general means; it thus enables us to focus on the remaining aspects that are specific to the problem at hand. To give an example, it is certainly not the case that all modal formulas are canonical, but Sahlqvist's theorem considerably simplifies completeness proofs by taking care of the canonical part of the axiomatization. A second benefit of embedding modal logic in its mathematical context is that it may lead to a better understanding of notions from modal logic. Taking an example from coalgebra, the notion of a bounded morphism between Kripke models (or frames), becomes much more natural once we understand that it coincides with the natural coalgebraic notion of a homomorphism.

Our main aim with this chapter is to give the reader an impression of both the algebraic and the coalgebraic perspective on modal logic. Our focus will be on concepts and ideas, but we will also mention important techniques and landmark results; proofs, or rather proof sketches, are given as much as possible. Despite its over-average length, a text of this size cannot come close to being comprehensive; our main selection criterion has been to focus on *generality* of methods and results. Unfortunately, even some important topics have fallen prey to this, most particularly, the *algebras of relations*, even though they played and continue to play a crucial role in the history of algebraic logic. Fortunately, these kinds of BAOs are well documented elsewhere, see for instance HENKIN, MONK & TARSKI [57] for cylindric algebras, or HIRSCH & HODKINSON [58] for relation algebras. A second topic receiving only fragmented attention is *historical context*. While we do attribute results as much as possible, readers with an interest in the (fascinating!) history of modal logic, will not find much to suit their taste here. Rather, they should consult GOLDBLATT [44], or perhaps the historical notes of BLACKBURN, DE RIJKE & VENEMA [13]. Finally, a warning: in this chapter we assume familiarity with basic notions from category theory (such as functors, duality), universal algebra (such as congruences, free algebras), and more specifically, Boolean algebras. Readers encountering unfamiliar concepts in this chapter are advised to consult some text book in universal algebra or category theory. For convenience, in an appendix we have summed up all the material that we consider to be background knowledge.

2 BASICS OF MODAL LOGIC

In this section we briefly review the basic definitions of modal logic. Starting with syntax, we take a fairly general approach towards modal languages and allow modal connectives of arbitrary finite rank. A *modal similarity type* is a set τ of modal connectives, together with an arity function $ar : \tau \to \omega$ assigning to each symbol $\nabla \in \tau$ a *rank* or *arity* $ar(\nabla)$. Given a modal similarity type τ and a set X of variables we inductively define the set $Fma_\tau(X)$ of *modal τ-formulas in X* by the following rule:

$$\varphi ::= x \in X \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \nabla(\varphi_1, \ldots, \varphi_n)$$

with $\nabla \in \tau$ and $n = ar(\nabla)$. We will use standard abbreviations such as $\to$ and $\leftrightarrow$; we also define the *dual* operator Δ of $\nabla \in \tau$ as $\Delta(\varphi_1, \ldots, \varphi_n) := \neg\nabla(\neg\varphi_1, \ldots, \neg\varphi_n)$. Unary modalities are usually called *diamonds*, and their duals, *boxes*; to denote these modalities we reserve (possibly indexed) symbols of the shape $\Diamond$ and $\Box$, respectively.

Throughout this chapter we will work with an arbitrary but fixed modal similarity type τ. Often, we will provide proofs only for the *basic modal similarity type* which consists of a single diamond that will always simply be denoted as $\Diamond$ (its dual as $\Box$). Unless explicitly stated otherwise, we are always dealing with a fixed, countably infinite set X of variables; in order not to clutter up notation we will suppress explicit references to X as much as possible.