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MODAL CONSEQUENCE RELATIONS

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1 INTRODUCTION

Logic is generally defined as the science of reasoning. Mathematical logic is mainly concerned with forms of reasoning that lead from true premises to true conclusions. Thus we say that the argument from $\sigma_0; \sigma_1; \cdots; \sigma_{n-1}$ to δ is *logically correct* if whenever σ_i is true for all $i < n$, then so is δ. In place of 'argument' one also speaks of 'inference'. The language object '$\sigma_0; \sigma_1; \cdots; \sigma_{n-1}/\delta$' is is called a *rule*, of which arguments are instances. A rule is *valid* if all its instances are. Central to this approach is the notion of a *consequence relation*, which is a relation between sets of formulae and formulae. A consequence relation $\vdash$ specifies which arguments are valid; the argument from a set Σ to a formula δ is valid in $\vdash$ iff $\langle \Sigma, \delta \rangle \in \vdash$, for which we write $\Sigma \vdash \delta$. δ is a *tautology* of $\vdash$ if $\varnothing \vdash \delta$, for which we also write $\vdash \delta$.

In the early years, research into modal logic was concerned with the question of finding the correct inference rules. This research line is still there but has been marginalized by the research into modal *logics*, where a logic is just a set of formulae; this set is the set of tautologies of a certain consequence relation, but many consequence relations share the same tautologies. The shift of focus in the research has to do in part with the precedent set by predicate logic: predicate logic is standardly axiomatized in a Hilbert-style fashion, which fixes the inference rules and leaves only the axioms as a parameter. Another source may have been the fact that there is a biunique correspondence between varieties of modal algebras and axiomatic extensions of K, which allowed for rather deep investigations into the space of logics, using the machinery of equational theories. This research led to deep results on the structure of the lattice of modal logics and benefits also the research into consequence relations. Recently, however, algebraic logic has provided more and more tools that allow to extend the algebraic method to the study of consequence relations in general (see for example [60] and [14]). In particular the investigations into the Leibniz operator initiated by Blok and Pigozzi in [5] have brought new life into the discussion and allow to see a much broader picture than before.

Now, even if one is comfortable with classical logic, it is not immediately clear what the correct inferences are in modal logic. The first problem is that it is not generally agreed what the meaning of the modal operator(s) is or should be. In fact, rather than a drawback, the availability of very many different interpretations is the strength of modal logic; it gives flexibility, however at the price that there is not one modal logic, there are uncountably many. For example, $\Box$ as metaphysical necessity satisfies **S5**, $\Box$ as provability in **PA** satisfies **G**, $\Box$ as future necessity (arguably) satisfies **S4.3**, and so on. This is in part because the interpretation decides which algebras are suitable (intended) and which ones are not. However, there is another parameter of variation, and this is the notion of truth itself. In the most popular interpretation, truth is truth at a world; but we could also understand it as truth in every world of the structure. The two give rise to two distinct consequence relations, the *local* and the *global*, which very often do not coincide even though they always have the same set of tautologies. If truth is defined to be truth at every world under all substitutions we finally arrive at the maximal consequence relation compatible with a logic, in which a rule is derived iff it is admissible for that logic. It is this plurality of interpretations that gives rise to the different topics of this contribution and provides the underlying thread that connects them.

The paper is organised as follows. We shall first review basic concepts from universal algebra and basic logical notions such as consequence relations, rules, the deduction theorem and interpolation; then we shall briefly look at modal consequence relations and the structure of the lattice they form; finally, we turn to the notion of a splitting. This concludes Section 2. In Section 3 we shall look at local and global consequence relations. The first part will deal with consequence relations from an algebraic perspective; the second part studies global consequence relations in more detail and the third part outlines the connection between semisimple varieties of modal algebras and weak transitivity. The next section deals with reductions of polymodal and polyadic modal logics to monomodal logic. It reviews results that establish that the lattices of polymodal and polyadic logics can be naturally embedded into the lattice of monomodal logics preserving and reflecting a good deal of properties. This justifies ex post the almost exclusive study of monomodal logics in spite of the practical usefulness of polymodal and polyadic logics. Section 5 looks at interpolation. In detail, it shall give an algebraic characterisation of interpolation and ways of establishing interpolation for logics. Next we shall look at Beth-definability and fixed point theorems and finally at uniform interpolation. Section 6 is devoted to admissible rules. In particular, it deals with questions of axiomatisability of the set of admissible rules, and with the problem of deciding whether a given rule is admissible in a logic. Finally, in Section 7 we take a brief look at more general notions of a rule, like multiple conclusion rules.

2 BASIC THEORY OF MODAL CONSEQUENCE RELATIONS

This chapter makes heavy use of notions from universal algebra. The reader is referred to Chapter 6 for background information concerning universal algebra and in particular the theory of BAOs and how they relate to (general) frames. We shall quickly review some terminology. A **signature** is a pair $\langle F, \nu \rangle$, where F is a set of so-called **function symbols** or **connectives** and $\nu : F \to \omega$ a function assigning to each symbol an arity. **Terms** are expressions of this language based on variables. We shall also refer to ν alone as a signature. We shall assume that the reader is acquainted with basic notions of universal algebra, such as a ν-**algebra**. Given a map $v : X \to A$ from a set X of variables into the underlying set of A, there is at most one homomorphic extension $\overline{v} : \mathfrak{Tm}_\nu(X) \to \mathfrak{A}$, where $\mathfrak{Tm}_\nu(X)$ denotes the algebra of terms in the signature ν over the set X (whose underlying set is $\mathrm{Tm}_\nu(X)$). On a ν-algebra $\mathfrak{A}$, terms induce **term functions** in the obvious way. If we allow to expand the signature by a constant $\underline{a}$ for every $a \in A$, the term functions induced by this enriched language on $\mathfrak{A}$ are called **polynomials**. In what is to follow, terms will also be called **formulae**, F will always contain $\top$, $\wedge$ and $\neg$, and $\nu(\top) = 0$, $\nu(\neg) = 1$ and $\nu(\wedge) = 2$. Moreover, F will additionally contain connectives $\Box_i$, $i < \kappa$, called **modal operators**, which are unary unless otherwise stated. κ need not be finite. The relation corresponding to $\Box_i$ will standardly be denoted by $\lhd_i$. The set of variables is $V := \{p_i : i \in \omega\}$. Sets of formulae are denoted in the usual way using the semicolon notation: $\Delta; \chi$ abbreviates $\Delta \cup \{\chi\}$. We write $\mathrm{var}(\varphi)$ for the set of variables occurring in φ, and $\mathrm{sf}(\varphi)$ for the set of subformulae of φ. Similarly, $\mathrm{var}(\Delta)$ and $\mathrm{sf}(\Delta)$ are used for sets of formulae. A **substitution** is defined by a map $s : V \to \mathrm{Tm}_\nu(V)$. $s(\varphi)$ or φ^s denotes the effect on φ of performing the substitution s.