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Willem Blok and modal logic

Abstract. We present our personal view on W.J. Blok's contribution to modal logic.

Introduction

Willem Johannes Blok started his scientific career in 1973 as an algebraist with an investigation of varieties of interior (a.k.a. closure or topological Boolean or topoboolean) algebras. In the years following his PhD in 1976 he moved on to study more general varieties of modal algebras (and matrices) and, by the end of the 1970s, this algebraist was recognised by the modal logic community as one of the most influential modal logicians.

Wim Blok worked on modal algebras till about 1979–80, and over those seven years he obtained a number of very beautiful and profound results that had and still have a great impact on the development of the whole discipline of modal logic.

In this short note we discuss only three areas of modal logic where Wim Blok's contribution was, in our opinion, the most significant:

1. splittings of the lattice of normal modal logics and the position of Kripke incomplete logics in this lattice [3, 4, 6],

2. the relation between modal logics containing **S4** and extensions of intuitionistic propositional logic [1],

3. the position of tabular, pretabular, and locally tabular logics in lattices of modal logics [1, 6, 8, 10].

As usual in mathematics and logic, it is not only the results themselves that matter, but also the power and beauty of the methods developed to obtain them. In this respect, one of the most important achievements of Wim Blok was a brilliant demonstration of the fact that various techniques and results that originated in universal algebra can be used to prove significant and deep theorems in modal logic. Perhaps the most impressive were applications of

Presented by ; *Received*

Studia Logica **0**: 1–**??**, 2006.

Jónsson's lemma [25] on subdirectly irreducible algebras in congruence distributive varieties and McKenzie's splitting technique [32] to solve problems formulated in purely logical terms.

It is of interest to observe, however, that Wim Blok himself was not concerned too much with modal logic while working on his PhD thesis in 1973–75. The 263 page thesis entitled 'Varieties of interior algebras' was published in 1976 and contained just a four page section on the relation to modal logic. Wim Blok wrote: 'Although no mention will be made of modal logic anywhere in this paper, it seems appropriate to say a few words about the connection of interior algebras with these logics, in order to facilitate an interpretation of the mathematical results of our work in logical terms.' In fact, Wim Blok's thesis is a purely algebraic study of the structure of the lattice of subvarieties of the variety of interior algebras. He motivates it by referring to J.C.C. McKinsey and A. Tarski's seminal papers 'The algebra of topology' [33] (which showed, among many other things, that the variety of interior algebras is generated by topological spaces regarded as interior algebras) and 'On closed elements in closure algebras' [34] (which established connections between interior and Heyting algebras).

Later, in his 'Reminiscences about modal logic in the seventies' [9],* Wim Blok himself described the connection between the results obtained in his PhD thesis and modal logic as follows: 'It took a while before I saw the implications for intermediate and modal logics. ... But it soon emerged that the new algebraic methods made it possible to settle many questions concerning logics—at that point concerning extensions of IPC and S4.' In this context he also explained why, around 1979, his interest moved from modal algebras and logics to abstract algebraic logic: "One problem was understanding the exact connection between the algebraic theory and the logics—for me these questions are at the root of later work on 'algebraization' of logics." This direction of Wim Blok's work goes beyond the scope of our note; the reader is referred to [**?**] for details. It is worth mentioning, however, that throughout his later work in algebra and algebraic logic, he always used modal logics and algebras as a source of significant examples.

Refer to proper article in volume

In about 1975, Wim Blok became interested in Kripke semantics for modal logics, the main alternative to algebraic semantics, through discussions with Johan van Benthem who was also working on his PhD thesis at that time. Being an algebraist, Wim Blok wanted to understand the difference between these two semantics. In 1975–76, when he started working in

*These 'Reminiscences' were written by Wim in 2000 for Rob Goldblatt, as a background for the historical article [22] that Rob was writing at that time.

this area, the first examples of Kripke incomplete logics had already been found, and the main research problem was to investigate the position of Kripke incomplete logics in the lattice of all normal modal logics [21]. After some 'preliminary' results on uncountable families of logics validating the same class of Kripke frames [2, 6], Wim Blok observed, through discussions with Wolfgang Rautenberg, that there might be a connection between the degree of Kripke incompleteness of a modal logic and the algebraic notion of splittings [32]. This observation resulted in a comprehensive and surprisingly transparent classification of normal modal logics according to their degree of Kripke incompleteness through the notion of splitting logics [3, 4].

As follows from the discussion above, this article could be written either in the language of algebra or in the language of logic. Given the enormous impact Wim Blok's results had on modal logic, we decided to present them from the modal logic point of view. However, the connections with algebra and the algebraic techniques applied in their proofs will be also explicitly formulated.

Modal logics and modal algebras

In this section, we introduce normal modal logics and their connection with varieties of modal algebras. We also explain Jónsson's lemma and the notion of splittings as the basic technical tools of Blok's work.

The *propositional* (*uni*)*modal language* $\mathcal{ML}$ consists of a countable set of propositional variables $p_1, p_2, \dots$, the Boolean connectives $\wedge$ and $\neg$, and the modal (necessity) operator $\Box$. Other logical connectives like $\to$ and $\Diamond$ are defined as standard abbreviations. A *normal modal logic* is a set of $\mathcal{ML}$-formulas containing all tautologies of classical logic, the distribution axiom

$$\Box(p_1 \to p_2) \to (\Box p_1 \to \Box p_2),$$

and closed under the rules of *modus ponens*, necessitation $\varphi/\Box\varphi$, and uniform substitution (of formulas instead of variables). The minimal normal modal logic is denoted by $\mathbf{K}$, and the maximal one, that is, the set of all $\mathcal{ML}$-formulas, by $\mathcal{ML}$.

If L_i, for $i \in I$, are normal modal logics then $\bigcap_{i\in I} L_i$ and the closure $\bigoplus_{i\in I} L_i$ of $\bigcup_{i\in I} L_i$ under the rules above are normal modal logics as well. The set $\mathrm{NExt}\,\mathbf{K}$ of all normal modal logics (normal extensions of $\mathbf{K}$) forms a complete distributive lattice with meet $\cap$, join $\oplus$, zero $\mathbf{K}$ and unit $\mathcal{ML}$ (with $\subseteq$ as the corresponding lattice order).

The lattice $\mathrm{NExt}\,\mathbf{K}$ is anti-isomorphic to the lattice of subvarieties of the variety of modal algebras, where a *modal algebra* $\mathfrak{A} = (A, \wedge, \neg, \Box, 1)$ is a Boolean algebra $(A, \wedge, \neg, 1)$ extended with a unary operator $\Box$ such that

$$\Box 1 \;=\; 1 \quad \text{and} \quad \Box(a \wedge b) \;=\; \Box a \wedge \Box b$$

for all $a, b \in A$. This anti-isomorphism can be established by regarding the $\mathcal{ML}$-formulas as terms and showing that a set $L \subseteq \mathcal{ML}$ is a normal modal logic iff there exists a class $\mathcal{V}$ of modal algebras such that

$$L \;=\; \mathsf{Th}\,\mathcal{V} \;=\; \{\varphi \in \mathcal{ML} \mid \forall \mathfrak{A} \in \mathcal{V}\ (\mathfrak{A} \models \varphi = 1)\}. \qquad (1)$$

In other words, the lattice of normal modal logics is isomorphic to the lattice of equational theories containing the equational theory of all modal algebras. For a normal modal logic L, we denote by $\mathcal{V}(L)$ the variety of L-algebras, i.e.,

$$\mathcal{V}(L) \;=\; \{\mathfrak{A} \mid \forall \varphi \in L\ (\mathfrak{A} \models \varphi = 1)\}.$$

Using the same argument as above one can observe that the lattice $\mathrm{NExt}\,L$ of all normal modal logics containing L is anti-isomorphic to the lattice of varieties contained in $\mathcal{V}(L)$.

This observation makes it possible to use tools and methods of universal algebra in order to investigate normal modal logics. When Blok started working on his PhD, two important new techniques had just been introduced: Jónsson's lemma [25] for congruence distributive varieties, and McKenzie's splitting technique [32] for lattices. As the variety of modal algebras is congruence distributive, both techniques can be applied to modal logics.

THEOREM 1 (Jónsson's lemma for modal algebras). *The subdirectly irreducible members of a variety of modal algebras generated by a class $\mathcal{K}$ of algebras are included in $HSP_U(\mathcal{K})$, where H means taking the closure under homomorphic images, S the closure under subalgebras, and P_U means taking the closure under ultraproducts.*

The significance of this result for investigating lattices of normal modal logics can be seen from the fact that every variety is generated by its *subdirectly irreducible algebras* (*s.i. algebras*, for short), and that the finite s.i. modal algebras are easily described: they are exactly the algebras induced by finite rooted Kripke frames.

We remind the reader that a *Kripke frame* is a structure $\mathfrak{F} = (W, R)$ with a nonempty set W of 'worlds' (or points) and a binary relation R on it. The modal algebra induced by $\mathfrak{F}$ is

$$\mathfrak{F}^+ \;=\; (2^W, \cap, -, W, \Box),$$

where $\cap$ and $-$ are set-theoretic intersection and complement, and, for every $a \subseteq W$,

$$\Box a \;=\; \{x \in W \mid \forall y \in W\ (xRy \rightarrow y \in a)\}.$$

A Kripke frame $\mathfrak{F} = (W, R)$ is said to be *rooted* if there exists $r \in W$, a *root* of $\mathfrak{F}$, such that, for every $w \in W$, there is an R-path from r to w.

DEFINITION 2 (splittings). Let $\mathfrak{A}$ be a s.i. modal algebra and L_0 a normal modal logic with $\mathfrak{A} \in \mathcal{V}(L_0)$. We say that $\mathfrak{A}$ *splits* $\mathrm{NExt}\, L_0$ if there exists a logic L_1 such that, for every $L \in \mathrm{NExt}\, L_0$, either $L \supseteq L_1$ or $\mathfrak{A} \in \mathcal{V}(L)$, but not both. In this case $\mathfrak{A}$ is called a *spitting algebra* of $\mathrm{NExt}\, L_0$, and L_1 is called a *splitting logic* of $\mathrm{NExt}\, L_0$ and denoted by $\mathrm{NExt}\, L_0/\mathfrak{A}$. If $\mathfrak{F}$ is a Kripke frame such that $\mathfrak{F}^+$ is a splitting algebra of $\mathrm{NExt}\, L_0$, then the corresponding splitting logic will also be denoted by $\mathrm{NExt}\, L_0/\mathfrak{F}$.

One can show that if L_0 is generated (as in (1)) by the class of its *finite* algebras—in other words, if L_0 has the *finite model property*,—then only finite s.i. algebras from $\mathcal{V}(L_0)$ can split $\mathrm{NExt}\, L_0$. It turns out that the s.i. splitting algebras of $\mathrm{NExt}\, L_0$ and the corresponding splitting logics bear a lot of information about the structure of $\mathrm{NExt}\, L_0$ and, quite surprisingly, about seemingly unrelated properties of modal logics such as Kripke completeness.

The first to 'split lattices of logics,' even before McKenzie's general introduction of splittings, was Jankov [24] who associated with every finite s.i. Heyting algebra a *characteristic formula* axiomatising the corresponding splitting logic of the lattice of superintuitionistic logics. The importance of splittings for studying lattices of modal logics was first recognised by Blok and Rautenberg [36, 37].

Blok proved in [1] that in $\mathrm{NExt}\, \mathbf{S4}$ every finite s.i. algebra is a splitting algebra and that many important modal logics containing $\mathbf{S4}$ are (joins of) splitting logics of $\mathrm{NExt}\, \mathbf{S4}$. For example, the logic $\mathbf{S5}$ is the splitting logic $\mathrm{NExt}\, \mathbf{S4}/$  where  is the Kripke frame which is the chain of two reflexive points.

Using Jónsson's lemma one can show that, in $\mathrm{NExt}\, \mathbf{S4}$, the variety $\mathcal{V}(L_1)$ of a splitting logic $L_1 = \mathrm{NExt}\, \mathbf{S4}/\mathfrak{A}$ has a particularly transparent description, namely,

$$\mathcal{V}(\mathrm{NExt}\, \mathbf{S4}/\mathfrak{A}) \;=\; \{\mathfrak{B} \in \mathcal{V}(\mathbf{S4}) \mid \mathfrak{A} \notin HS\mathfrak{B}\}.$$

In [7], Blok noticed that for $\mathrm{NExt}\, \mathbf{K}$ the situation is different: some finite s.i. algebras split $\mathrm{NExt}\, \mathbf{K}$, while some finite s.i. algebras do not split. For example, the (algebra induced by the) frame consisting of a single irreflexive point

• splits NExt **K**, with the corresponding splitting logic being **D** (the normal modal logic of all frames (W, R) satisfying the seriality condition $\forall x \exists y\ xRy$). On the other hand, the frame consisting of a single reflexive point ∘ does not split NExt **K**, which follows from the well-known fact that **K** is determined by the class of all finite rooted frames without cycles. A characterisation of those finite rooted frames that split NExt **K** is one of the main ingredients of Blok's famous theorem on the degree of Kripke incompleteness to be discussed in the next section.

The degree of Kripke incompleteness

Soon after Kripke semantics for modal logic had been introduced, it was conjectured that every normal modal logic is determined by its Kripke frames—i.e., is *Kripke complete*. As Blok says in [4], 'it took eight years before it [this conjecture] could be decided; in [21] and [39], Fine and Thomason refuted it by presenting examples of (finitely axiomatizable) incomplete modal logics.' The next challenging problem was to uncover the phenomenon of Kripke incompleteness. Are incomplete logics exceptional and rare? Where are they located in the lattice of normal modal logics? In [3, 4, 6], Blok gave a comprehensive and rather surprising answer to these questions, using the notion of splittings.

For each normal modal logic L, Fine [21] defined its *degree of* (*Kripke*) *incompleteness* $\delta(L)$ by taking

$$\delta(L) \ = \ \left|\{L' \in \mathrm{NExt}\,\mathbf{K} \mid \forall \mathfrak{F}\ (\mathfrak{F}^+ \models L \ \text{ iff } \ \mathfrak{F}^+ \models L')\}\right|.$$

In other words, the degree of incompleteness of L is the number of normal modal logics which cannot be distinguished from L by means of Kripke frames. L is said to be *intrinsically complete* if $\delta(L) = 1$. The results of Thomason and Fine mentioned above show that there exist normal modal logics with the degree of incompleteness ≥ 2.

To understand the behaviour of δ basically means to answer the question how rare Kripke incomplete logics are and what position in the lattice of normal modal logics they occupy. Blok started his analysis of the function δ in late 1976 with the remarkable result that all consistent logics in the lattices $\mathrm{NExt}\,\mathbf{K} \oplus \Box p \to p$ and $\mathrm{NExt}\,\mathbf{K} \oplus \Box^m p \to \Box^{m+1} p$, $m > 0$, have degree of incompleteness $2^{\aleph_0}$ [6].

Soon afterwards, Blok made the first key observation to obtain a complete description of δ, namely, that a join of splitting logics of NExt **K** is intrinsically complete iff it is Kripke complete. We illustrate this observation by the example of the logic **D** which, as we have seen above, is Kripke complete and can be represented as the splitting logic NExt $\mathbf{K}/\bullet$. Now suppose that there is a logic $L \neq \mathbf{D}$ with the same Kripke frames as $\mathbf{D}$. Then $\bullet$ is not a frame for L. As $\bullet$ splits NExt $\mathbf{K}$ with the corresponding splitting logic $\mathbf{D}$, we must have then $L \supsetneqq \mathbf{D}$. But this leads to a contradiction because $\mathbf{D}$ is Kripke complete, and so there must exist a frame validating $\mathbf{D}$ but not L.

Blok proved in [4] that indeed every join of splitting logics has the finite model property, hence is Kripke complete, and therefore intrinsically complete. Moreover, he gave a characterisation of splitting frames of NExt $\mathbf{K}$.

THEOREM 3. (i) *A finite rooted frame* $\mathfrak{F}$ *splits* NExt $\mathbf{K}$ *iff it is cycle free.*

(ii) *Every join of splitting logics of* NExt $\mathbf{K}$ *has the finite model property, and is intrinsically complete.* ($\mathbf{K}$ *itself can be regarded as the empty join of splitting logics.*)

Conversely, if a consistent logic is not a join of splitting logics then its degree of incompleteness turns out to be $2^{\aleph_0}$. Blok proved this using a variant of the 'veiled recession frame' (the logic of which he had axiomatised earlier in [5]) and, as usual, a number of subtle applications of Jónsson's lemma. Thus, Blok arrived at a complete description of the function δ:

THEOREM 4. $\delta(L) = 2^{\aleph_0}$ *iff* L *is not a join of splitting logics and* $L \subsetneqq \mathcal{ML}$. *Otherwise,* $\delta(L) = 1$.

Another, purely lattice-theoretic, measure of the complexity of NExt $\mathbf{K}$ is given by the function $\varkappa$ which associates with each $L \in$ NExt $\mathbf{K}$ the number of its immediate predecessors (or *covers*). We remind the reader that $L' \in$ NExt $\mathbf{K}$ is called an *immediate predecessor* of L if $L' \subsetneqq L$ and there does not exist any L'' such that $L' \subsetneqq L'' \subsetneqq L$. It is not difficult to see that $\varkappa(L) \leq \aleph_0$, for every join of splitting logics. Using the techniques similar to those involved in his analysis of the function δ, Blok obtained the following characterisation of $\varkappa$:

THEOREM 5. $\varkappa(L) = 2^{\aleph_0}$ *iff* L *is not a join of splitting logics and* $L \subsetneqq \mathcal{ML}$. *Otherwise,* $\varkappa(L) \leq \aleph_0$.

It is a pity that the full paper containing these remarkable results has never been published. Blok first submitted an abstract to the Bulletin of the Section of Logic of the Polish Academy of Sciences, and was informed by the editors that 'publication had been postponed because one could not believe the results' [9]. It appeared a bit later [3]. The full paper of sixty pages was submitted to the Annals of Pure and Applied Logic at about the same time. What happened next is explained by Blok in his 'Reminiscences' [9]: 'I received a referee's report [...] in the summer of 1979 that was quite favourable but demanded (probably justifiably so) thorough rewriting of the paper. I did start the rewriting process at one point, but it was never completed, and the paper remained unpublished.' Fortunately, good people (Wolfgang Rautenberg one of them) have kept the preprint [4] and passed it on to the next generation of modal logicians. Full proofs of Blok's theorems are now available in [12, 26] (see also [42, 41]).

These results have had a considerable impact on the research of various notions of completeness for (multi)modal logics. An analogue of Blok's theorem for neighbourhood semantics was proved by Chagrova [15]; see also [17]. Wolter [40] investigated the behaviour of the functions δ and $\varkappa$ in the lattice of so-called subframe logics (which is a sublattice of NExt $\mathbf{K}$). Recently, Litak [28] investigated the degree of incompleteness for some (algebraically motivated) weaker notions of completeness.

A recent, quite surprising result of Chagrov (see [41]) applies Blok's classification to the following algorithmic problem. Consider a normal modal logic L represented as $L = \mathbf{K} \oplus \Gamma$, where Γ is a finite set of modal formulas. Is there an algorithm which decides, given a modal formula φ, whether $L = \mathbf{K} \oplus \varphi$? In other words, is the *axiomatisation problem* for L decidable? Chagrov shows that this problem is decidable for L iff either $L = \mathcal{ML}$ or L is a join of splitting logics.

Modal and superintuitionistic logics: there and back again

Another deep and surprising result obtained by Blok is what is now known as the *Blok–Esakia isomorphism* between the lattice NExt $\mathbf{Grz}$ of normal extensions of the Grzegorcsyk logic and the lattice Ext $\mathbf{Int}$ of extensions of intuitionistic propositional logic. In his PhD thesis [1] Blok described it as 'our slightly unexpected result that the lattice of subvarieties of the variety of Heyting algebras is isomorphic to the lattice of subvarieties of $\underline{B}_i^*$' where $\underline{B}_i^*$ is a certain variety of interior algebras introduced by Blok which turned out to be precisely $\mathcal{V}(\mathbf{Grz})$.

The historical background of this result of Blok (and actually his whole PhD thesis) is a really amazing:

- *Intuitionistic logic* was constructed in the 1920s by Kolmogorov, Glivenko, and Heyting as a formalisation of Brouwer's (1907–08) ideas of mathematical intuitionism. We denote propositional intuitionistic logic by $\mathbf{Int}$.

- Lewis (1918, 1932) introduced his *propositional modal logics*, in particular **S4**, in an attempt to cope with the paradoxes of material implication.
- Hausdorff (1914) and Kuratowski (1922) conceived *topological spaces* as a mathematical abstraction for studying such properties of spaces as compactness, continuity, connectedness.

Who could imagine in the 1920s that these three lines of research were destined to meet?

- In the 1930–40s, Stone, Tarski, Tsao Chen, and McKinsey showed that topological spaces provide a sound and complete semantics for both **Int** and **S4**.
- Orlov (1928) and Gödel (1933), trying to give a classical interpretation of **Int**, introduced a new modal operator 'it is provable' and ended up with the same modal logic **S4**.

Orlov and Gödel suggested that **Int** could be embedded into **S4** by the translation T prefixing the modal box $\Box$ (read as 'provable') to every subformula of an intuitionistic formula. Using the fact that **S4** is determined by interior (a.k.a. topological Boolean) algebras and **Int** by the algebras of open elements of interior algebras (known as *Heyting* or *pseudo-Boolean algebras*), McKinsey and Tarski [35] proved that T is indeed an embedding of **Int** to **S4**. Dummett and Lemmon [16] generalised this result to the class Ext **Int** of all *superintuitionistic logics*, *si-logics* for short (a.k.a. *intermediate logics*), by showing that every si-logic $L = \mathbf{Int} + \Gamma$ (where Γ is an arbitrary set of formulas in the language of **Int** and $+$ means taking the closure under *modus ponens* and substitution) is embedded by the translation T into the modal logic $\tau(L) = \mathbf{S4} \oplus T(\Gamma)$.

Blok started working on his PhD by trying to determine the structure of the free interior algebra on one generator. This problem (originally posed by Birkhoff) was suggested to Blok by Ph. Dwinger probably because the structure of the free Heyting algebra on one generator had been already described by Rieger (1957) and Nishimura (1960). Blok took a more general approach and 'started thinking about equational classes in the context of varieties of Heyting algebras and closure algebras' [9]. Independently of Blok the relationship between the lattices Ext **Int** and NExt **S4** was actively investigated by Maksimova and Rybakov [31] and Esakia [18, 19] in the USSR. This research resulted in a nice theory of 'modal companions of si-logics' which is briefly outlined below.

Say that a modal logic $M \in \mathrm{NExt}\,\mathbf{S4}$ is a *modal companion* of a si-logic L if L is embedded in M by the translation T in the sense that, for every intuitionistic formula φ,

$$\varphi \in L \quad \text{iff} \quad T(\varphi) \in M.$$

If M is a modal companion of L then L is called the *si-fragment* of M and denoted by $\boldsymbol{\rho} M$.

Blok (as well as the other researchers mentioned above) observed that, for every $M \in \mathrm{NExt}\,\mathbf{S4}$, we have $\boldsymbol{\rho} M = \{\varphi \mid T(\varphi) \in M\}$. Moreover, the variety $\mathcal{V}(\boldsymbol{\rho} M)$ of Heyting algebras for $\boldsymbol{\rho} M$ consists of the algebras of open elements of the interior algebras from $\mathcal{V}(M)$. (In terms of Kripke frames this operation means collapsing all clusters in **S4**-frames into single points.) According to [16], $\boldsymbol{\rho\tau} L = L$ for every si-logic L. So $\boldsymbol{\rho}$ is a surjection. Clearly, $\boldsymbol{\tau} L$ is the smallest modal companion of L in $\mathrm{NExt}\,\mathbf{S4}$. The next observation made by Blok was that

$$\boldsymbol{\sigma} L \;=\; \boldsymbol{\tau} L \oplus \Box(\Box(p \to \Box p) \to p) \to p$$

is the largest modal companion of L in $\mathrm{NExt}\,\mathbf{S4}$. The variety $\mathcal{V}(\boldsymbol{\sigma} L)$ is obtained by taking the 'Boolean closure' of every Heyting algebra in $\mathcal{V}(L)$. In terms of finite Kripke frames this means simply that we regard every such frame for **Int** as a frame for **S4**.

The logic $\mathbf{S4} \oplus \Box(\Box(p \to \Box p) \to p) \to p$ is known as the *Grzegorcsyk logic* [23] and denoted by **Grz**. Blok proved that **Grz** is in fact the join of splittings of $\mathrm{NExt}\,\mathbf{S4}$ by the frames ② and  , where ② is the two-point cluster.

Thus, the set of modal companions of every consistent si-logic L forms the interval

$$\boldsymbol{\rho}^{-1}(L) \;=\; \{M \in \mathrm{NExt}\,\mathbf{S4} \mid \boldsymbol{\tau} L \subseteq M \subseteq \boldsymbol{\tau} L \oplus \mathbf{Grz}\}$$

which, by the way, contains an infinite descending chain of logics.

This investigation of the relationship between the lattices $\mathrm{NExt}\,\mathbf{S4}$ and $\mathrm{Ext}\,\mathbf{Int}$ culminated in the following result:

THEOREM 6. (i) *The map* $\boldsymbol{\rho}$ *is a homomorphism of the lattice* $\mathrm{NExt}\,\mathbf{S4}$ *onto the lattice* $\mathrm{Ext}\,\mathbf{Int}$.

(ii) *The map* $\boldsymbol{\tau}$ *is an isomorphism of* $\mathrm{Ext}\,\mathbf{Int}$ *into* $\mathrm{NExt}\,\mathbf{S4}$.

(iii) *The map* $\boldsymbol{\sigma}$ *is an isomorphism of* $\mathrm{Ext}\,\mathbf{Int}$ *onto* $\mathrm{NExt}\,\mathbf{Grz}$.

(iv) *All these maps preserve infinite joins and intersections of logics.*

The most surprising and difficult part of this result is item (iii) which is known as the *Blok–Esakia theorem.* For further references and details of the theory of modal companions of si-logics (in particular, preservation theorems under the maps $\boldsymbol{\rho}$, $\boldsymbol{\tau}$, and $\boldsymbol{\sigma}$) the reader can consult [14, 12].

Tabular, pretabular, and locally tabular logics

Blok always considered his work in modal logic as an investigation of the structure of lattices of logics rather than individual systems. He wrote in [8]: 'Much of the literature on modal logics has been engaged in introducing new logics and comparing them with existing ones regarding their strength. Such investigations are really part of the more ambitious attempt to provide a description of the lattice of all modal logics.'

One of the first questions we usually ask when investigating a lattice of logics is whether each logic in this lattice is characterised by a *finite* algebra, frame, matrix, etc. (which can then be regarded as a finite 'truth-table' similar to the truth-table for classical propositional logic). Logics of this sort are called *tabular.* In many respects, tabular logics are easy to deal with. For example, the satisfiability problem for tabular modal or si-logics is NP-complete. Every modal or si-logic with the finite model property can be represented as an intersection of tabular logics.

The first results on the tabularity of modal and si-logics were obtained by Gödel (1932) and Dugundji (1940) who showed that **Int** and the Lewis modal logics **S1**–**S5** are not tabular.

The problem of characterising tabular modal and si-logics has become popular since the mid 1960s, when Dick de Jongh observed (but did not publish) that all tabular si-logics are finitely axiomatisable. For normal modal logics this fact follows from a general algebraic result of Baker (1977), and for quasi-normal modal logics it was proved by Blok and Köhler [11]. It is not difficult to check that a logic $L \in \mathrm{NExt}\,\mathbf{K}$ is tabular iff $\boldsymbol{tab}_n \in L$ for some $n < \omega$, where $\boldsymbol{tab}_n$ is the formula

$$\neg(\varphi_1 \wedge \Diamond(\varphi_2 \wedge \Diamond(\varphi_3 \wedge \cdots \wedge \Diamond\varphi_n)\dots)) \wedge \bigwedge_{m=0}^{n-1} \neg\Diamond^m(\Diamond\varphi_1 \wedge \dots \wedge \Diamond\varphi_n)$$

and $\varphi_i = p_1 \wedge \cdots \wedge p_{i-1} \wedge \neg p_i \wedge p_{i+1} \wedge \cdots \wedge p_n$.

Using Jónsson's lemma, Blok proved in his PhD thesis [1] that every tabular logic in $\mathrm{NExt}\,\mathbf{S4}$ is a finite join of splitting logics and has finitely many immediate predecessors, which are also tabular; later, in [8], he extended this result to tabular logics in NExt **K4**. (The same fact for si-logics was established by Kuznetsov [27].) These results mean, in particular, that, given some fixed tabular logic L and an arbitrary formula φ, one can effectively decide whether $\mathbf{K4} \oplus \varphi = L$. Indeed, let $L_1, \ldots, L_n$ be all the immediate predecessors of L. Then we have $\mathbf{K4} \oplus \varphi = L$ iff $\varphi \in L$ and $\varphi \notin L_i$, for $i = 1, \ldots, n$, which can be effectively checked because all the L_i are tabular.

Strange as it may seem, the situation is quite different in NExt **K**. It follows from Theorem 5 and the fact that no tabular logic is a join of splitting logics that every consistent tabular logic in this lattice has a *continuum* of immediate predecessors. And by Chagrov's theorem above, the problem whether $\mathbf{K} \oplus \varphi = L$, for some fixed tabular logic L, is undecidable.

It seems that Blok [1, 8] was the first to give a correct proof of the following characterisation of tabular logics in NExt **K4**: a logic is tabular iff it has finitely many extensions. This result, wrote Blok in [8], made, in principle, a description of the upper part of the lattice NExt **K4** possible. Again, in view of Theorem 5 above, nothing like this is possible for NExt **K**.

The tabularity criteria formulated above are not effective in the sense that they do not provide us with an algorithm for deciding, given a formula φ, whether, say, the logic $\mathbf{K4} \oplus \varphi$ is tabular. Moreover, as was shown by Chagrov (see [12] and references therein) no effective tabularity criterion exists for large classes like NExt **K**. However, if we restrict attention to 'sufficiently strong' logics, e.g., to the class NExt **S4**, the tabularity problem turns out to be decidable. The key idea, proposed by Kuznetsov in the 1970s, is to consider the so-called pretabular logics.

A logic $L \in \mathrm{NExt}\, L_0$ is said to be *pretabular* in the lattice $\mathrm{NExt}\, L_0$, if L is not tabular but every proper extension of L in $\mathrm{NExt}\, L_0$ is tabular. In other words, a pretabular logic in $\mathrm{NExt}\, L_0$ is a maximal non-tabular logic in $\mathrm{NExt}\, L_0$. The fact that every non-tabular logic in NExt **K** is contained in a pretabular one follows from Zorn's lemma.

If there is a simple description of all pretabular logics in a lattice, then we obtain an effective (modulo the description) tabularity criterion for the lattice. Indeed, take for definiteness the lattice NExt **K4**. How to determine, given a formula φ, whether $\mathbf{K4} \oplus \varphi$ is tabular? We may launch two parallel processes: one of them generates all derivations in $\mathbf{K4} \oplus \varphi$ and stops after finding a derivation of $\boldsymbol{tab}_n$, for some $n < \omega$; another process checks if φ belongs to a pretabular logic in NExt **K4** and stops if this is the case. The termination of the first process means that $\mathbf{K4} \oplus \varphi$ is tabular, and if the second one comes to a stop then this logic is not tabular.

Unfortunately, it is impossible to describe in an effective way all pretabular logics in NExt **K4**: Blok [8] constructed a continuum of them. However,

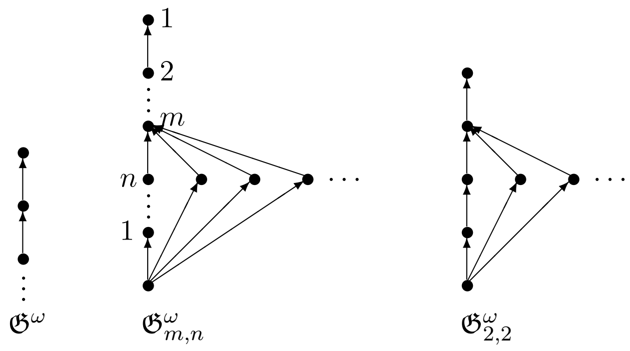

Figure 1.

for some smaller lattices such descriptions do exist. Maksimova [30] and Esakia and Meskhi [20] showed, e.g., that there are only *five* (rather simple) pretabular logics in NExt **S4**. It follows that the tabularity property is decidable in this lattice.

Blok [8] (see also [13]) characterised all pretabular logics in NExt **GL**, where $\mathbf{GL} = \mathbf{K} \oplus \Box(\Box p \to p) \to \Box p$ is the *Gödel–Löb provability logic*:

THEOREM 7. *The set of pretabular logics in* NExt **GL** *is denumerable. It consists of the logics of the frames $\mathfrak{G}^\omega$ and $\mathfrak{G}^\omega_{m,n}$ (for $m \geq 0$, $n \geq 1$) shown in Fig. 1.*

Using this result one can show that the property of tabularity is decidable in NExt **GL** as well (for details see [12]).

The question whether tabularity is decidable in NExt **K4** is still open. One way to a positive solution is opened by yet another important result Blok obtained in [8] by connecting pretabular and locally tabular logics in NExt **K4**.

We remind the reader that a logic L is called *locally tabular* (or *locally finite*) if, for every natural number $n \geq 0$, there are only finitely many pairwise nonequivalent formulas in L built from the variables $p_1, \ldots, p_n$. In algebraic terms this means that all finitely generated free algebras (alias Lindenbaum algebras) for L are finite. It follows immediately that locally tabular logics have the finite model property, and so are decidable if finitely axiomatisable.

A nice syntactical and semantical characterisation of locally tabular logics in NExt **K4** was obtained by Blok [10, 1] and Maksimova [29] (see also [38]). Namely, a logic $L \in$ NExt **K4** is locally tabular iff L has no Kripke frames of depth n for some $n \geq 1$ iff $\boldsymbol{bd}_n \in L$ for some $n \geq 1$, where

$$\boldsymbol{bd}_1 = \Diamond\Box p_1 \to p_1, \quad \boldsymbol{bd}_{n+1} = \Diamond(\Box p_{n+1} \wedge \neg \boldsymbol{bd}_n) \to p_{n+1}.$$

Clearly, every tabular logic is locally tabular.

Suppose now that we have an algorithm for deciding, given a formula φ, whether $\mathbf{K4} \oplus \varphi$ is locally tabular. If this hypothetical algorithm says that $L = \mathbf{K4} \oplus \varphi$ is not locally tabular then L is not tabular either. Otherwise, we can effectively find some number n such that $\boldsymbol{bd}_n \in L$. And then we use Blok's [8] remarkable theorem according to which there are only *finitely many* pretabular logics containing $\boldsymbol{bd}_n$. All these pretabular logics have rather simple Kripke frames which can be easily axiomatised, so all of them are decidable. What remains to be done is to run Kuznetsov's algorithm described above.

Nobody knows, however, whether local tabularity is decidable in NExt $\mathbf{K4}$. This and other exciting open problems arising from Wim Blok's work (say, a characterisation of the degree of Kripke incompleteness for logics in NExt $\mathbf{K4}$) remain a great challenge for the new generation of modal logicians.

Acknowledgements

We are grateful to Alexander Chagrov, Rob Goldblatt, Ian Hodkinson, Johan van Benthem, and the editors of this special issue for their comments and suggestions.

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