Ontology Languages (COMP321)
Solutions for Exercise 5

1. Translate the following assertions to concept inclusions in the description logic $\mathcal{SHOIQ}$. State which symbols are used as concept names, role names, and nominals.

   - Every student at Liverpool University is a Human being;
   - Liverpool University has at least 10,000 students;
   - Every citizen of France is a European;
   - The EU consists of at least 10 states;
   - The domain of the relation "citizen of" consists of human beings;
   - The range of the relation "citizen of" consists of states;
   - There are at least 50,000,000 French citizens.

   Ensure that it follows from your translation that France is a state.

2. Translate the assertions from (1) into FOPL.

3. Recall that an inclusion $C \sqsubseteq D$ follows from a TBox $\mathcal{T}$, in symbols $\mathcal{T} \models C \sqsubseteq D$, if every $\mathcal{I}$ that satisfies $\mathcal{T}$ satisfies $C \sqsubseteq D$ as well.

   Consider the following statements.

   - $\{\forall r.B \sqsubseteq A\} \models A \sqsubseteq B$;
   - $\{A \sqsubseteq B, B \sqsubseteq E\} \models A \sqsubseteq E$;
   - $\{\top \sqsubseteq \exists r.(A \sqcap B), E \sqsubseteq A\} \models E \sqsubseteq B$;
   - $\{\top \sqsubseteq \exists r.(A \sqcap B)\} \models A \sqsubseteq B$;
   - $\{\top \sqsubseteq \exists r.(\{a\} \sqcap B)\} \models \{a\} \sqsubseteq B$;
   - $\{A \sqsubseteq \exists r.B, A \sqsubseteq \forall r.E\} \models A \sqsubseteq \exists r.(B \sqcap E)$.

(a) Translate all inclusions in those statements into FOPL.

(b) Check whether the statements hold. If a statement does not hold, provide an interpretation that satisfies the TBox but not the concept inclusion on the right hand side.

Solution for 1.

One should work with France, EU, and LU (for Liverpool University) as individual names, European, State, Human_being as concept names, and consists_of and citizen_of, and student_at should be regarded as role names. One possible translation is now as follows:

- $\exists \mathsf{student\_at}.\{\mathsf{LU}\} \sqsubseteq \mathsf{Human\_being}$;
- $\{\mathsf{LU}\} \sqsubseteq (\geq 10,000\ \mathsf{student\_at}^{-}\ .\top)$;
- $\exists \mathsf{citizen\_of}.\{\mathsf{France}\} \sqsubseteq \mathsf{European}$;
- $\{\mathsf{EU}\} \sqsubseteq (\geq 10\ \mathsf{consists\_of}\ \mathsf{State})$;
- $\exists \mathsf{citizen\_of}.\top \sqsubseteq \mathsf{Human\_being}$;
- $\exists \mathsf{citizen\_of}^{-}.\top \sqsubseteq \mathsf{State}$;
- $\{\mathsf{France}\} \sqsubseteq (\geq 50,000,000\ \mathsf{citizen\_of}^{-}\ \top)$

By using the inverse of citizen_of for the final sentence, one can ensure that it follows that France is a state.

Solution for 2.

- $\forall x.(\mathsf{student\_at}(x, \mathsf{LU}) \rightarrow \mathsf{Human\_being}(x))$;
- $\exists x_1 \cdots \exists x_{10,000}.(\bigwedge_{i<j\leq 10,000} \neg(x_i = x_j) \wedge \bigwedge_{i\leq 10,000} \mathsf{student\_at}(x_i, \mathsf{LU}))$;
- $\forall x.(\mathsf{citizen\_of}(x, \mathsf{France}) \rightarrow \mathsf{European}(x))$;
- $\exists x_1 \cdots \exists x_{10}.(\bigwedge_{i<j\leq 10} \neg(x_i = x_j) \wedge \bigwedge_{i\leq 10} \mathsf{consists\_of}(\mathsf{EU}, x_i) \wedge \mathsf{State}(x_i))$
- $\forall x.(\exists y.\mathsf{citizen\_of}(x, y) \rightarrow \mathsf{Human\_being}(x))$;

- $\forall x.(\exists y.\mathsf{citizen\_of}(y,x) \rightarrow \mathsf{State}(x))$;
- $\exists x_1 \cdots \exists x_{50,000,000}.(\bigwedge_{i<j\leq 50,000,000} \neg(x_i = x_j) \wedge \bigwedge_{i\leq 50,000,000} \mathsf{citizen\_of}(x_i, \mathsf{France}))$.

Solution for 3.

(a) $\forall r.B \sqsubseteq A$ corresponds to

$$\forall x.((\forall y.(r(x,y) \rightarrow B(y)) \rightarrow A(x))$$

$A \sqsubseteq B$ corresponds to

$$\forall x.(A(x) \rightarrow B(x))$$

$\top \sqsubseteq \exists r.(A \sqcap B)$ corresponds to

$$\forall x.\exists y.(r(x,y) \wedge A(y) \wedge B(y))$$

$\top \sqsubseteq \exists r.(\{a\} \sqcap B)$ corresponds to

$$\forall x.\exists y.(r(x,y) \wedge (y = a) \wedge B(y))$$

$\{a\} \sqsubseteq B$ corresponds to

$$B(a)$$

or, equivalently,

$$\forall x.(x = a \rightarrow B(x))$$

$A \sqsubseteq \exists r.B$ corresponds to

$$\forall x.(A(x) \rightarrow \exists y.(r(x,y) \wedge B(y)))$$

$A \sqsubseteq \forall r.E$ corresponds to

$$\forall x.(A(x) \rightarrow \forall y.(r(x,y) \wedge E(y)))$$

$A \sqsubseteq \exists r.(B \sqcap E)$ corresponds to

$$\forall x.(A(x) \rightarrow \exists y.(r(x,y) \wedge B(y) \wedge E(y)))$$

(b)

$\{\forall r.B \sqsubseteq A\} \models \{A \sqsubseteq B\}$ does NOT hold. An interpretation $\mathcal{I}$ proving this is given by

- $\Delta^{\mathcal{I}} = \{a\}$;
- $A^{\mathcal{I}} = \{a\}$;
- $B^{\mathcal{I}} = \emptyset$;
- $r^{\mathcal{I}} = \emptyset$.

Then $\mathcal{I} \models \forall r.B \sqsubseteq A$, but $a \in A^{\mathcal{I}}$ and $a \notin B^{\mathcal{I}}$ and so $\mathcal{I} \not\models A \sqsubseteq B$.

$\{A \sqsubseteq B, B \sqsubseteq E\} \models A \sqsubseteq E$ holds.

$\{\top \sqsubseteq \exists r.(A \sqcap B), E \sqsubseteq A\} \models E \sqsubseteq B$ does NOT hold. An interpretation $\mathcal{I}$ proving this is given by

- $\Delta^{\mathcal{I}} = \{a, b\}$;
- $A^{\mathcal{I}} = \{a, b\}$;
- $E^{\mathcal{I}} = \{a, b\}$;
- $B^{\mathcal{I}} = \{b\}$;
- $r^{\mathcal{I}} = \{(a, b), (b, b)\}$.

Then $\mathcal{I} \models \top \sqsubseteq \exists r.(A \sqcap B)$ and $\mathcal{I} \models E \sqsubseteq A$, but $a \in E^{\mathcal{I}}$ and $a \notin B^{\mathcal{I}}$. So $\mathcal{I} \not\models E \sqsubseteq B$.

$\{\top \sqsubseteq \exists r.(A \sqcap B)\} \models A \sqsubseteq B$ does NOT hold. An interpretation $\mathcal{I}$ proving this is given by

- $\Delta^{\mathcal{I}} = \{a, b\}$;
- $A^{\mathcal{I}} = \{a, b\}$;
- $B^{\mathcal{I}} = \{b\}$;
- $r^{\mathcal{I}} = \{(a, b), (b, b)\}$.

Then $\mathcal{I} \models \top \sqsubseteq \exists r.(A \sqcap B)$ but $a \in A^{\mathcal{I}}$ and $a \notin B^{\mathcal{I}}$. So $\mathcal{I} \not\models A \sqsubseteq B$.

$\{\top \sqsubseteq \exists r.(\{a\} \sqcap B)\} \models \{a\} \sqsubseteq B$ holds because for every interpretation $\mathcal{I}$ satisfying the TBox $\mathcal{T} = \{\top \sqsubseteq \exists r.(\{a\} \sqcap B)\}$ we have $a^{\mathcal{I}} \in B^{\mathcal{I}}$.

$\{A \sqsubseteq \exists r.B, A \sqsubseteq \forall r.E\} \models A \sqsubseteq \exists r.(B \sqcap E)$ holds: Let $\mathcal{I}$ be an interpretation satisfying the TBox $\mathcal{T} = \{A \sqsubseteq \exists r.B, A \sqsubseteq \forall r.E\}$. Let $a \in A^{\mathcal{I}}$. We have to show there is a $b \in (B \sqcap E)^{\mathcal{I}}$ with $(a, b) \in r^{\mathcal{I}}$. But $\mathcal{I} \models A \sqsubseteq \exists r.B$ and so there exists b such that $b \in B^{\mathcal{I}}$ and $(a, b) \in r^{\mathcal{I}}$. By $\mathcal{I} \models A \sqsubseteq \forall r.E$, we obtain $b \in (B \sqcap E)^{\mathcal{I}}$, as required.