# On the performance of approximate equilibria in congestion games

George Christodoulou* Elias Koutsoupias† Paul G. Spirakis ‡

**Abstract**

We study the performance of approximate Nash equilibria for linear congestion games. We consider how much the price of anarchy worsens and how much the price of stability improves as a function of the approximation factor $\epsilon$. We give (almost) tight upper and lower bounds for both the price of anarchy and the price of stability for atomic and non-atomic congestion games. Our results not only encompass and generalize the existing results of exact equilibria to $\epsilon$-Nash equilibria, but they also provide a unified approach which reveals the common threads of the atomic and non-atomic price of anarchy results. By expanding the spectrum, we also cast the existing results in a new light. For example, the Pigou network, which gives tight results for exact Nash equilibria of selfish routing, remains tight for the price of stability of $\epsilon$-Nash equilibria.

## 1 Introduction

A central concept in Game Theory is the notion of equilibrium and in particular the notion of Nash equilibrium. Algorithmic Game Theory has studied extensively and with remarkable success the computational issues of Nash equilibria. As a result, we understand almost completely the computational complexity of exact Nash equilibria (they are PPAD-complete for games described explicitly [11, 5] and PLS-complete for games described succinctly [13]). The results established a long suspected drawback of Nash equilibria, namely that they cannot be computed effectively, thus upgrading the importance of approximate Nash equilibria. We don't understand completely the computational issues of approximate Nash equilibria [15, 13, 12, 26], but they provide a more reasonable equilibrium concept: It makes sense to assume that an agent is willing to accept a situation that is almost optimal to him.

In another direction, a large body of research in Algorithmic Game Theory concerns the degree of performance degradation of systems due to the selfish behavior of its users. Central to this area is the notion of price of anarchy (PoA) [14, 19] and the price of stability (PoS) [1]. The first notion compares the social cost of the worst-case equilibrium to the social optimum, which could be obtained if every agent followed obediently a central authority. The second notion is very similar but it considers the best Nash equilibrium instead of the worst one.

A natural question then is how the performance of a system is affected when its users are approximately selfish: What is the *approximate price of anarchy* and the *approximate price of stability*? Clearly, by allowing the players to be almost rational (within an $\epsilon$ factor), we expand the equilibrium concept and we expect the price of anarchy to get worse. On the other hand, the price of stability should improve. The question is how they change as functions of the parameter $\epsilon$. This is exactly the question that we address in this work.

We study two fundamental classes of games: the class of congestion games [20, 17] and the class of non-atomic congestion games [16]. The latter class of games includes the selfish routing games which played a central role in the development of the price of anarchy [21, 22]. The former class played also an important role in the development of the area of the price of anarchy, since it relates to the task allocation problem, which was the first problem to be studied within the framework

*Max-Planck-Institut für Informatik, Saarbrücken, Germany. Email: {gchristo}@mpi-inf.mpg.de
†Department of Informatics, University of Athens. Email: elias@di.uoa.gr
‡Computer Engineering and Informatics Department, Patras University, Greece. Email: {spirakis}@cti.gr

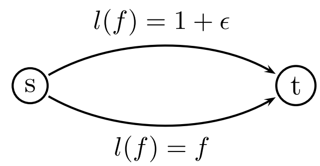


Figure 1: The Pigou network.

of the price of anarchy [14]. Although the price of anarchy and stability of these games for exact equilibria was established long ago [21, 8, 7, 2]—and actually Tim Roughgarden [23, 25] addressed partially the question for the price of anarchy of approximate equilibria—our results add an unexpected understanding of the issues involved.

While these two classes of games are conceptually very similar, dissimilar techniques were employed to answer the questions concerning the PoA and PoS. Moreover, the qualitative aspects of the answers were quite different. For instance, the Pigou network of two parallel links captures the hardest network situation for the price of anarchy for the selfish routing. In fact, Roughgarden [24] proved that the Pigou network is the worst case scenario for a very broad class of delay functions. On the other hand, the lower bound for congestion games is different and somewhat more involved [2, 8].

For the selfish routing games, the price of stability is not different than the price of anarchy because these games have a unique Nash (or Wardrop as it is called in these games) equilibrium. On the other hand, for the atomic congestion games, the problem proved more challenging [7, 4]. New techniques exploiting the potential of these games needed in order to come up with an upper bound. The lower bound is quite complicated and, unlike the selfish routing case, it has a dependency on the number of players (it attains the maximum value at the limit).

The main difference between the two classes is the "integrality" of atomic congestion games: In congestion games, when a player considers switching to another strategy, he has to take into account the extra cost that he will add to the edges (or facilities) of the new strategy. The number of players on the new edges increases by one and this changes the cost. On the other hand, in the selfish routing games the change of strategies has no additional cost. A simple—although not entirely rigorous—way to think about it, is to consider the effects of a tiny amount of flow that ponders whether to change path: it will not really affect the flow on the new edges (at least for continuous cost functions).

Is integrality the reason which lies behind the difference of these two classes of games? It seems so for the exact case. But our work could be interpreted as revealing that the uniqueness of the Nash equilibrium is also an important factor. Because when we move to the wider class of $\epsilon$-Nash equilibria, the uniqueness is dropped and the problems look quite similar qualitatively; the integrality difference is still there, but it only manifests itself in different quantitative or algebraic differences.

### 1.1 Our contribution and related work

Our work encompasses and generalizes some fundamental results in the area of the price of anarchy [21, 8, 7, 2] (see also the recently published book [18] for background information). Our techniques not only provide a unifying approach but they cast the existing results in a new light. For instance, the Pigou network (Figure 1) is still the tight example for the price of stability, but not the price of anarchy. Instead for the price of anarchy, the network of Figure 2 is tight; in fact, this network is tight only for $\epsilon \leq 1$; a more complicated network is required for larger $\epsilon$.

We consider $\epsilon$ approximate Nash equilibria. We use the *multiplicative definition of approximate equilibria*: In congestion games, a player does not switch to a new strategy as long as his current cost is less than $1 + \epsilon$ times the new cost. In the selfish routing games, we use exactly the same definition: the flow is at an equilibrium when the cost on its paths is less than $1 + \epsilon$ times the cost of every

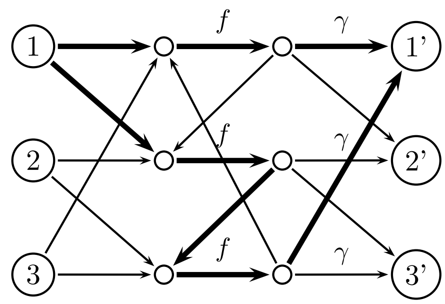


Figure 2: Lower bound for selfish routing. There are 3 distinct edge latency functions: $l(f) = f$, $l(f) = \gamma$ (a constant which depends on $\epsilon$), $l(f) = 0$ (omitted in the picture). There are 3 commodities of rate 1 with source $i$ and destination $i'$. The two paths for the first commodity are shown in bold lines.

alternate path. There have been other definitions for approximate Nash equilibria in the literature. The most-studied is the additive case [15, 11]. In [6], they consider approximate equilibria of the multiplicative case and they study convergence issues for congestion games. Our definition differs slightly and our results can be naturally adapted to the definition of [6].

There is a large body of work on the price of anarchy in various models [18]. More relevant to our work are the following publications: In [2, 8], it is proved that the price of anarchy of congestion games for pure equilibria is $\frac{5}{2}$. Later in [7], it is showed that the ratio 5/2 is tight even for correlated equilibria, and consequently for mixed equilibria. For symmetric games, it is something less: $\frac{5n-2}{2n+1}$ [8], where $n$ is the number of players. For weighted congestion games, the price of anarchy is $1+\phi \approx 2.618$ [2]. Later in [7], it was proved that the same ratio holds even for correlated equilibria. In [7, 4], it was shown that the price of stability of linear congestion games is $1 + \sqrt{3}/3$. For the selfish routing paradigm, the price of anarchy (and of stability) for linear latencies is 4/3 [21] (see also [9] for a simplified version of this proof), and the results are extended to non-atomic games in [22]. The most relevant work is [23, 25] which gives tight bounds for the price of anarchy of approximate equilibria when $\epsilon \leq 1$. We extend this to every positive $\epsilon$ using different techniques.

In this work, we give (almost) tight upper and lower bounds for the PoA and PoS of atomic and non-atomic linear congestion games. Our results are summarized in the Table 1 (where atomic refers to congestions games).

| | Atomic | Non-atomic |
|---|---|---|
| Anarchy | $(1+\epsilon)\frac{z^2+3z+1}{2z-\epsilon}$ where $z = \lfloor \frac{1+\epsilon+\sqrt{5+6\epsilon+\epsilon^2}}{2} \rfloor$ (Section 3) | $(1+\epsilon)^2$ (especially for $\epsilon \leq 1$: $\frac{4(1+\epsilon)}{3-\epsilon}$) (Section 4) |
| Stability | $\frac{1+\sqrt{3}}{\epsilon+\sqrt{3}}$ (Section 5) | $\frac{4}{(3-\epsilon)(1+\epsilon)}$ (Section 6) |

Table 1: The upper bounds (with pointers to relevant sections).

The results in the above table include the upper bounds. We have matching lower bounds except for the atomic PoS (and for non-integral values $\epsilon > 1$ for the PoA of non-atomic case). The price of stability reduces to 1 for $\epsilon \geq 1$, which means that the optimal is a 1-Nash equilibrium, for both the atomic and non-atomic case. Also, the price of anarchy is approximately $(1+\epsilon)(3+\epsilon)$ and $(1+\epsilon)^2$ for large $\epsilon$, the atomic and non-atomic case respectively. The price of anarchy for $\epsilon \leq 1$ has been established before in [23, 25].

The interesting case is probably when $\epsilon$ is small. For $\epsilon \leq 1/3$ the results are summarized in the Table 2:

| | Atomic | Non-atomic |
|---|---|---|
| Anarchy | $\frac{5(1+\epsilon)}{2-\epsilon}$ | $\frac{4(1+\epsilon)}{3-\epsilon}$ |
| Stability | $\frac{1+\sqrt{3}}{\epsilon+\sqrt{3}}$ | $\frac{4}{(3-\epsilon)(1+\epsilon)}$ |

Table 2: The upper bounds for $\epsilon \leq 1/3$.

A useful tool, interesting in its own right, is a generalization of the notion of potential function for both the atomic and non-atomic case (Theorems 6 and 10) for the case of $\epsilon$-Nash equilibria. Remarkably, the parameter $\epsilon$ appears only in the linear part of the (quadratic) potential function.

Our approach is similar to [8, 7], but it is much more involved technically and requires a deeper understanding of the potential function issues involved. We want also to draw attention to our techniques in bounding the approximate price of anarchy for the selfish routing which differ considerably from the techniques of [21] and others [18]. The main difference is that we move from a domain with unique equilibrium to a domain with a set of solutions.

## 2 Definitions

A congestion game [20], also called an exact potential game [17], is a tuple $(N, E, (\mathcal{S}_i)_{i\in N}, (f_e)_{e\in E})$, where $N = \{1, \ldots, n\}$ is a set of $n$ players, $E$ is a set of facilities, $\mathcal{S}_i \subseteq 2^E$ is a set of pure strategies for player $i$: a pure strategy $A_i \in \mathcal{S}_i$ is simply a subset of facilities and $l_e$ is a cost (or latency) function, one for each facility $e \in E$. The cost of player $i$ for the pure strategy profile $A = (A_1, \ldots, A_n)$ is $c_i(A) = \sum_{e\in A_i} l_e(n_e(A))$, where $n_e(A)$ is the number of players who use facility $e$ in the strategy profile $A$.

**Definition 1.** A pure strategy profile $A$ is an $\epsilon$ equilibrium iff for every player $i \in N$

$$c_i(A) \leq (1+\epsilon)c_i(A_i, A_{-i}), \qquad \forall A_i \in \mathcal{S}_i \tag{1}$$

We believe that the multiplicative definition of approximate equilibria makes more sense in the framework that we consider. This is because the costs of the players usually vary in this setting and a uniform $\epsilon$ does not make much sense. Given that the price of anarchy is a ratio, we need a definition that is insensitive to scaling.

The social cost of a pure strategy profile $A$ is the sum of the players cost

$$SC(A) = \textsc{Sum}(A) = \sum_{i\in N} c_i(A)$$

The pure approximate price of anarchy, is the social cost of the worst case $\epsilon$ equilibrium over the optimal social cost

$$PoA = \max_{A \text{ is a } \epsilon\text{-Nash}} \frac{SC(A)}{\text{OPT}},$$

while the pure approximate price of stability, is the social cost of the worst case $\epsilon$ equilibrium over the optimal social cost

$$PoS = \min_{A \text{ is a } \epsilon\text{-Nash}} \frac{SC(A)}{\text{OPT}}.$$

Instead of defining formally the class of nonatomic congestion games, we prefer to focus on the more illustrative–more restrictive though–class of selfish routing games. The difference in the two models is that in a non-atomic game, there does not exist any network and the strategies of the players are just subsets of facilities (as in the case of atomic congestion games) and they do not necessarily form a path in a network. The most desirable results are obtained when the upper bounds hold for general non-atomic games and matching lower bounds hold for the special case of selfish routing. Our results almost follow this pattern, with a few exceptions of lower bounds. This is because we put emphasis on the simplicity and we didn't attempt to extend them to the selfish routing case.

Let $G = (V, E)$ be a directed graph, where $V$ is a set of vertices and $E$ is a set of edges. In this network we consider $k$ commodities: source-node pairs $(s_i, t_i)$ with $i = 1 \dots k$, that define the sources and destinations. The set of simple paths in every pair $(s_i, t_i)$ is denoted by $\mathcal{P}_i$, while with $\mathcal{P} = \cup_{i=1}^k \mathcal{P}_i$ we denote their union. A flow $f$, is a mapping from the set of paths to the set of nonnegative reals $f : \mathcal{P} \to \mathbb{R}^+$. For a given flow $f$, the flow on an edge is defined as the sum of the flows of all the paths that use this edge $f_e = \sum_{P\in\mathcal{P}, e\in P} f_P$. We relate with every commodity $(s_i, t_i)$ a traffic rate $r_i$, as the total traffic that needs to move from $s_i$ to $t_i$. A flow $f$ is feasible, if for every commodity $\{s_i, t_i\}$, the traffic rate equals the flow of every path in $\mathcal{P}_i$, $r_i = \sum_{P\in\mathcal{P}_i} f_P$. Every edge introduces a delay in the network. This delay depends on the load of the edge and is determined by a delay function, $l_e(\cdot)$. An instance of a routing game is denoted by the triple $(G, r, l)$. The latency of a path $P$, for a given flow $f$, is defined as the sum of all the latencies of the edges that belong to $P$, $l_P(f) = \sum_{e\in P} l_e(f_e)$. The social cost that evaluates a given flow $f$, is the total delay due to $f$

$$C(f) = \sum_{P\in\mathcal{P}} l_P(f) f_P.$$

The total delay can also be expressed via edge flows $C(f) = \sum_{e\in E} l_e(f_e) f_e$.

From now on, when we are talking about flows, we mean feasible flows. In [3, 10], it is shown that there exists a (unique) equilibrium flow, known as Wardrop equilibrium[27]. In analogy to their definition, we define the $\epsilon$ Wardrop equilibrium flows, as follows

**Definition 2.** A feasible flow $f$, is an $\epsilon$-Nash (or Wardrop) equilibrium, if and only if for every commodity $i \in \{1, \dots, k\}$ and $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, $l_{P_1}(f) \le (1+\epsilon) l_{P_2}(f)$.

In this work we restrict our attention to *linear latency functions*: $l_e(x) = a_e x + b_e$, where $a_e$ and $b_e$ are nonnegative constants. Our results naturally extend to mixed and correlated equilibria. We also believe that they can be also extended to more general latency functions such as polynomials.

## 3 Congestion Games – PoA

In this section we study the dependency on the parameter $\epsilon$, of the price of anarchy for the case of atomic congestion games. For large $\epsilon$ the price of anarchy is roughly $(1+\epsilon)^2$. The same holds the non-atomic case as we are going to establish in the next section.

We will need the following arithmetic lemma.

**Lemma 1.** *For every $\alpha, \beta, z \in \mathbb{N}$:*

$$\beta(\alpha + 1) \le \frac{1}{2z+1}\alpha^2 + \frac{z^2+3z+1}{2z+1}\beta^2$$

*Proof.* Consider the function $f(\alpha, \beta)$ which we obtain when we subtract the left part of the statement's inequality from the right part and multiply the result by $2z + 1$.

$$\begin{aligned} f(\alpha,\beta) &= a^2 + (z^2+3z+1)\beta^2 - (2z+1)\beta(\alpha+1) \\ &= \left(\alpha - \frac{2z+1}{2}\beta\right)^2 + \frac{(8z+3)\beta^2 - (8z+4)\beta}{4}. \end{aligned}$$

For $\beta = 0$, and for any $\beta \geq 2$, $f(\alpha,\beta)$ is clearly positive. For $\beta = 1$ it takes the form $f(\alpha,1) = (\alpha - z)(\alpha - z - 1) \geq 0$, and the lemma follows. □

Our first theorem gives an upper bound for the price of anarchy for congestion games; this is tight, as we are going soon to establish. This result generalizes the bound in [2, 8] to approximate equilibria. The proof is for linear latency functions of the form $l_e(x) = x$, but it can be easily extended to latencies of the form $l_e(x) = a_e x + b_e$, with nonnegative $a_e, b_e$.

**Theorem 1** (Atomic-PoA-Upper-Bound)**.** *For any positive real $\epsilon$, the approximate price of anarchy of general congestion games with linear latencies is at most*

$$(1+\epsilon)\frac{z^2+3z+1}{2z-\epsilon},$$

*where $z \in \mathbb{N}$ is the maximum integer with $\frac{z^2}{z+1} \leq 1+\epsilon$ (or equivalently for $z = \lfloor \frac{1+\epsilon+\sqrt{5+6\epsilon+\epsilon^2}}{2} \rfloor$).*

*Proof.* Let $A = (A_1, \ldots, A_n)$ be an $\epsilon$-approximate pure Nash, and $P = (P_1, \ldots, P_n)$ be the optimum allocation. From the definition of $\epsilon$-equilibria (Inequality (1)) we get

$$\sum_{e \in A_i} n_e(A) \leq (1+\epsilon) \sum_{e \in P_i} (n_e(A) + 1).$$

If we sum up for every player $i$ and use Lemma 1, we get

$$\begin{aligned} \textsc{Sum}(A) &= \sum_{i \in N} c_i(A) \\ &= \sum_{i \in N} \sum_{e \in A_i} n_e(A) \\ &= \sum_{e \in E} n_e^2(A) \leq (1+\epsilon) \sum_{e \in E} n_e(P)\,(n_e(A)+1) \\ &\leq \frac{1+\epsilon}{2z+1} \sum_{e \in E} n_e^2(A) + \frac{(1+\epsilon)(z^2+3z+1)}{2z+1} \sum_{e \in E} n_e^2(P) \\ &= \frac{1+\epsilon}{2z+1} \textsc{Sum}(A) + \frac{(1+\epsilon)(z^2+3z+1)}{2z+1} \textsc{opt}. \end{aligned}$$

From this we obtain the theorem

$$\textsc{Sum}(A) \leq (1+\epsilon)\frac{z^2+3z+1}{2z-\epsilon}\textsc{opt}.$$

□

The above is a typical proof in this work. All our upper bound proofs have similar form. The proofs of the price of stability are more challenging however, as they require the use of appropriate generalizations of the potential function. We now show that the above upper bound is tight.

**Theorem 2** (Atomic-PoA-Lower-Bound)**.** *For any real positive $\epsilon$, there are instances of congestion games with linear latencies, for which the approximate price of anarchy of general congestion games with linear latencies, is at least*

$$(1+\epsilon)\frac{z^2+3z+1}{2z-\epsilon},$$

*where $z \in \mathbb{N}$ is the maximum integer with $\frac{z^2}{z+1} \leq 1+\epsilon$.*

*Proof.* Let $z \in \mathbb{N}$ be the maximum integer with $\frac{z^2}{z+1} \leq 1+\epsilon$. We will construct an instance with $z+2$ players and $2z+4$ facilities. There are two types of facilities:

- $z+2$ facilities of type $\alpha$, with latency $l_e(x) = x$ and
- $z+2$ facilities of type $\beta$ with latency $l_e(x) = \gamma x = \frac{(z+1)^2-(1+\epsilon)(z+2)}{(1+\epsilon)(z+1)-z^2}x$.

Player $i$ has two alternative pure strategies, $S_i^1$ and $S_i^2$.

- The first strategy is to play the two facilities $\alpha_i$ and $\beta_i$, i.e. $S_i^1 = \{\alpha_i, \beta_i\}$.
- The second strategy is to play every facility of type $\alpha$ except for $\alpha_i$ and $z+1$ facilities of type $\beta$ starting at facility $\beta_{i+1}$. More precisely, the second strategy has the facilities

  $$S_i^2 = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{z+2}, \beta_{i+1}, \ldots, \beta_{i+1+z}\},$$

  where the indices may require computations ( $\mod z+2$).

First we prove that playing the second strategy $S^2 = (S_1^2, \ldots, S_n^2)$ is a $\epsilon$-Nash equilibrium. The cost of player $i$ is

$$c_i(S^2) = (z+1)^2 + \gamma z^2,$$

as there are exactly $z+1$ players using facilities of type $\alpha$ and exactly $z$ players using facilities of type $\beta$.

If player $i$ unilaterally switches to the other available strategy $S_i^1$ he has cost

$$c_i(S_i^1, S_{-i}^2) = (z+2) + \gamma(z+1) = \frac{c_i(S^2)}{1+\epsilon},$$

which shows that $S^2$ is an $\epsilon$-Nash equilibrium.

The optimum allocation is the strategy profile $S^1$, where every player has cost $c_i(S^1) = 1+\gamma$ and so the price of anarchy is

$$\frac{c_i(S^2)}{c_i(S^1)} = \frac{(z+1)^2+\gamma z^2}{1+\gamma} = (1+\epsilon)\frac{z^2+3z+1}{2z-\epsilon}.$$

Notice that the parameter $z$ is an integer because it expresses a number of facilities. □

The above theorems (lower and upper bound) employ, for any positive real $\epsilon$, an integer $z(\epsilon)$, which is the maximum integer that satisfies $\frac{z^2}{z+1} \leq 1+\epsilon$. So for $\epsilon \in [0, 1/3]$, $z(\epsilon) = 1$ and the price of anarchy is $\frac{5(1+\epsilon)}{2-\epsilon}$, for $\epsilon \in [1/3, 5/4]$, $z(\epsilon) = 2$ and the price of anarchy is $\frac{11(1+\epsilon)}{4-\epsilon}$ and so on. Roughly the price of anarchy grows as $(1+\epsilon)(3+\epsilon)$.

## 4 Selfish Routing – PoA

In this section we estimate the price of anarchy for non-atomic congestion games and consequently for its special case, the selfish routing. Our results generalize the results in [21, 22] to the case of approximate equilibria. The proof has the same form with the proof of the atomic case in the previous section.

Again, we will need an arithmetic lemma. The main change now is that we deal with continuous values instead of integrals.

**Lemma 2.** *For every reals $\alpha, \beta, \lambda$ it holds,*

$$\beta\alpha \leq \frac{1}{4\lambda}\alpha^2 + \lambda\beta^2,$$

*where*

*Proof.* Simply because $\alpha^2 + 4\lambda^2\beta^2 - 4\lambda\alpha\beta = (\alpha - 2\lambda\beta)^2 \geq 0$. □

**Theorem 3** (Selfish-PoA-Upper-Bound)**.** *For any positive real $\epsilon$, and for every $\lambda \geq 1$, the approximate price of anarchy of non-atomic congestion games with linear latencies is at most*

$$\frac{4\lambda^2(1+\epsilon)}{4\lambda - 1 - \epsilon}.$$

*Proof.* Let $f$ be an $\epsilon$-approximate Nash flow, and $f^*$ be the optimum flow (or any other feasible flow). From the definition of approximate Nash equilibria (Inequality (1)), we get that for every path $p$ with non-zero flow in $f$ and any other path $p'$:

$$\sum_{e \in p} l_e(f_e) \leq (1+\epsilon) \sum_{e \in p'} l_e(f_e^*).$$

We sum these inequalities for all pairs of paths $p$ and $p'$ weighted with the amount of flow of $f$ and $f^*$ on these paths.

$$\sum_{p,p'} f_p f_{p'}^* \sum_{e \in p} l_e(f_e) \leq (1+\epsilon) \sum_{p,p'} f_p f_{p'}^* \sum_{e \in p'} l_e(f_e^*)$$
$$\sum_{p'} f_{p'}^* \sum_{e \in E} l_e(f_e) f_e \leq (1+\epsilon) \sum_{p} f_p \sum_{e \in E} l_e(f_e^*) f_e^*$$
$$(\sum_{p'} f_{p'}^*) \sum_{e \in E} l_e(f_e) f_e \leq (1+\epsilon) (\sum_{p} f_p) \sum_{e \in E} l_e(f_e^*) f_e^*$$

But $\sum_p f_p = \sum_{p'} f_{p'}^*$ is equal to the total rate for the feasible flows $f$ and $f^*$. Simplifying, we get

$$\sum_{e \in E} l_e(f_e) f_e \leq (1+\epsilon) \sum_{e \in E} l_e(f_e) f_e^*.$$

This is the generalization to approximate equilibria of the inequality established by Beckmann, McGuire, and Winston [3] for exact Wardrop equilibria.

Since we consider linear functions of the form $l_e(f_e) = a_e f_e + b_e$, we get

$$\sum_{e \in E} \left(a_e f_e^2 + b_e f_e\right) \leq (1+\epsilon) \sum_{e \in E} a_e f_e f_e^* + (1+\epsilon) \sum_{e \in E} b_e f_e^*.$$

Applying Lemma 2, we get

$$\sum_{e\in E}\left(a_e f_e^2 + b_e f_e\right) \le (1+\epsilon)\sum_{e\in E} a_e\left(\frac{1}{4\lambda}f_e^2 + \lambda f_e^{*2}\right) + (1+\epsilon)\sum_{e\in E} b_e f_e^*.$$

from which we get

$$\sum_{e\in E}\left(a_e\left(1-(1+\epsilon)\frac{1}{4\lambda}\right)f_e^2 + b_e f_e\right) \le \lambda(1+\epsilon)\sum_{e\in E} a_e f_e^{*2} + (1+\epsilon)\sum_{e\in E} b_e f_e^*,$$

and for $\lambda \ge 1$

$$\frac{4\lambda-1-\epsilon}{4\lambda}SC(f) \le (1+\epsilon)\lambda SC(f^*).$$

This gives price of anarchy at most of

$$\frac{4\lambda^2(1+\epsilon)}{4\lambda-1-\epsilon},$$

for every $\lambda \ge 1$. □

The expression $\frac{4\lambda^2(1+\epsilon)}{4\lambda-1-\epsilon}$ of the theorem is minimized for $\lambda = (1+\epsilon)/2$ when $\epsilon \ge 1$ (and $\lambda = 1$ when $\epsilon \le 1$). We therefore obtain the following two corollaries by substituting $\lambda = 1$ and $\lambda = (1+\epsilon)/2$. The first corollary was proved before in [23, 25] using different techniques.

**Corollary 1.** *For any nonnegative real $\epsilon \le 1$, the approximate price of anarchy of non-atomic congestion games with linear latencies is at most*

$$\frac{4(1+\epsilon)}{3-\epsilon}.$$

**Corollary 2.** *For any positive real $\epsilon \ge 1$, the approximate price of anarchy of non-atomic congestion games with linear latencies is at most*

$$(1+\epsilon)^2.$$

We now show that the above upper bounds are tight. To be precise, we show that Corollary 1 is tight and that Corollary 2 is partially tight—only for integral values of $\epsilon$.

The following theorem for the case of $\epsilon \le 1$ was first shown in [23, 25]. We include a different proof here for completeness and because it is similar to the generalization for $\epsilon > 1$, in Theorem 5.

**Theorem 4** (Selfish-PoA-Lower-Bound for $\epsilon \le 1$)**.** *For any nonnegative real $\epsilon \le 1$, there are instances of congestion games with linear latencies, for which the approximate price of anarchy of general congestion games with linear latencies, is at least*

$$\frac{4(1+\epsilon)}{3-\epsilon}.$$

*Proof.* We will construct an instance with 3 commodities, each of them with unit flow, and 6 facilities (a slightly more involved network case appears in Figure 2. There are two types of facilities:

- 3 facilities of type $\alpha$, with latency $l(x) = x$ and
- 3 facilities of type $\beta$ with constant latency $l(x) = \gamma = \frac{2(1-\epsilon)}{1+\epsilon}$.

Commodity $i$ has two alternative pure strategies, $S_i^1$ and $S_i^2$.

- The first strategy is to choose both the facilities $\alpha_i$ and $\beta_i$, i.e. $S_i^1 = \{\alpha_i, \beta_i\}$
- As a second alternative, players of commodity $i$ may choose every facility of type $\alpha$ except for $\alpha_i$; we denote this set by $S_i^2 = \{\alpha_{-i}\}$.

First we prove that playing the second strategy $S^2 = (S_1^2, S_2^2, S_3^2)$ is a $\epsilon$-Nash equilibrium. The cost of every player in commodity $i$ is $c_i(S^2) = 4$, as there are exactly $z+1$ players using facilities of type $\alpha$ and exactly $z$ players using facilities of type $\beta$.

If player $i$ unilaterally switches to the other available strategy $S_i^1$ he gets

$$c_i(S_i^1, S_{-i}^2) = 2 + \gamma = \frac{c_i(S^2)}{1+\epsilon}$$

and so $S^2$ is an $\epsilon-$approximate equilibrium.

In the optimum case, the players use strategy profile $S^1$, where commodity $i$ has cost $c_i(S^1) = 1+\gamma$ and so the price of anarchy is

$$\frac{c_i(S^2)}{c_i(S^1)} = \frac{4}{1+\gamma} = \frac{4(1+\epsilon)}{3-\epsilon}.$$

□

For larger $\epsilon$ ($\epsilon > 1$), we have:

**Theorem 5** (Selfish-PoA-Lower-Bound for $\epsilon \geq 1$)**.** *For any real positive $\epsilon$, there are instances of congestion games with linear latencies, for which the approximate price of anarchy of general congestion games with linear latencies, is at least*

$$(1+\epsilon)\frac{z(z+1)}{2z-\epsilon} = (1+\epsilon)\frac{z^2+z}{2z-\epsilon},$$

*where $z = \lfloor 1+\epsilon \rfloor$.*

*Proof.* Let $z = \lfloor 1+\epsilon \rfloor$. We will construct an instance with $z+2$ commodities and $2z+4$ facilities. There are two types of facilities:

- $z+2$ facilities of type $\alpha$, with latency 1 and
- $z+2$ facilities of type $\beta$ with latency $\gamma = \frac{(z+1)^2-(1+\epsilon)(z+1)}{(1+\epsilon)z-z^2}$.

Commodity $i$ has two alternative pure strategies, $S_i^1$ and $S_i^2$.

- The first strategy is to choose both the facilities $\alpha_i$ and $\beta_i$, i.e. $S_i^1 = \{\alpha_i, \beta_i\}$.
- As a second alternative, commodity $i$ may choose every facility of type $\alpha$ except for $\alpha_i$ and $z$ facilities of type $\beta$ as defined in the following:

  $$S_i^2 = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{z+2}, \beta_{i+1}, \ldots, \beta_{z+i+1}\},$$

  where the indices are computed ( $\mod z+2$).

First we prove that playing the second strategy $S^2 = (S_1^2, \ldots, S_n^2)$ is a $\epsilon$-Nash equilibrium. The cost of commodity $i$ is

$$c_i(S^2) = (z+1)^2 + \gamma z^2,$$

as there are exactly $z+1$ commodities using facilities of type $\alpha$ and exactly $z$ players using facilities of type $\beta$.

If commodity $i$ unilaterally switches to the other available strategy $S_i^1$, its cost becomes

$$c_i(S_i^1, S_{-i}^2) = (z+1) + \gamma z = \frac{c_i(S^2)}{1+\epsilon},$$

which shows that $S^2$ is an $\epsilon-$approximate equilibrium.

The optimum is the strategy profile $S^1$, where every commodity has cost $c_i(S^1) = 1+\gamma$. It follows that the price of anarchy is

$$\frac{c_i(S^2)}{c_i(S^1)} = \frac{(z+1)^2 + \gamma z^2}{1+\gamma} = (1+\epsilon)\frac{z(z+1)}{2z-\epsilon}.$$

□

## 5 Atomic Games – PoS

An upper bound of the price of stability is perhaps more difficult to obtain because we have to find a way to characterize the best $\epsilon$-Nash equilibrium. We don't know how to do this, so we use an indirect approach: We identify a property such that every profile satisfying the property is guaranteed to be an $\epsilon$-Nash equilibrium. We then upper bound the price of anarchy of all profiles satisfying this property. To this end, we generalize the notion of potential [17]; a characteristic property of congestion games is that they possess a potential function.

We define the $\epsilon$-potential function of a profile $A$ to be q

$$\Phi^\epsilon(A) = \frac{1}{2}\sum_{e\in E}(a_e n_e(A) + b_e)n_e(A) + \frac{1}{2}\frac{1-\epsilon}{1+\epsilon}\sum_{e\in E}(a_e + b_e)n_e(A).$$

For $\epsilon = 0$, this reduces to the classical potential function for congestion games. More importantly, it generalizes the following interesting property to $\epsilon$-Nash equilibrium.

**Theorem 6.** *Any profile $A$ which is a local minimum of $\Phi^\epsilon$, is an $\epsilon$-Nash equilibrium.*

*Proof.* Consider a profile $A = (A_1, \dots, A_n)$. We want to compute the change in the $\epsilon$-potential function when player $i$ changes from strategy $A_i$ to a strategy $P_i \in \mathcal{S}_i$. The resulting profile $(P_i, A_{-i})$ has

$$n_e(P_i, A_{-i}) = \begin{cases} n_e(A)+1, & e \in P_i - A_i \\ n_e(A)-1, & e \in A_i - P_i \\ n_e(A), & \text{otherwise.} \end{cases}$$

From this we can compute the difference

$$\Phi^\epsilon(P_i, A_{-i}) - \Phi^\epsilon(A) = \sum_{e\in P_i - A_i}\left(a_e n_e(A) + \frac{1}{1+\epsilon}(a_e + b_e)\right) - \sum_{e\in A_i - P_i}\left(a_e n_e(A) + \frac{1}{1+\epsilon}(-a_e\epsilon + b_e)\right).$$

We can rewrite this as

$$\Phi^\epsilon(P_i, A_{-i}) - \Phi^\epsilon(A) = \sum_{e\in P_i}\left(a_e n_e(A) + \frac{1}{1+\epsilon}(a_e + b_e)\right) - \sum_{e\in P_i\cap A_i} a_e - \sum_{e\in A_i}\left(a_e n_e(A) + \frac{1}{1+\epsilon}(-a_e\epsilon + b_e)\right). \quad (2)$$

Suppose now that profile $A$ is a local minimum of $\Phi^\epsilon$. This translates to $\Phi^\epsilon(P_i, A_{-i}) \geq \Phi^\epsilon(A)$ for all $i$. The cost for player $i$ before the change is $c_i(A) = \sum_{e\in A_i}(a_e n_e(A) + b_e)$ and after the change is $c_i(P_i, A_{-i}) = \sum_{e\in P_i}(a_e n_e(P_i, A_{-i}) + b_e)$. We want to show that $A$ is an $\epsilon$-Nash equilibrium: $c_i(A) \leq (1+\epsilon)c_i(P_i, A_{-i})$.

The $\epsilon$-potential consists of two parts that can be used to bound the cost of player $i$ at profile $A$ and $(P_i, A_{-i})$:

$$\begin{aligned} c_i(A) &= \sum_{e\in A_i}(a_e n_e(A) + b_e) \\ &\leq \sum_{e\in A_i}(1+\epsilon)\left(a_e n_e(A) + \frac{1}{1+\epsilon}(-a_e\epsilon + b_e)\right), \end{aligned}$$

(which holds because $n_e(A) \geq 1$ when $e \in A_i$), and

$$\begin{aligned} c_i(P_i, A_{-i}) &= \sum_{e\in P_i}(a_e(n_e(A)+1) + b_e) - \sum_{e\in P_i\cap A_i} a_e \\ &\geq \sum_{e\in P_i}\left(a_e n_e(A) + \frac{1}{1+\epsilon}(a_e + b_e)\right) - \sum_{e\in P_i\cap A_i} a_e \end{aligned}$$

(which holds for $\epsilon \geq 0$).

It follows immediately that $c_i(A) \leq (1+\epsilon)c_i(P_i, A_{-i})$. Consequently, $A$ is an $\epsilon$-Nash equilibrium. □

First we present an easy upper bound, that uses only the previous theorem.

**Proposition 1.** *For linear congestion games, the price of stability is at most* $\frac{2}{1+\epsilon}$.

*Proof.* Let $A$ be the allocation that minimizes the $\epsilon$ potential $\Phi^\epsilon$, and let $P$ be the optimum allocation.

We have

$$\Phi^\epsilon(A) \leq \Phi^\epsilon(P)$$

and so

$$\textsc{Sum}(A) + \frac{1-\epsilon}{1+\epsilon}\sum_{e\in E}(a_e + b_e)n_e(A) \leq \textsc{Sum}(P) + \frac{1-\epsilon}{1+\epsilon}\sum_{e\in E}(a_e + b_e)n_e(P), \tag{3}$$

from which we get

$$\textsc{Sum}(A) \leq \frac{2}{1+\epsilon}\textsc{Sum}(P).$$

□

The previous theorem gives us an easy way to bound the price of stability. Clearly this is not tight: for $\epsilon = 0$, it doesn't provide us the right answer $1 + \sqrt{3}/3$ [7, 4], although it gives us a good estimation. To get a better upper bound we need to work harder.

**Lemma 3.** *For integers* $\alpha, \beta$ *and for* $\gamma = \frac{(3+2\sqrt{3})(e-3+2\sqrt{3})}{3e+3+2\sqrt{3}}$

$$\gamma\beta^2 + \frac{1-\gamma\epsilon}{1+\epsilon}\beta - \frac{\gamma-\epsilon}{1+\epsilon}\alpha + (1-\gamma)\beta\alpha \leq \frac{(2\sqrt{3}-3)(e-1)}{3e+3+2\sqrt{3}}\alpha^2 + 2\frac{3+\sqrt{3}}{3e+3+2\sqrt{3}}\beta^2$$

*Proof.* Let $f$ be the expression that we take if we substitute $\gamma = \frac{\left(3+2\sqrt{3}\right)\left(e-3+2\sqrt{3}\right)}{3e+3+2\sqrt{3}}$, and then substract the first part from the second and divide by $\frac{\left(2\sqrt{3}-3\right)(e-1)}{3e+3+2\sqrt{3}}$. We can study $f$ as a function of integers $\alpha$ and $\beta$.

$$f(\alpha,\beta) = 1/4\left(2\sqrt{3}+3-4b-2b\sqrt{3}+2a\right)^2 + 1/8\left(5+3\sqrt{3}\right)\left(8\beta-3-3\sqrt{3}\right).$$

We want to prove that $f(\alpha,\beta) \geq 0$, for every $\alpha,\beta \in \mathbb{N}$. One can easily verify that

$$f(\alpha,0) = \left(3+a+2\sqrt{3}\right)a \geq 0,$$

$$f(\alpha,1) = \alpha(\alpha-1) \geq 0,$$

while for $\beta \geq 2$ it gets only positive values. □

We can now prove the most important result of this section.

**Theorem 7** (Atomic-PoS-Upper-Bound). *For any positive real $\epsilon \leq 1$, the approximate price of stability of general congestion games with linear latencies is at most*

$$\frac{\sqrt{3}+1}{\sqrt{3}+\epsilon}.$$

*Proof.* Let $A$ be the allocation that minimizes the $\epsilon$ potential $\Phi^\epsilon$, and let $P$ be the optimum allocation. Since $A$ is a local minimum of $\Phi^\epsilon$, if we sum (2) for all players $i$, we get

$$\sum_{e\in E} n_e(A)\left(a_e n_e(A) + \frac{1}{1+\epsilon}(-a_e\epsilon + b_e)\right) \leq \sum_{e\in E} n_e(P)\left(a_e n_e(A) + \frac{1}{1+\epsilon}(a_e+b_e)\right) - \sum_{i\in N}\sum_{e\in P_i\cap A_i} a_e.$$

For simplicity let's assume $a_e = 1, b_e = 0$, although the results hold in general. We get

$$\sum_{e\in E}\left(n_e^2(A) - \frac{\epsilon}{1+\epsilon}n_e(A)\right) \leq \sum_{e\in E} n_e(P)\left(n_e(A) + \frac{1}{1+\epsilon}\right) \tag{4}$$

If we multiply (3) with $\gamma$ and (4) with $(1-\gamma)$, for $\gamma = \frac{\left(3+2\sqrt{3}\right)\left(e-3+2\sqrt{3}\right)}{3e+3+2\sqrt{3}}$ and add them, we get

$$\begin{aligned}
\sum_{e\in E} n_e^2(A) &\leq \gamma\beta^2 + \frac{1-\gamma\epsilon}{1+\epsilon}\sum_{e\in E} n_e(P) - \frac{\gamma-\epsilon}{1+\epsilon}\sum_{e\in E} n_e(A) + (1-\gamma)\sum_{e\in E} n_e(P)n_e(A) \\
&\leq \frac{\left(2\sqrt{3}-3\right)(1-\epsilon)}{3\epsilon+3+2\sqrt{3}}\sum_{e\in E} n_e^2(A) + \frac{6+2\sqrt{3}}{3\epsilon+3+2\sqrt{3}}\sum_{e\in E} n_e^2(P)
\end{aligned}$$

and so

$$\sum_{e\in E} n_e^2(A) \leq \frac{\sqrt{3}+1}{\sqrt{3}+\epsilon}\sum_{e\in E} n_e^2(P).$$

□

Theorem 6 implies that the socially optimal allocation is 1−equilibrium. So for $\epsilon \geq 1$, trivially the price of stability is 1. The following theorem shows that this is tight, in the sense that the social cost of the best $(1-\delta)$-approximate equilibrium, is strictly greater than the social optimum.

**Theorem 8.** *There exist instances of congestion games, (even with two parallel links), where a the optimum allocation is not a $(1-\delta)-$approximate equilibrium, for any arbitrarily small positive $\delta$. This means that the price of stability for $(1-\delta)$-approximate equilibria is strictly greater than 1.*

*Proof.* Consider a game with two facilities (*parallel links*) $e_1, e_2$ and $n$ players. The facilities have latencies $l_{e_1}(x) = (2n-1) \cdot x - \gamma$, for some arbitrary small positive $\gamma$ and $l_{e_2}(x) = x$.

Consider the allocation $P$, that is produced when one player, (say the first), chooses the first link and the rest of the players use the second link. This has cost

$$\text{SUM}(P) = 2n - 1 - \gamma + (n-1)^2,$$

which is optimal: Any other allocation, in which $k \neq 1$ players use the first link and $n-k$ the second, has cost

$$(2n-1-\gamma)k^2 + (n-k)^2 \geq (2n-1-\gamma) + (n-1)^2.$$

In strategy profile $P$, the first player has cost $2n-1-\gamma$, while the rest of the players have cost $n-1$ each. If the first player unilaterally deviates to the second link he will have cost $n$. This means that OPT is a $(1-\frac{1+\gamma}{n})$-approximate equilibrium. Therefore, for any $\delta$, there is an instance with sufficiently large number of players $n(\delta)$, where OPT is not a $(1-\delta)$-approximate equilibrium. □

We now give an almost matching lower bound for the price of stability. The upper and lower bounds are not equal but they match at the extreme values of $\epsilon = 0$ and $\epsilon = 1$. For $\epsilon = 0$, we get the known price of stability [7, 4]. The price of stability decreases as a function of $\epsilon$, and drops to 1 for $\epsilon = 1$.

**Theorem 9** (Atomic-PoS-Lower-Bound)**.** *There are linear congestion games whose approximate Nash equilibria (even their dominant equilibria as the proof reveals) have price of stability of the* SUM *social cost approaching*

$$2\,\frac{3+\epsilon+\theta\epsilon^2+3\,\epsilon^3+2\,\epsilon^4+\theta+\theta\epsilon}{6+2\,\epsilon+5\,\theta\epsilon+6\,\epsilon^3+4\,\epsilon^4-\theta\epsilon^3+2\,\theta\epsilon^2},$$

*where* $\theta = \sqrt{3\,\epsilon^3+3+\epsilon+2\,\epsilon^4}$.

*Proof.* We describe a game of $n+\lambda$ players with parameters $\alpha$, $\beta$, and $\lambda$ which we will fix later to obtain the desired properties. Each player $i$ has two strategies $A_i$ and $P_i$, where the strategy profile $(A_1, \ldots, A_n)$ will be the equilibrium and $(P_1, \ldots, P_n)$ will have optimal social cost.

There are also $\lambda$ players that have fixed strategies; they don't have any alternative. They play a fixed facility $f_\lambda$.

There are 3 types of facilities:

- $n$ facilities $\alpha_i$, $i = 1, \ldots, n$, each with cost function $l(x) = \alpha x$. Facility $\alpha_i$ belongs only to strategy $P_i$.
- $n(n-1)$ facilities $\beta_{ij}$, $i, j = 1, \ldots, n$ and $i \neq j$, each with cost $l(x) = \beta x$. Facility $\beta_{ij}$ belongs only to strategies $A_i$ and $P_j$.
- 1 facility $f_\lambda$ with unit cost $l(x) = x$.

We will first compute the cost of every player and every strategy profile. By symmetry, we need only to consider the cost $cost_A(k)$ of player 1 and the cost $cost_P(k)$ of player $N$ of the strategy profile $(A_1, \ldots, A_k, P_{k+1}, \ldots, P_n)$. Therefore,

$$cost_A(k) \;\; = \;\; (2n-k-1)\beta + (\lambda + k).$$

Similarly, we compute

$$cost_P(k) \quad = \quad \alpha + (n+k-1)\beta.$$

We now want to select the parameters $\alpha$ and $\beta$ so that the strategy profile $(A_1, \ldots, A_n)$ is $(1+\epsilon)$-dominant. Equivalently, at every strategy profile $(A_1, \ldots, A_k,\ P_{k+1}, \ldots, P_n)$, player $i$, $i = 1, \ldots, k$, has no reason to switch to strategy $P_i$, because it's $(1+\epsilon)$ times less profitable. We use dominant strategies because it is easier to guarantee that there is no other equilibrium. This is expressed by the constraints

$$(1+\epsilon)\cdot cost_A(k) \le cost_P(k-1), \quad \text{for every } k = 1, \ldots, n. \tag{5}$$

All these constraints are linear in $k$ and they are satisfied by equality when

$$\alpha = \frac{(1+\epsilon)\,(2\,n\epsilon - \epsilon + \epsilon\lambda + n + 2\,\lambda + 1)}{2+\epsilon}$$

and

$$\beta = \frac{1+\epsilon}{2+\epsilon},$$

as one can verify with straightforward, albeit tedious, substitution.

In summary, for the above values of the parameters $\alpha$ and $\beta$, we obtain the desired property that the strategy profile $(A_1, \ldots, A_n)$ is a $(1+\epsilon)$-dominant strategy. If we increase $\alpha$ by any small positive $\delta$, inequality (5) becomes strict and the $(1+\epsilon)$-dominant strategy is unique (and therefore unique Nash equilibrium).

We now want to select the value of the parameter $m$ so that the price of anarchy[1] of this equilibrium is as high as possible. The price of anarchy is

$$\frac{cost_A(N) + \lambda(\lambda+n)}{cost_P(0) + \lambda^2}$$

which for the above values of $\alpha$ and $\beta$ can be simplified to

$$\frac{3\,n^2 + 2\,n^2\epsilon - n - n\epsilon + 4\,n\lambda + 2\,n\lambda\epsilon + 2\,\lambda^2 + \epsilon\lambda^2}{4\,n^2\epsilon - n\epsilon + 3\,n\lambda\epsilon + 2\,n^2 + 2\,n\lambda + 2\,n^2\epsilon^2 - n\epsilon^2 + n\epsilon^2\lambda + 2\,\lambda^2 + \epsilon\lambda^2}.$$

If we optimize the parameter $\lambda$ and take the limit of $n$ to infinity we get the theorem. □

## 6 Selfish Routing – PoS

Here we follow the ideas of the previous section to define an appropriate potential function for $\epsilon$-Nash equilibrium for the selfish routing problem or more generally non-atomic congestion games. It is easier to deal with the more general case of non-atomic congestion games rather than the selfish routing case, since we don't have to concern ourselves with the underlying network. In fact, our approach reveals how little we really need to establish results that encompass many influential results in the literature.

Consider a flow $f$ for the selfish routing with flow $f_e$ through every edge $e$. We define the $\epsilon$-potential function

$$\Phi^\epsilon(f) = \sum_{e\in E}\left(\frac{1}{2}\,a_e f_e^2 + \frac{1}{1+\epsilon}\,b_e f_e\right).$$

We will show that the global minimum of $\Phi^\epsilon(f)$ is an $\epsilon$-Nash equilibrium:

[1] Since this is the unique $1+\epsilon$ Nash Equilibrium of this game, the terms price of anarchy and price of stability are equivalent.

**Theorem 10.** *In a non-atomic congestion game, the flow $f$ which minimizes the $\epsilon$-potential function is an $\epsilon$-Nash equilibrium. Furthermore, for any other flow $f'$ the following inequality holds:*

$$\sum_{e \in E} \left( a_e f_e^2 + \frac{1}{1+\epsilon}\, b_e f_e \right) \le \sum_{e \in E} \left( a_e f_e f'_e + \frac{1}{1+\epsilon}\, b_e f'_e \right).$$

*Proof.* Consider a flow $f$ and two paths $p$ and $p'$ of the same commodity. Suppose that the flow $f$ on path $p$ is positive. We want to compute how much $\Phi^\epsilon(f)$ changes when we shift a small amount $\delta > 0$ of flow from path $p$ to path $p'$. More precisely, if $f'$ denotes the new flow, we compute the following limit

$$\lim_{\delta \to 0} \frac{\Phi^\epsilon(f') - \Phi^\epsilon(f)}{\delta} = \sum_{e \in p'} \left( a_e f_e + \frac{1}{1+\epsilon}\, b_e \right) - \sum_{e \in p} \left( a_e f_e + \frac{1}{1+\epsilon}\, b_e \right) \tag{6}$$

If $f$ minimizes $\Phi^\epsilon$, then the above quantity is nonnegative. But we can bound the cost of paths $p$ and $p'$ with the two terms of this quantity as follows:

$$l_p(f) = \sum_{e \in p} (a_e f_e + b_e) \le (1+\epsilon) \sum_{e \in p} (a_e f_e + \frac{1}{1+\epsilon}\, b_e)$$

and

$$l_{p'}(f) = \sum_{e \in p'} (a_e f_e + b_e) \ge \sum_{e \in p'} (a_e f_e + \frac{1}{1+\epsilon}\, b_e).$$

It follows that $l_p(f) \le (1+\epsilon) l_{p'}(f)$, which implies that $f$ is an $\epsilon$-Nash equilibrium.

For the second part, we observe that the expression (6), which is nonnegative for $f$ which minimizes $\Phi^\epsilon$, implies that for every path $p$ on which $f$ is positive and every other path $p'$ we must have

$$\sum_{e \in p} \left( a_e f_e + \frac{1}{1+\epsilon}\, b_e \right) \le \sum_{e \in p'} \left( a_e f_e + \frac{1}{1+\epsilon}\, b_e \right).$$

Consider now another flow $f'$ which satisfies the rate constraints for the commodities and let us sum the above inequalities for all paths $p$ and $p'$ weighted with the amount of flow in $f$ and $f'$. More precisely:

$$\sum_{p,p'} f_p f'_{p'} \sum_{e \in p} \left( a_e f_e + \frac{1}{1+\epsilon}\, b_e \right) \le \sum_{p,p'} f_p f'_{p'} \sum_{e \in p'} \left( a_e f_e + \frac{1}{1+\epsilon}\, b_e \right)$$

$$\sum_{p'} f'_{p'} \sum_{e \in E} \left( a_e f_e^2 + \frac{1}{1+\epsilon}\, b_e f_e \right) \le \sum_{p} f_p \sum_{e \in E} \left( a_e f_e f'_e + \frac{1}{1+\epsilon}\, b_e f'_e \right)$$

But $\sum_{p'} f'_{p'} = \sum_p f_p$ is equal to the sum of the rates for all commodities. If we remove from the expression this common factor, the second part of the theorem follows. ☐

From Lemma 2, if we substitute $\lambda$ with $1/(1+\epsilon)$, we get that for any reals $\alpha, \beta$, and $\epsilon \in [0,1]$

$$\alpha\beta \le \frac{1+\epsilon}{4}\alpha^2 + \frac{1}{1+\epsilon}\beta^2. \tag{7}$$

**Theorem 11** (Selfish-PoS-Upper-Bound)**.** *The price of stability is at most $\frac{4}{(3-\epsilon)(1+\epsilon)}$.*

*Proof.* Let $f$ be the potential minimizer of $\Phi^\epsilon$ and $f^*$ be the optimum flow. From Theorem (10) and (7) we get that

$$\sum_{e\in E} a_e f_e^2 + \frac{1}{1+\epsilon} b_e f_e \leq \sum_{e\in E} a_e(\frac{1+\epsilon}{4} f_e 2 + \frac{1}{1+\epsilon} f_e^{*2}) + \frac{1}{1+\epsilon} b_e f_e^*$$

or

$$\sum_{e\in E} a_e \frac{3-\epsilon}{4} f_e^2 + \frac{1}{1+\epsilon} b_e f_e \leq \frac{1}{1+\epsilon} C(f^*),$$

and since $1/(1+\epsilon) \geq (3-\epsilon)/4$, we get

$$\frac{3-\epsilon}{4} C(f) \leq \frac{1}{1+\epsilon} C(f^*),$$

which gives the desired result:

$$C(f) \leq \frac{4}{(3-\epsilon)(1+\epsilon)} C(f^*).$$

□

We now establish that the Pigou network (extended to take into account the parameter $\epsilon$, Figure 1) gives tight results.

**Theorem 12** (Selfish-PoS-Lower-Bound)**.** *The price of stability is at least* $\frac{4}{(3-\epsilon)(1+\epsilon)}$.

*Proof.* Consider the Pigou network of Figure 1. There is a unit of flow that wants to move from $s$ to $t$. Clearly, the only $(1+\epsilon)$-Wardrop flow is to choose the lower edge, for $\epsilon < 1$. This gives a social cost of 1.

On the other hand the optimum is to route $(1+\epsilon)/2$ of the traffic from the lower edge and $(1-\epsilon)/2$ of the traffic from the upper edge. This gives a social opt of $\frac{(1+\epsilon)(1-\epsilon)}{2} + \frac{(1+\epsilon)}{2}\frac{(1+\epsilon)}{2} = \frac{(1+\epsilon)(3-\epsilon)}{4}$, and so the price of stability is $\frac{4}{(3-\epsilon)(1+\epsilon)}$ as needed. □

**Acknowledgements** The authors would like to thank Ioannis Caragiannis for many helpful discussions and Tim Roughgarden for useful pointers to literature.

# References

[1] Elliot Anshelevich, Anirban Dasgupta, Jon M. Kleinberg, Éva Tardos, Tom Wexler, and Tim Roughgarden. The price of stability for network design with fair cost allocation. In *45th Symposium on Foundations of Computer Science (FOCS)*, pages 295–304, 2004.

[2] Baruch Awerbuch, Yossi Azar, and Amir Epstein. The price of routing unsplittable flow. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC)*, pages 57–66, 2005.

[3] M. Beckmann, C. B. McGuire, and C. B. Winston. *Studies in the Economimices of Transportation.* Yale University Press, 1956.

[4] I. Caragiannis, M. Flammini, C. Kaklamanis, P. Kanellopoulos, and L. Moscardelli. Tight bounds for selfish and greedy load balancing. In *Proceedings of the 33rd International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 311–322, 2006.

[5] Xi Chen and Xiaotie Deng. Settling the complexity of two-player nash equilibrium. In *47th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 261–272, 2006.

[6] Steve Chien and Alistair Sinclair. Convergence to approximate nash equilibria in congestion games. In *Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms(SODA)*, pages 169–178, 2007.

[7] George Christodoulou and Elias Koutsoupias. On the price of anarchy and stability of correlated equilibria of linear congestion games. In *Algorithms - ESA 2005, 13th Annual European Symposium*, pages 59–70, 2005.

[8] George Christodoulou and Elias Koutsoupias. The price of anarchy of finite congestion games. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC)*, pages 67–73, 2005.

[9] José R. Correa, Andreas S. Schulz, and Nicolás E. Stier Moses. Selfish routing in capacitated networks. *Math. Oper. Res.*, 29(4):961–976, 2004.

[10] S. C. Dafermos and F. T. Sparrow. The traffic assignment problem for a general network. *Journal of Research of the National Bureau of Standards, Series B*, 73B(2):91–118, 1969.

[11] Constantinos Daskalakis, Paul W. Goldberg, and Christos H. Papadimitriou. The complexity of computing a nash equilibrium. In *Proceedings of the 38th Annual ACM Symposium on Theory of Computing (STOC)*, pages 71–78, 2006.

[12] Constantinos Daskalakis, Aranyak Mehta, and Christos H. Papadimitriou. A note on approximate nash equilibria. In *WINE*, 2006.

[13] A. Fabrikant, C. Papadimitriou, and K. Talwar. On the complexity of pure equilibria. In *STOC*, 2004.

[14] E. Koutsoupias and C. H. Papadimitriou. Worst-case equilibria. In *Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science (STACS)*, pages 404–413, 1999.

[15] Richard J. Lipton, Evangelos Markakis, and Aranyak Mehta. Playing large games using simple strategies. In *ACM Conference on Electronic Commerce (EC)*, pages 36–41, 2003.

[16] I. Milchtaich. Congestion games with player-specific payoff functions. *Games and Economic Behavior*, 13:111–124, 1996.

[17] D. Monderer and L. Shapley. Potential games. *Games and Economics Behavior*, 14:124–143, 1996.

[18] N. Nisan, T. Roughgarden, E. Tardos, and V.V. Vazirani. *Algorithmic Game Theory*. Cambridge University Press, 2007.

[19] Christos H. Papadimitriou. Algorithms, games, and the internet. In *Proceedings on 33rd Annual ACM Symposium on Theory of Computing (STOC)*, pages 749–753, 2001.

[20] R. W. Rosenthal. A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory*, 2:65–67, 1973.

[21] T. Roughgarden and E. Tardos. How bad is selfish routing? *J. ACM*, 49(2):236–259, 2002.

[22] T. Roughgarden and E. Tardos. Bounding the inefficiency of equilibria in nonatomic congestion games. *Games and Economic Behavior*, 47(2):389–403, 2004.

[23] Tim Roughgarden. *Selfish Routing.* PhD. Thesis, Cornell University, May 2002.

[24] Tim Roughgarden. The price of anarchy is independent of the network topology. *J. Comput. Syst. Sci.*, 67(2):341–364, 2003.

[25] Tim Roughgarden. *Selfish Routing and the Price of Anarchy.* The MIT Press, 2005.

[26] H. Tsaknakis and P. G. Spirakis. An optimization approach for approximate nash equilibria. In *WINE*, volume 4858, pages 47–56. LNCS , Springer, 2007.

[27] J. G. Wardrop. Some theoretical aspects of road traffic research. In *Proceedings of the Institute of Civil Engineers, Part II*, volume 1, pages 325–378, 1952.