Sheaves, Objects and Distributed Systems

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Sources


Some Slogans

Objects Give Rise to Sheaves
Morphisms Represent Inheritance
Systems are Diagrams
Behaviour is Limit
Interconnection is Colimit
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Complete Heyting Algebras

Sheaves can be thought of as consistent systems of observations.

Observations are made at ‘locations’, which have ‘intersections’ and ‘unions’
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Examples

\[ \omega + 1 \] Topological spaces

This can be seen as discrete time.

\[ \Omega(P) \] The set of downwards-closed subsets of a partially ordered set \( P \)

For example, if \( P \) is a monoid, the prefix order is \( x \leq y \) iff \( xz = y \) for some \( z \); then \( \Omega(P) \) is the set of traces.
Examples

- Topological spaces

- $\omega + 1$ The set $\{0, 1, 2, \ldots, \omega\}$
  This can be seen as discrete time.

- $\Omega(P)$ The set of downwards-closed subsets of a partially ordered set $P$
  - For example, if $P$ is a monoid, the prefix order is $x \leq y$ iff $xz = y$ for some $z$;
    then $\Omega(P)$ is the set of traces.
Examples

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Examples

\( \omega + 1 \) The set \( \{0, 1, 2, \ldots, \omega\} \)
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- For example, if \( P \) is a monoid, the prefix order is \( x \leq y \) iff \( xz = y \) for some \( z \); then \( \Omega(P) \) is the set of traces.
Sheaves are Presheaves

\[ F(C_1) \rightarrow F(C_2) \]

\[ C_1 \rightarrow C_2 \]
Sheaves are Presheaves with Pasting

Consistent families

\[ F(C_1 \cup C_2) \]

\[ F(C_1) \quad F(C_2) \]

\[ F(C_1 \cap C_2) \]
Sheaves are Presheaves with Pasting

Consistent families have unique amalgamations
Example: Continuous Functions

For a topological space $\mathcal{O}$,

$$R(C) = \{ f : C \to R \mid f \text{ is continuous} \}$$
Example: Substitutions

Problem: $Y = 2X$ and $Y \leq Z \leq 2Y$

$$\{X, Y, Z\} : \quad X \leftarrow 3, \quad Y \leftarrow 6, \quad Z \leftarrow 8$$

$$\{X, Y\} : \quad X \leftarrow 3, \quad Y \leftarrow 6$$

$$\{Y, Z\} : \quad Y \leftarrow 6, \quad Z \leftarrow 8$$

$$\{Y\} : \quad Y \leftarrow 6$$
Example: Trading Nails and Screws

\[ N \parallel S : (N \text{ gives } n \parallel S \text{ gives } s); (N \text{ gets } s \parallel S \text{ gets } n) \]

\[ \begin{align*}
N : & \text{ give n; get s} \\
S : & \text{ give s; get n} \\
B : & \text{ in; out}
\end{align*} \]
Example: Partial Truth

Let $\Omega(X)$ be the (downward-closed) subsets of $X$.
If $X \subseteq Y$, restriction $\Omega(Y) \to \Omega(X)$ is given by $S \subseteq Y \mapsto S \cap X \subseteq X$.

Amalgamation:

- $X_1 \cup X_2 : S_1 \cup S_2$
- $X_1 : S_1$
- $X_2 : S_2$
- $X_1 \cap X_2 : S_1 \cap X_1 \cap X_2 \cap S_2$
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\begin{align*}
X_1 \cup X_2 : & \quad S_1 \cup S_2 \\
X_1 : & \quad S_1 \\
X_2 : & \quad S_2 \\
X_1 \cap X_2 : & \quad S_1 \cap X_1 \cap X_2 \cap S_2
\end{align*}
\]
Sheaves as Transition Systems


Sheaves as Transition Systems


Semantics of an Object-oriented Language (Cîrstea)
Labelled Transition Systems

Let $\mathcal{M} = (M, \cdot, \varepsilon)$ be a monoid.

**Definition**

A labeled transition system over $\mathcal{M}$ is a pair $(T, \rightarrow)$, with $\rightarrow \subseteq T \times M \times T$ such that

$t \xrightarrow{\varepsilon} t'$ iff $t = t'$

$t \xrightarrow{m n} t''$ iff $t \xrightarrow{m} t'$ and $t' \xrightarrow{n} t''$ for some $t' \in T$.

Morphisms $(h, f) : (\mathcal{M}, T) \to (\mathcal{N}, U)$ preserve transitions.
Let $\mathcal{M} = (M, \cdot, \varepsilon)$ be a monoid.

**Definition**

A *labelled transition system over* $\mathcal{M}$ is a pair $(T, \xrightarrow{\cdot})$, with $\xrightarrow{\cdot} \subseteq T \times M \times T$ such that

- $t \xrightarrow{\varepsilon} t'$ iff $t = t'$
- $t \xrightarrow{mn} t''$ iff $t \xrightarrow{m} t'$ and $t' \xrightarrow{n} t''$ for some $t' \in T$.

Morphisms $(h, f) : (\mathcal{M}, T) \rightarrow (\mathcal{N}, U)$ preserve transitions.
Example: Coffee Dispenser

Let $M = \{c, d, r\}^*$. Let $T = Bool \times \{0..20\}$ with transitions:

- $(false, N) \xrightarrow{c} (true, N)$ for all $0 \leq N \leq 20$, and
- $(true, N) \xrightarrow{d} (false, N - 1)$ for all $0 < N \leq 20$,
- $(B, N) \xrightarrow{r} (B, 20)$ for all $B \in Bool$ and all $0 \leq N \leq 20$. 
Example: Coin Slot

Let $\mathcal{N} = \{c, d\}^*$ and $U = Bool$ with transitions

- $false \xrightarrow{c} true$
- $true \xrightarrow{d} false$.

Here is a morphism from the coffee dispenser to the coin slot. The monoid homomorphism on labels is defined by

- $c \mapsto c$
- $d \mapsto d$
- $r \mapsto \varepsilon$

and on states $Bool \times \{0..20\} \rightarrow Bool$ is just the first projection.
Example: Coin Slot

Let $\mathcal{N} = \{c, d\}^*$ and $U = \mathit{Bool}$ with transitions

- $\text{false} \xrightarrow{c} \text{true}$
- $\text{true} \xrightarrow{d} \text{false}$.

Here is a morphism from the coffee dispenser to the coin slot. The monoid homomorphism on labels is defined by

\[
\begin{align*}
  c & \mapsto c \\
  d & \mapsto d \\
  r & \mapsto \varepsilon
\end{align*}
\]

and on states $\mathit{Bool} \times \{0..20\} \rightarrow \mathit{Bool}$ is just the first projection.
$\mathcal{M}, T$ determines the sheaf $F$ on $\Omega(M)$:

$F(X) = \text{LTS}(X, T)$

E.g., for $X = \{\varepsilon, a, aa, ab\}$ and $f : X \to T$:

$f(\varepsilon) \xrightarrow{a} f(a) \xrightarrow{a} f(aa)$ and

$f(\varepsilon) \xrightarrow{a} f(a) \xrightarrow{b} f(ab)$
\( \mathcal{M}, T \) determines the sheaf \( F \) on \( \Omega(\mathcal{M}) \):
\[
F(X) = LTS(X, T)
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E.g., for \( X = \{\varepsilon, a, aa, ab\} \) and \( f : X \rightarrow T \):
\[
\begin{align*}
   f(\varepsilon) &\xrightarrow{a} f(a) &\xrightarrow{a} f(aa) \\
   f(\varepsilon) &\xrightarrow{a} f(a) &\xrightarrow{b} f(ab)
\end{align*}
\]
Examples

\[ \varepsilon \]

\[ a \]

\[ ab \]

\[ \{aa, ab\} \]
Examples

\[ \varepsilon \]

\[ a \]

\[ a \]

\[ \{aa,ab\} \]

\[ ab \]
Sheaves $\rightarrow$ Labelled Transition Systems

A sheaf $F$ determines a labelled transition system with

$T = \sum_{m \in M} F(m\downarrow)$

and transitions $(m, e) \xrightarrow{n} (m', e')$ iff

$m' = mn$ and $e'\uparrow_{m\downarrow} = e$. 
A sheaf $F$ determines a labelled transition system with 
\[ T = \sum_{m \in M} F(m\downarrow) \]
and transitions \((m, e) \xrightarrow{n} (m', e')\) iff 
\(m' = mn\) and \(e' \mid_{m\downarrow} = e\).
 Behaviour as Limit

A money box with a coin slot has states $\text{Bool} \times \omega$ and transitions

- $(\text{false}, N) \xrightarrow{c} (\text{true}, N)$ for all $N \geq 0$,
- $(\text{true}, N) \xrightarrow{d} (\text{false}, N + 1)$ for all $N \geq 0$, and
- $(B, N) \xrightarrow{m} (B, 0)$ for all $N \geq 0$ and $B \in \text{Bool}$.

The monoid homomorphism to the coin slot is given by

\begin{align*}
 c & \mapsto c \\
 d & \mapsto d \\
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\end{align*}
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The monoid homomorphism to the coin slot is given by

\[\begin{align*}
c & \mapsto c \\
d & \mapsto d \\
m & \mapsto \varepsilon \end{align*}\]
Behaviour as Limit

Behaviour is limit:

\[ (f_1, g_1) \quad (f_2, g_2) \]

The limit of the monoid morphisms is \{cc, dd, m, r\} with, e.g.,

\[ cc \, r \, m \, dd \, m = cc \, m \, r \, dd \, m \]
Behaviour as Limit

Behaviour is limit:

The limit of the monoid morphisms is \{cc, dd, m, r\}* with, e.g.,

\[ cc \, r \, m \, dd \, m = cc \, m \, r \, dd \, m \]
A sheaf with values in a category $L$ is a functor $F$ from a complete Heyting algebra to $L$ such that if $X = \bigcup_{i \in I} X_i$, then

\[ F(X) \to \prod_{i \in I} F(X_i) \to \prod_{i,j \in I} F(X_i \cap X_j) \]

is an equaliser diagram (where all the arrows arise from the obvious restrictions by the universal property of the target product).
Behaviour-as-Limit is Sheafy

Let $X$ be a preorder category, and let $\delta : X \rightarrow \text{LTS}$. Define $\delta^* : \Omega(X) \rightarrow \text{LTS}$ by

$$\delta^*(X) = \lim(\delta|_X);$$

then $\delta^*$ is a sheaf of transition systems.

Or

Let $G$ be a directed graph; let $\mathcal{O}(G)$ be the subsets of nodes closed ‘under edges’.
Then $\delta : G \rightarrow L$ gives $\delta^* : \mathcal{O}(G) \rightarrow L$, where

$$\delta^*(X) = \lim(\delta|_X).$$
Let $X$ be a preorder category, and let $\delta : X \to \text{LTS}$. Define $\delta^* : \Omega(X) \to \text{LTS}$ by $\delta^*(X) = \lim(\delta|_X)$; then $\delta^*$ is a sheaf of transition systems.

Or

Let $G$ be a directed graph; let $\mathcal{O}(G)$ be the subsets of nodes closed ‘under edges’. Then $\delta : G \to L$ gives $\delta^* : \mathcal{O}(G) \to L$, where

$$\delta^*(X) = \lim(\delta|_X).$$
Independence (Winskel & Nielsen: |)

\[ t \xrightarrow{m} t_1 \sim t \xrightarrow{m} t_2 \Rightarrow t_1 = t_2 \]

\[ t \xrightarrow{m} t_1 \mid t \xrightarrow{n} t_2 \Rightarrow \]

\[ (\exists u) \ t \xrightarrow{m} t_1 \mid t_1 \xrightarrow{n} u \land t \xrightarrow{n} t_2 \mid t_2 \xrightarrow{m} u \]

\[ t \xrightarrow{m} t_1 \mid t_1 \xrightarrow{n} u \Rightarrow \]

\[ (\exists t_2) \ t \xrightarrow{m} t_1 \mid t_2 \xrightarrow{n} u \land t \xrightarrow{n} t_2 \mid t_2 \xrightarrow{m} u \]

\[ t \xrightarrow{m} t_1 \sim t_2 \xrightarrow{m} u \mid w \xrightarrow{n} w' \Rightarrow t \xrightarrow{m} t_1 \mid w \xrightarrow{n} w' \]

where \( \sim \) is the equivalence relation freely generated by \( \prec \), which is defined by \( t \xrightarrow{m} t_1 \prec t_2 \xrightarrow{m} u \) iff there is an \( n \) with \( t \xrightarrow{m} t_1 \mid t \xrightarrow{n} t_2, t \xrightarrow{m} t_1 \mid t_1 \xrightarrow{m} u \) and \( t \xrightarrow{n} t_2 \mid t_2 \xrightarrow{n} u \).
Independence (Sheaf Remix)

Let $F$ be a sheaf of transition systems. Transitions $t_1 \xrightarrow{m} t_1'$ and $t_2 \xleftarrow{n} t_2'$ at $F(C)$ are independent iff $C = C_1 \cup C_2$ and $m|_{C_1} = \varepsilon$ and $n|_{C_2} = \varepsilon$. 
Hidden Algebra (Goguen, et al.)

Algebraic specification of systems with state.

- sorts are partitioned: *visible* and *hidden*
- operations take at most one hidden argument
  (but cf. Goguen and Roşu)
- concurrent connection gives composition of subcomponents.
Hidden Algebra (Goguen, et al.)

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Algebraic specification of systems with state.

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  (but cf. Goguen and Roșu)
- concurrent connection gives composition of subcomponents.
Streams of Natural Numbers

th STREAM is
  pr DATA .

  hsort State .

  op head : State -> Nat .
  op tail : State -> State .

endth
th SENDER is
using STREAM .

op input : State -> Nat .
op put : Nat State -> State .

var N : Nat .
var S : State .
eq head(put(N,S)) = head(S) .
eq input(put(N,S)) = N .
eq head(tail(S)) = input(S) .
eq input(tail(S)) = input(S) .
endth
Sending Values on a Stream

th SENDER is
  using STREAM .  a meromorphism

    op input : State -> Nat .
    op put : Nat State -> State .

    var N : Nat .
    var S : State .
    eq  head(put(N,S)) = head(S) .
    eq  input(put(N,S)) = N .
    eq  head(tail(S)) = input(S) .
    eq  input(tail(S)) = input(S) .

endth
We think of $\text{STREAM}$ as a subobject of $\text{SENDER}$. The inclusion of $\text{STREAM}$ into $\text{SENDER}$ is a meromorphism.
th SUM[S :: STREAM] is

  op sum : State -> Nat .
  op add : State -> State .

  var N : Nat .
  var S : State .
  eq sum(add(S)) = head(S) + sum(S) .
  eq sum(tail(S)) = sum(S) .
  eq head(add(S)) = head(S) .
endth
Adding Values on a Stream

th SUM[S :: STREAM] is meromorphism

op sum : State -> Nat .
op add : State -> State .

var N : Nat .
var S : State .
eq sum(add(S)) = head(S) + sum(S) .
eq sum(tail(S)) = sum(S) .
eq head(add(S)) = head(S) .
endth
Parameterised Theories

\[ \text{SENDER} \rightarrow \text{SUM[SENDER]} \]

\[ \text{STREAM} \rightarrow \text{SUM[S :: STREAM]} \]
Concurrent Connection

th SENDER || SUM is

pr SUM[SENDER] .

var N : Nat .
var S : State .

eq input(add(S)) = input(S) .

eq sum(put(N,S)) = sum(S) .

eq put(N, add(S)) = add(put(N,S)) .
endth
Concurrent Connection

th SENDER || SUM is

pr SUM[SENDER] .

var N : Nat .
var S : State .
eq input(add(S)) = input(S) .
eq sum(put(N,S)) = sum(S) .
eq put(N, add(S)) = add(put(N,S)) .
endth
A Distributed System
Sheaves of Theories and Models

Concurrent connection is limit for object specifications with meromorphisms.

This extends to models — also as limit.

A natural view of models of such system specifications is sheaves of theories with models:

\[
\begin{array}{c}
\text{SENDER} \\
\downarrow \\
\text{STREAM} \\
\downarrow \\
\text{ADDER}
\end{array}
\]
Dynamic Systems…

…as a transition system (Colman Reilly, TCD):

**State set**: tuples \((G, A, b)\), where

- \(G\) is a graph,
- \(\delta : G \to \text{Set}\), and
- \(b\) is an element of the *limit* of \(\delta\) (i.e., an element of \(\delta^*\)).

**Transitions**: \((G, \delta, b) \leftrightarrow (G', \delta', b')\) iff

- there is a span \(G \leftarrow G_0 \rightarrow G'\), and
- \(\delta|_{G_0} = \delta'|_{G_0}\).
Example: Adding to a Linked List
Structured Dynamic Systems

**State set:** tuples \((G, \delta, b)\), where

- \(G\) is a graph,
- \(\delta : G \to \text{TSys}\), and
- \(b\) is an element of the *limit* of \(\delta\).

**Transitions:** \((G, \delta, b) \leftrightarrow (G', \delta', b')\) iff

- there is a span \(G \leftrightarrow G_0 \leftrightarrow G'\),
- \(\delta|_{G_0} = \delta'|_{G_0}\), and
- \(b|_{G_0} \leftrightarrow b'|_{G_0}\) in \(\text{Lim}(\delta|_{G_0})\).
Labelled Structured Dynamic Systems

**State set:** tuples \((G, \delta, b)\), where
- \(G\) is a graph,
- \(\delta : G \rightarrow \text{LTSys}\), and
- \(b\) is an element of the limit of \(\delta\) (i.e., an element of \(\delta^*\)).

**Transitions:** \((G, \delta, b) \xrightarrow{l} (G', \delta', b')\) iff
- there is a span \(G \leftarrow G_0 \rightarrow G'\),
- \(\delta |_{G_0} = \delta' |_{G_0}\), and
- \(b |_{G_0} \xrightarrow{l |_{G_0}} b' |_{G_0}\) in \(\text{Lim}(\delta |_{G_0})\).
- \((l \in \text{labels}(\text{Lim}(\delta |_{G_0}))).\)
Evolving Specifications

**State set:** pairs \((G, \delta)\), where

- \(G\) is a graph,
- \(\delta\) assigns to nodes:
  - a theory \(T\),
  - with distinguished sort \(s\),
  - a model \(B\) of \(T\), and
  - an element \(b \in B_s\).
- Edges use the Grothendieck construction.

...
would correspond to an operation

\[ \text{joinCR} : \text{Chatter} \; \text{Chatroom} \rightarrow \text{ChatterInChatroom} \]
Sheaves allow a structured approach to distributed systems
Similar to presheaf models for labelled transition systems
Generalises to sheaves of structures
- dynamic systems
- two-level rewriting: global and local
- work in progress...