3-Coloring is NP-Complete

- 3-Coloring is in NP
  - Certificate: for each node a color from $\{1, 2, 3\}$
  - Certifier: Check if for each edge (u, v), the color of u is different from that of v
- Hardness: We will show 3-SAT $\leq_P$ 3-Coloring

Reduction Idea

Start with 3-SAT formula ϕ with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$. Create graph G_ϕ such that G_ϕ is 3-colorable iff ϕ is satisfiable

- need to establish truth assignment for $x_1, \ldots, x_n$ via colors for some nodes in G_ϕ.
- create triangle with node True, False, Base
- for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
- If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i
- For each clause $C_j = (a \vee b \vee c)$, create a small gadget graph
    - gadget graph connects to nodes corresponding to a, b, c
    - needs to implement OR

OR-Gadget Graph

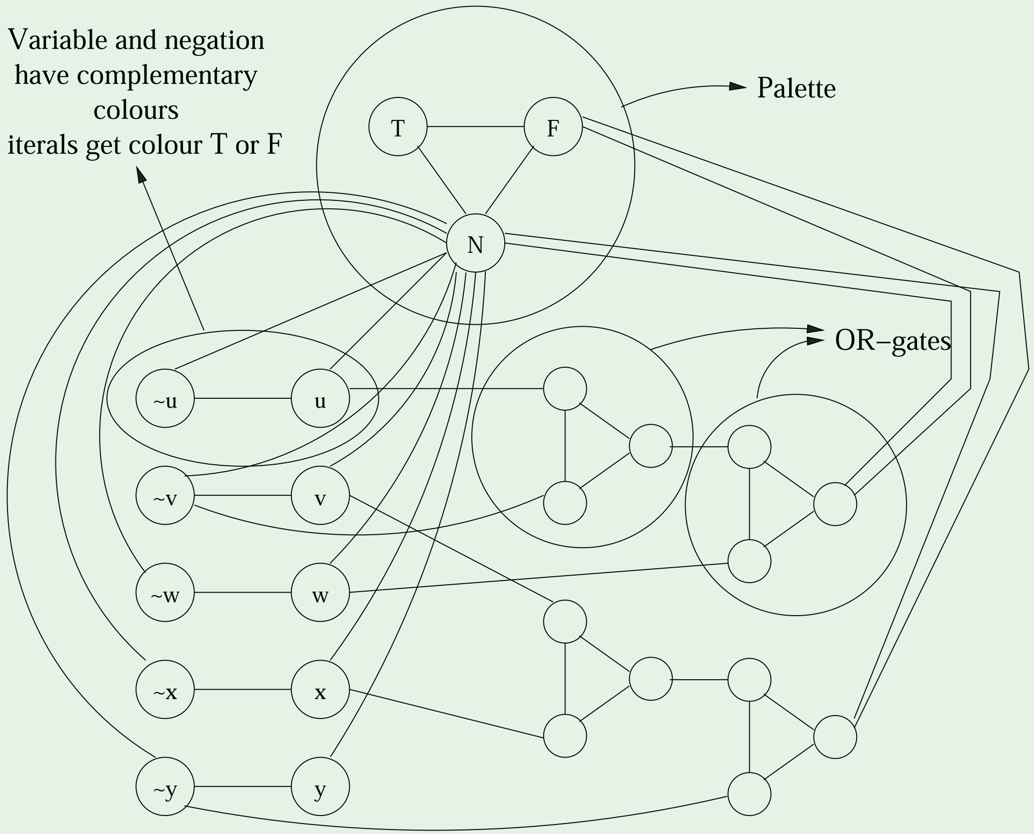
Property: if a, b, c are colored False in a 3-coloring then output node of OR-gadget has to be colored False.

Property: if one of a, b, c is colored True then OR-gadget can be 3-colored such that output node of OR-gadget is colored True.

- create triangle with node True, False, Base
- for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
- for each clause $C_j = (a \vee b \vee c)$, add OR-gadget graph with input nodes a, b, c and connect output node of gadget to both False and Base

Example

$$\varphi = (u \vee \neg v \vee w) \wedge (v \vee x \vee \neg y)$$



ϕ is satisfiable implies G_ϕ is 3-colorable

- if x_i is assigned True, color v_i True and $\bar{v}_i$ False
- for each clause $C_j = (a \vee b \vee c)$ at least one of a, b, c is colored True. OR-gadget for C_j can be 3-colored such that output is True.

G_ϕ is 3-colorable implies ϕ is satisfiable

- if v_i is colored True then set x_i to be True, this is a legal truth assignment
- consider any clause $C_j = (a \vee b \vee c)$. it cannot be that all a, b, c are False. If so, output of OR-gadget for C_j has to be colored False but output is connected to Base and False!