COMP202
Complexity of Algorithms
Fundamental Solution Techniques
Dynamic Programming
[See Chapter 5 in Goodrich and Tamassia.]
Learning Outcomes

At the conclusion of this set of lecture notes, you should:

1. Understand the general ideas behind the method of dynamic programming.

2. See the importance of the method of dynamic programming for solving problems having suitable structure, and be able to recognize when it can be applied.

3. Be familiar with some standard problems for which dynamic programming is a successful solution technique such as the \{0, 1\} Knapsack Problem and the Weighted Interval Scheduling Problem.
Dynamic Programming

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The main difference is that (possibly) repetitive recursive calls are replaced by a reference to already computed values stored in a *special table*. 
Dynamic programming is primarily used for optimization problems (maximizing or minimizing some function).

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However, dynamic programming is efficient only if the problem has a certain amount of structure that can be exploited.
An effective dynamic programming solution depends on the following factors:

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- **Sub-problem overlap**: Optimal solutions to some sub-problems can themselves contain sub-problems in common (and often do).
A typical approach to solve a problem using Dynamic Programming is to first find a *recursive* solution to the problem.
A typical approach to solve a problem using Dynamic Programming is to first find a recursive solution to the problem. Once this recursive solution is found, we then try to determine a way to compute solutions to the subproblems “from the bottom up”, so that we avoid recomputing solutions many times.
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Recall we have a set, \(S\), of \(n\) items where item \(i\) has benefit \(b_i\) and an integer weight \(w_i\).

We have the following objective:
Find a subset \(T \subseteq S\) that

\[
\text{maximizes } \sum_{i \in T} b_i
\]

subject to

\[
\sum_{i \in T} w_i \leq W_{\text{max}},
\]

where \(W_{\text{max}}\) is the maximum total allowed weight in the knapsack.
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*Exponential solution*: We can “easily” solve the \( \{0, 1\} \) knapsack problem in \( O(2^n) \) time by *testing all possible* subsets of the \( n \) items.
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Unfortunately, *exponential complexity* is unacceptable for large $n$, so we then want to focus on nice characterizations for sub-problems in order to use dynamic programming.
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If $I$ does use item $n$, then the optimal solution consists of item $n$, together with an optimal solution for the set $\{1, 2, \ldots, n-1\}$, but with a new maximum allowed weight of $W_{\text{max}} - w_n$. 
To more precisely define this recursive solution, let $S_k = \{1, 2, \ldots, k\}$ denote the set containing the first $k$ items, and define $S_0 = \emptyset$. 

Our desired solution is then $B[n, W_{max}]$. 

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Then we define $B[0, w] = 0$, for each $w \leq W_{\text{max}}$, and

$$B[k, w] = \begin{cases} B[k - 1, w] & \text{if } w < w_k \\ \max \{ B[k - 1, w], b_k + B[k - 1, w - w_k] \} & \text{otherwise.} \end{cases}$$
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Our desired solution is then \( B[n, W_{\text{max}}] \).
To make our solution as easy as possible to find, we will compute these values “from the bottom up”, i.e. first we compute $B[1, w]$ for each $w \leq W_{\text{max}}$.

Then we use those values to find $B[2, w]$ for all values of $w$, and so forth.
01KNAPSACK(S, W)

▷ Input: Set S of n items with weights \( w_i \), benefits \( b_i \), and a total weight \( W \).
▷ Output: A subset \( T \), of S with weight at most \( W \).

1 \hspace{1em} \textbf{for} w \leftarrow 0 \ \textbf{to} \ W
2 \hspace{1em} \textbf{do}
3 \hspace{2em} B[0, w] \leftarrow 0
4 \hspace{1em} \textbf{for} k \leftarrow 1 \ \textbf{to} \ n
5 \hspace{2em} \textbf{do}
6 \hspace{3em} \textbf{for} w \leftarrow W \ \textbf{downto} \ w_k
7 \hspace{4em} \textbf{do}
8 \hspace{5em} \textbf{if} B[k - 1, w - w_k] + b_k > B[k - 1, w] \ \textbf{then}
9 \hspace{6em} B[k, w] \leftarrow B[k - 1, w - w_k] + b_k
10 \hspace{5em} \textbf{else}
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**Theorem:** The 01KNAPSACK algorithm finds a *highest benefit subset* of \( S \) with total weight at most \( W \) in \( O(nW) \) time.

[Note that the algorithm can be easily modified to also return the set of items that gives the maximum benefit. How?]
Weighted Interval Scheduling

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Our goal was to maximize the number of intervals that we could schedule on this single machine.

This problem could be easily solved with a greedy algorithm.
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**Goal:** In this case, the aim is to schedule a subset of non-conflicting intervals having *maximum total value*. 


Despite the similarities to the Interval Scheduling Problem without weights (or where each interval has value 1), there is no natural greedy algorithm to solve the Weighted Interval Scheduling Problem. So we need some other approach.
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To help in what follows, define for an interval $j$, define $p(j)$ to be the largest index $i < j$ such that intervals $i$ and $j$ are disjoint (i.e. the largest value of $i$ such that $f_i \leq s_j$).
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We define \( p(j) = 0 \) if there is no disjoint interval \( i < j \).
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We note that if interval $n$ is in the solution, then no interval with index larger than $p(n)$ can be in the solution, because of the definition of $p(n)$.

So if $S$ contains interval $n$, then it also must contain an optimal solution to the sub-problem that consists of intervals $\{1, 2, \cdots, p(n)\}$. 
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If interval $n$ isn’t in the optimal solution $S$, then $S$ consists of an optimal solution of the subproblem with intervals \{1, 2, \cdots, n - 1\}. 

Weighted Interval Scheduling (cont.)
With the last reasoning in mind, we define $S_j$ to be an optimal solution to the subproblem consisting of the intervals $\{1, \ldots, j\}$, and $Opt(j)$ to be the value of $S_j$. (Also define $S_0 = \emptyset$ and $Opt(0) = 0$.)
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For any interval $j$, either $j \in S_j$ or $j \not\in S_j$. 
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The important thing to notice is that if $j \in S_j$ then

$$Opt(j) = v_j + Opt(p(j)).$$
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If $j \not\in S_j$ then

$$Opt(j) = Opt(j - 1).$$
With these observations, we have that

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We also see that (depending upon the value of \( \text{Opt}(j) \) above) either

\[ S_j = S_{p(j)} \cup \{j\} \text{ or } S_j = S_{j-1}. \]
This recursive solution can be used to find $Opt(n)$ (and the set $S_n$, containing the set of tasks that make up an optimal schedule).

But in order to avoid an exponential time solution, we instead compute the values in order $Opt(1), Opt(2), \cdots, Opt(n)$. 
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Doing so avoids repeating calculations that would otherwise be performed in the recursive calls.

**Theorem:** Finding $Opt(n)$ can be performed in time $O(n \log n)$. 