On Efficient Connectivity-Preserving Transformations in a Grid

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16th International Symposium on Algorithms for Sensor Systems, ALGOSENSORS 2020

Outline

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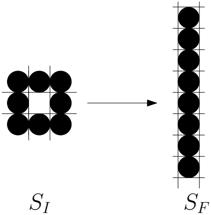
Introduction

- Recent developments for collective robotic systems.
- From the scale of milli or micro down to nano size of individual devices.
- The research area of *programmable matter* - materials that can algorithmically change its physical properties:
  - such as their shape, colour, conductivity and density.
- The implementation indicates whether the system is:
  - centralised.
  - decentralised.
- The need for the development of an algorithmic theory of such systems.

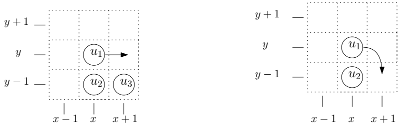
Settings

- 2D square grid.
- Each cell is occupied by a distinct device (node) on the grid.
- A number of nodes connected to each other, forming a shape S_I.
- Given a desired target shape S_F of the same order.
- **Goal:** transform S_I into S_F via a finite number of valid moves.



Models of individual moves

- Only one node moves in a single time-step.
- Such as, Dumitrescu *et al.* IJRR'04 and Michail *et al.* JCSS'19:
  - An individual device can move over and turn around its neighbours through empty space.



- Akitaya *et al.* ESA'19 consider transformations based on similar moves.
- To transform some pairs of connected shapes, $\Omega(n^2)$ moves are required for all models of constant-distance individual moves.

Cont.

-**This motivates the study of alternative types of moves that are reasonable with respect to practical implementations and allow for sub-quadratic reconfiguration time in the worst case.**

-**There are attempts to provide alternatives for more efficient reconfiguration.**

Parallel transformations

- Multiple nodes move together in a single time-step.
- Especially in distributed systems.
  - nodes can make independent decisions and move locally in parallel to other nodes.
- Theoretical studies, such as Daymude *et al.*, Natural Computing'18.
- Practical implementations, such as Rubesntein *et al.*, Science'14.
- It can be shown that a connected shape can transform into any other connected shape, by performing in the worst case $O(n)$ parallel moves around the perimeter of the shape.

Models of more powerful mechanism

- Equip nodes with strong actuation mechanisms.
- Reduce the inherent distance by a factor greater than a constant in a single time-step.
- Linear-strength mechanisms, such as:
  - Aloupis *et al.*, (Computational Geometry'13) provide a node with arms that are capable to extend and contract a neighbour.
  - Woods *et al.*, (ITCS'13) proposed an alternative linear-strength mechanism, where a node has the ability to rotate a whole line of consecutive nodes.
  - Czyzowicz *et al.*, (ESA'19) consider a single moving robot that transforms a static shape by carrying its tiles one at a time.
  - Recently, Almethen *et al.*, (TCS'20) introduce the line-pushing model.

The line-pushing model

[Almethen et al., TCS'20]:

- A new linear-strength mechanism, where a whole line of consecutive devices can, in a single time-step, move by one position in a given direction.
- Generalisation of other existing models of reconfiguration:
  - Focus on exploiting the power of parallelism.
- Pure theoretical interest.
- A practical framework.

The line-pushing model

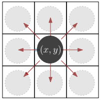
- Simulate the rotation and sliding models.
- All transformations of individual nodes, their universality and reversibility properties still hold true in the line-pushing model.
- Achieved sub-quadratic time transformations,
  - An $O(n \log n)$-time universal transformation which does not preserve connectivity.
  - A connectivity-preserving $O(n\sqrt{n})$-time transformation for the special case of transforming a diagonal into a straight line.

Our Contribution

- We build upon the findings of Almethen *et al.*, TCS'20.
- All transformations in the present study preserve connectivity of the shape during the transformations.
- An $O(n \log n)$-time connectivity-preserving transformations in which the *associated graphs* of (S_I, S_F) are isomorphic to a Hamiltonian line (defined later).
  - Asymptotically equal to the best known running time of connectivity-breaking transformation.
- An $O(n\sqrt{n})$-time connectivity-preserving universal transformation.

Preliminaries

- Each node is connected to a neighbour vertically, horizontally or diagonally.



A node is *connected* to a neighbour at any directions.

- A line L is a sequence of nodes occupying consecutive cells in one direction of the grid, that is, either vertically or horizontally but not diagonally.

- Each node is equipped with a linear-strength mechanism which enables it to move a whole line in a single time-step.

Definitions

- A **line move** is an operation of moving all nodes of L together in a single time-step towards a position adjacent to one of L's endpoints, in a given direction d of the grid, $d \in \{up, down, right, left\}$.

Definition (A permissible line move)

A line $L = (x, y), (x+1, y), \dots, (x+k-1, y)$ of length k, where $1 \leq k \leq n$, can push all its k nodes rightwards in a single move to positions $(x+1, y), (x+2, y), \dots, (x+k, y)$ iff there exists an empty cell at $(x+k, y)$. The "down", "left", and "up" moves are defined symmetrically, by rotating the whole shape 90°, 180° and 270° clockwise, respectively.

- **The problem**:
  → Transform S_I into S_F via a finite number of valid line moves.

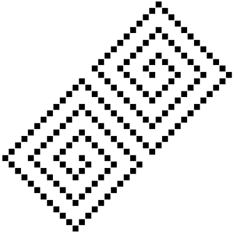
Definitions

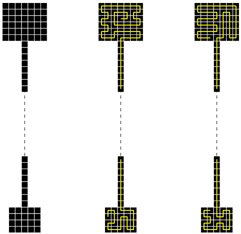
Definition

A graph $G(S) = (V, E)$ is *associated* with a shape S, where $u \in V$ iff u is a node of S and $(u, v) \in E$ iff u and v are neighbours in S.

Definition (Hamiltonian Shapes)

A shape S is called Hamiltonian iff $G(S) = (V, E)$ is isomorphic to a Hamiltonian path, i.e., a path starting from a node $u \in V$, visiting every node in V exactly once and ending at a node $v \in V$, where $v \neq u$. $\mathcal{H}$ denotes the family of all Hamiltonian shapes.





Definitions

- A configuration of the system is defined as a mapping $C\colon \mathbb{Z} \times \mathbb{Z} \to \{0, 1\}$, where $C(x, y) = 0$ if cell(x, y) is empty or $C(x, y) = 1$ if cell(x, y) is occupied by a node.

- A *rectangular path* P over the set of cells is defined as $P = [c_1, c_2, c_3, \ldots, c_k]$, where $c_i, c_{i+1} \in \mathbb{Z} \times \mathbb{Z}$ are two cells adjacent to each other either vertically or horizontally, for all $i \in \{1, 2, \ldots, k-1\}$.

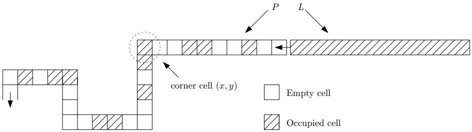
- Given any P, let C_P be the configuration of P defined as the subset of C (configuration of the system) restricted to the cells of P.

Transparency of Line Moves

Proposition

Let S be any shape, $L \subseteq S$ any line and P a rectangular path starting from a position adjacent to one of L's endpoints. There is a way to move L along P, while satisfying all the following properties:

1. **No delay:** *The number of steps is asymptotically equal to that of an optimum move of L along P in the case of C_P being empty (i.e., if no cells were occupied). That is, L is not delayed, independently of what C_P is.*
2. **No effect:** *After L's move along P, $C'_P = C_P$, i.e., the cell configuration has remained unchanged. Moreover, no occupied cell in C_P is ever emptied during L's move (but unoccupied cells may be temporarily occupied).*
3. **No break:** *S remains connected throughout L's move.*



Problem Definitions

HAMILTONIANCONNECTED. Given a pair of connected Hamiltonian shapes (S_I, S_F) of the same order, where S_I is the initial shape and S_F the target shape, transform S_I into S_F while preserving connectivity throughout the transformation.

DIAGONALTOLINECONNECTED. A special case of HAMILTONIANCONNECTED in which S_I is a diagonal line and S_F is a straight line.

UNIVERSALCONNECTED. Given *any* pair of connected shapes (S_I, S_F) of the same order, where S_I is the initial shape and S_F the target shape, transform S_I into S_F while preserving connectivity throughout the transformation.

An $O(n \log n)$-time Transformation for Hamiltonian Shapes

- Called *Walk-Through-Path* and solves HamiltonianConnected.

- Starts from one endpoint of the Hamiltonian path of S_I and applies a recursive successive doubling technique to transform S_I into a straight line S_L.

- Replace S_I with S_F reversely in *Walk-Through-Path* to go from S_I to S_F in the same asymptotic time.

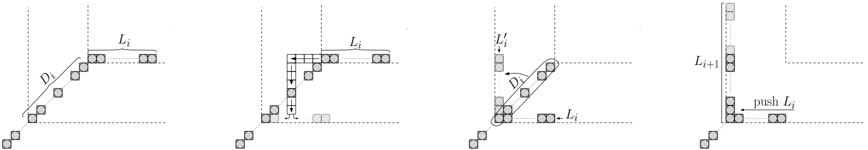


Figure: A snapshot of phase i of *Walk-Through-Path* applied on a diagonal. Light grey cells represent the ending positions of the corresponding moves depicted in each sub-figure.

An $O(n \log n)$-time Transformation for Hamiltonian Shapes

Algorithm 1: HamiltonianToLine(S)

$S = (u_0, u_1, \ldots, u_{|S|-1})$ is a Hamiltonian shape
Initial conditions: $S \leftarrow S_I$ and $L_0 \leftarrow \{u_0\}$

```
for i = 0, ..., log |S| do
    LineWalk(L_i)
    S_i ← select(2^i) // select the next terminal subset of 2^i consecutive nodes of S
    L'_i ← HamiltonianToLine(S_i) // recursive call on S_i
    L_{i+1} ← combine(L_i, L'_i) // combines L_i and L'_i into a new straight line L_{i+1}
end
```

Output: a straight line S_L

- Main challenge: make the above transformation work in the general case.
- Hamiltonian shapes do not necessarily provide free space.
- Moving a line through the remaining configuration of nodes.
- Does not break their and its own connectivity by *LineWalk* operation.

An $O(n \log n)$-time Transformation for Hamiltonian Shapes

Theorem

For any pair of Hamiltonian shapes $S_I, S_F \in \mathcal{H}$ of the same order n, Walk-Through-Path transforms S_I into S_F (and S_F into S_I) in $O(n \log n)$ moves, while preserving connectivity of the shape during its course.

$$T = \sum_{i=1}^{\log n} T(i) = \sum_{i=1}^{\log n} 2^{i-1}(i-1) - 2^i = \sum_{i=1}^{\log n - 1} (i-2)2^i - 2^{\log n} \leq \sum_{i=1}^{\log n - 1} i \cdot 2^i - n$$

$$\leq \sum_{j=1}^{\log n} \sum_{i=j}^{\log n} 2^i - n \ le \sum_{j=1}^{\log n} n - n \leq n \log n - n \leq O(n \log n).$$

An $O(n\sqrt{n})$-time Universal Transformation

- **Solves the** UniversalConnected **problem and is called UC-Box.**

- First, Compute a spanning tree T of the associated graph $G(S_I)$ of S_I.

- Enclose S_I into an $n \times n$ square box and divide it into $\sqrt{n} \times \sqrt{n}$ square sub-boxes.

- Each occupied sub-box contains one or more maximal sub-trees of T.

- Each such sub-tree corresponds to a sub-shape of S_I, called a *component*.

- Pick a leaf sub-tree T_l which is associated with component C_l occuping sub-box B_l.

- B_p is the sub-box adjacent to B_l containing the unique parent sub-tree T_p of T_l.

- Compress all nodes of C_l into B_p while keeping the nodes of C_p (the component of T_p) within B_p.

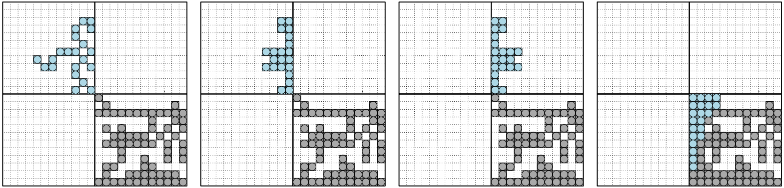
Algorithm 2: COMPRESS(S)

$S = (u_1, u_2, \ldots, u_{|S|})$ is a connected shape, T is a spanning tree of $G(S)$ **repeat**

```
    C_l ← pick(T_l) // select a leaf component associated with a leaf sub-tree
    Compress(C_l) // start compressing the leaf component
    if C_l collides then
        C'_r ← combine(C_r, C_l) or C'_m ← combine(C_m, C_l) // as described in text
    else
        C'_p ← combine(C_p, C_l) // combine C_l with a parent component
    end
    update(T) // update sub-trees and remove cycles after compression
```

until *the whole shape is compressed into a* $\sqrt{n} \times \sqrt{n}$ *square*

Output: a square shape S_C



An $O(n\sqrt{n})$-time Universal Transformation

- Main technical challenge: make this strategy work universally.
- S_I might occupy several sub-boxes of different configurations.
- Preserving connectivity during the transformation.
- We manage to upper bound the cost of each charging phase independently of the order of compressions.

Theorem

For any pair of connected shapes (S_I, S_F) of the same order n, UC-Box transforms S_I into S_F (and S_F into S_I) in $O(n\sqrt{n})$ steps, while preserving connectivity during its course.

Open Problems

- An $O(n \log n)$-time universal connectivity-preserving transformation.
- A general $\Omega(n \log n)$-time matching lower bound.
- A centralised parallel version in which more than one line can be moved concurrently in a single time-step.
- A distributed version of the parallel model.
  - The nodes operate autonomously through local control and under limited information.