The Complexity of Growing a Graph

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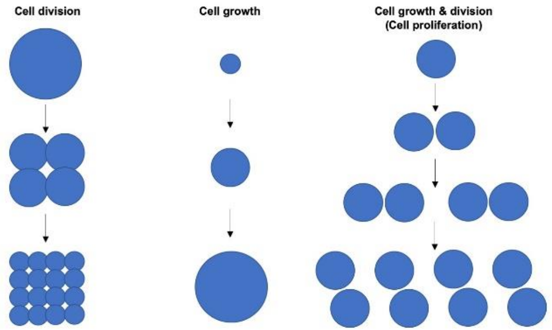
Why study growing graphs?

- Motivation: Networked systems that start from a single entity and grow into well defined structures

- Examples: social networks, sensor deployment, biological systems

- Common notion of a graph growth process which controls growth

Related Work

- No unified model seems to exist
- Literature does not focus on active growth
- Actively Dynamic Network literature: static network
- Network Costructors model: passive growth
- Random Graph Generators, Graph Editing…

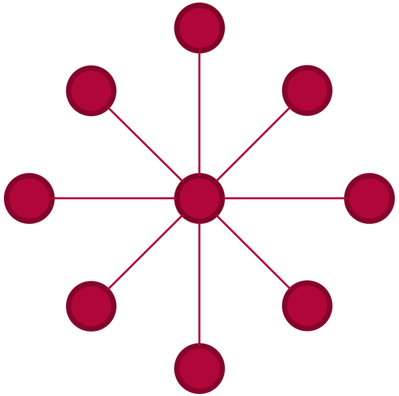
Related Work

- Systems that exhibit growth are very distinct and vary from abstract to geometrical
- *Woods et al* [ITCS’13] : Geometric and movement
- *Michail and Almalki* [Algosensors’22]: Geometric
- In this work: disregard geometry and focus on locality

Our Model

- Initial graph G_0 of a single vertex
- Centralized control
- Operations: Vertex Generation, Edge Activation at birth and Edge Deletion
- Discrete time-steps called slots
- Goal: Compute a growth process that grows G_0 into a target graph G
- Edge Activation Distance for locality constraint

Growth Example



Growth Example

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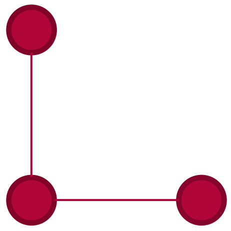

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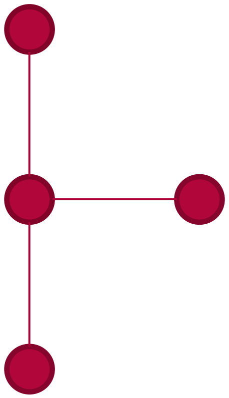

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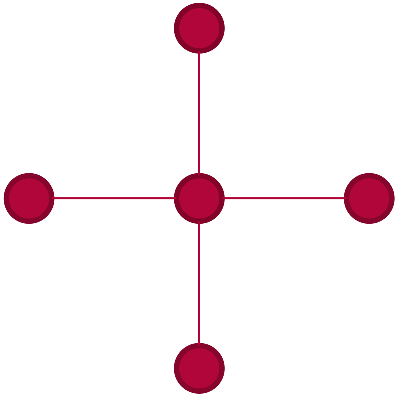

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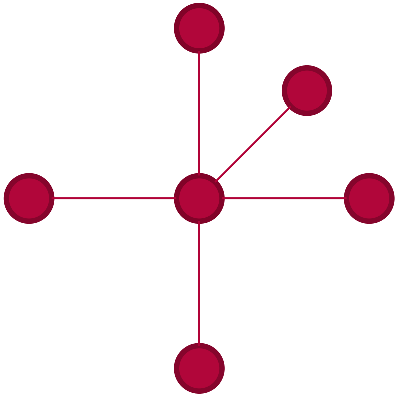

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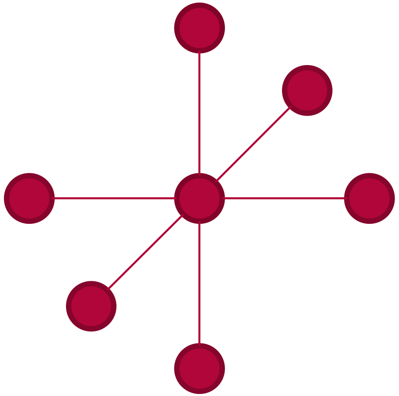

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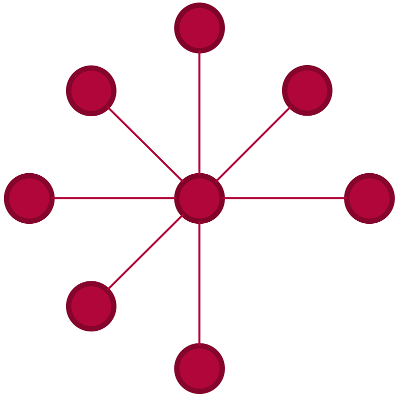

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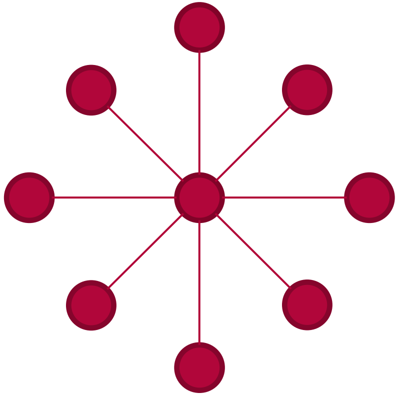

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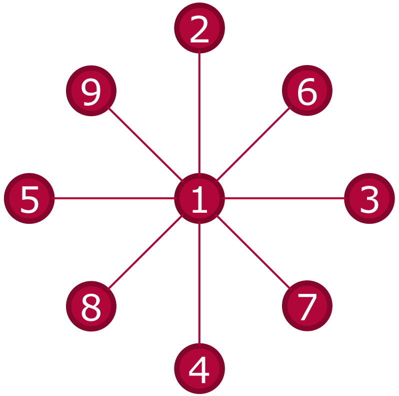


Growth Schedule

a) 1→2
b) 1→3
c) 1→4
d) 1→5
e) 1→6
f) 1→7
g) 1→8
h) 1→9

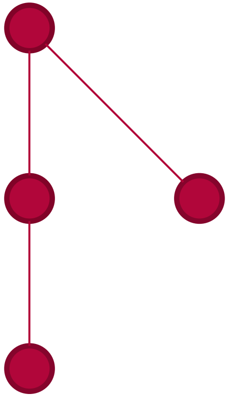
Faster Growth Example

Faster Growth Example

Growth Schedule

a) 1→2

Faster Growth Example



Growth Schedule

a) 1→2
b) 1→4, 2→3

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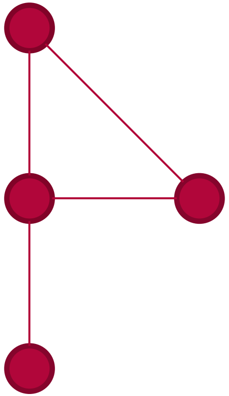

Growth Schedule

a) 1→2
b) 1→4, 2→3 (3,1)

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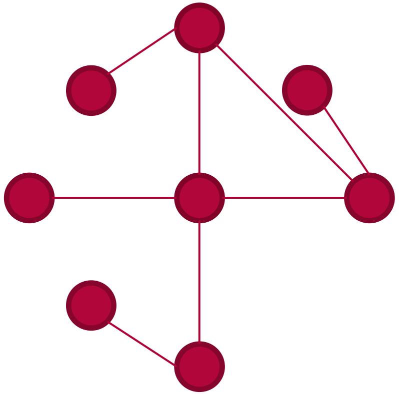

Growth Schedule

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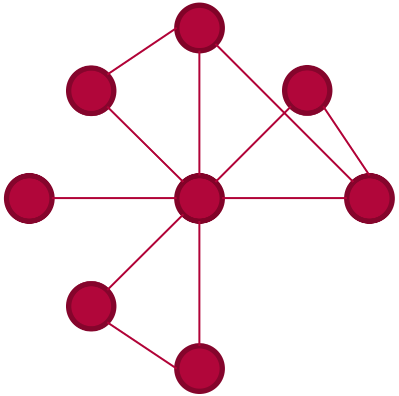

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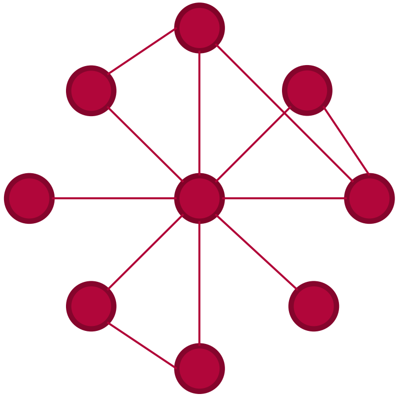

Growth Schedule

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d) 1→7

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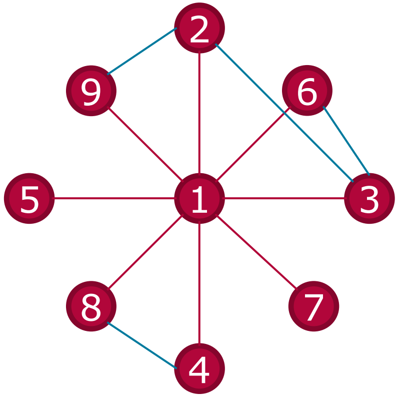

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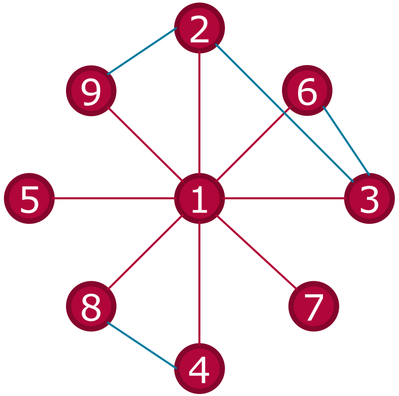

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d) 1→7, (2,3) (2,9) (3,6) (4,8)

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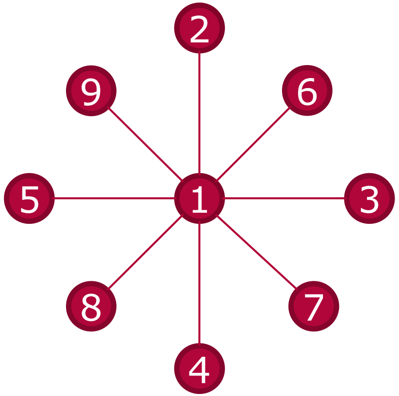

Growth Schedule

a) $1 \rightarrow 2$
b) $1 \rightarrow 4$, $2 \rightarrow 3$ (3,1)
c) $1 \rightarrow 5$, $2 \rightarrow 9$ (9,1), $3 \rightarrow 6$ (6,1), $4 \rightarrow 8$ (8,1)
d) $1 \rightarrow 7$, (2,3) (2,9) (3,6) (4,8)

The Problem

- Trade off between slots and excess edges

a) 1→2
b) 1→3
c) 1→4
d) 1→5
e) 1→6
f) 1→7
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h) 1→9

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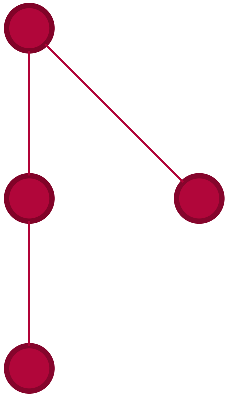
- **Graph Growth Problem**: Given an input graph G, compute in polynomial time a growth schedule with at most ***k*** slots and with at most ***l*** excess edges if it exists.

Edge Activation Distance

Growth Schedule

a) 1→2

Edge Activation Distance



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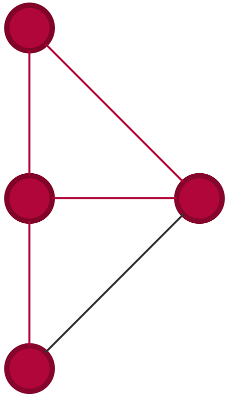

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Edge Activation Distance

Theorem

For $d = 1$, the trimming algorithm that computes in polynomial time an optimal growth schedule with k slots for any tree graph G.

Lemma

For $d \geq 3$, a graph $G = (V, E)$ with n nodes can be generated with a growth schedule with $\log n$ slots and with $O(n)$ excess edges.

- We focused on d=2 since its more natural and interesting
- Study two edge cases: Very fast schedules of very efficient schedules

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- Independency: The vertices generated in the same time slot form an independent set in the final graph

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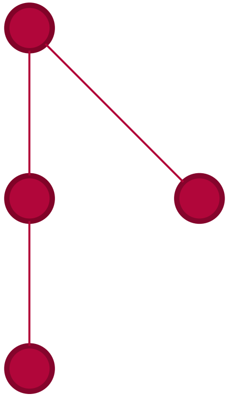

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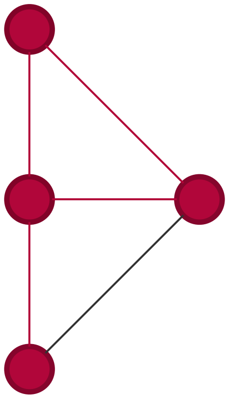

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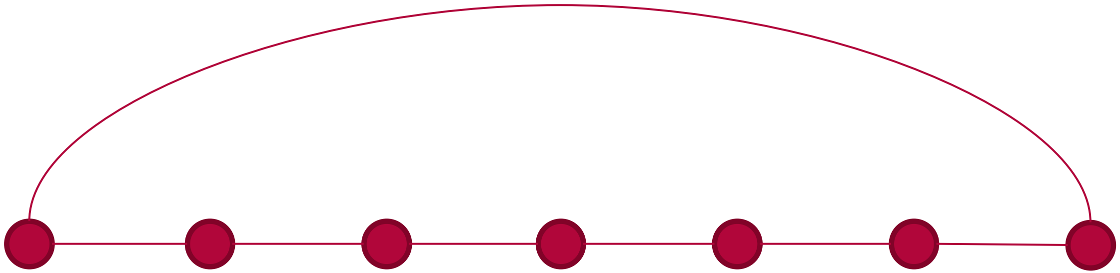

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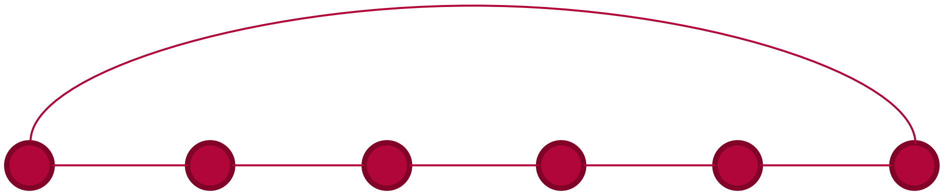

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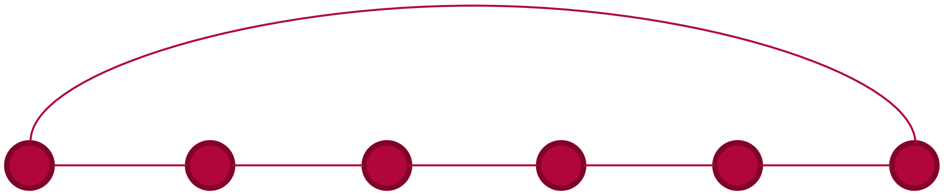

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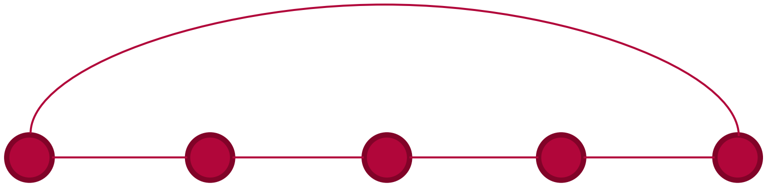

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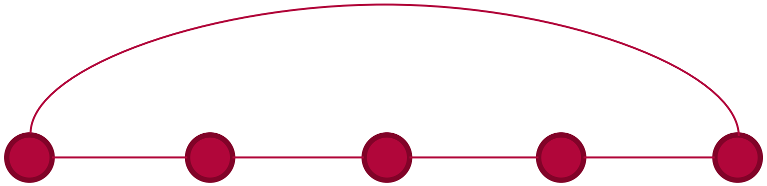

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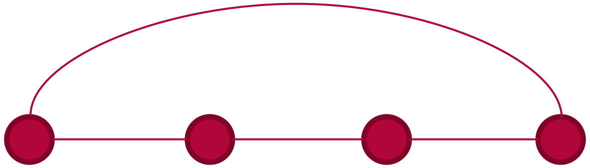

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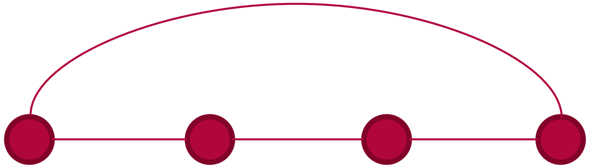

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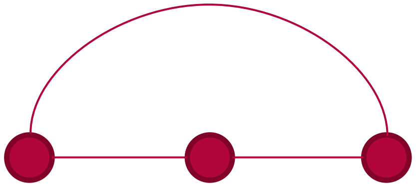

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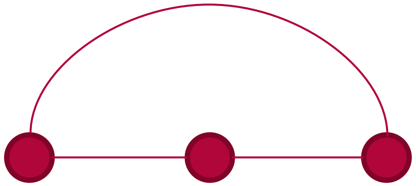

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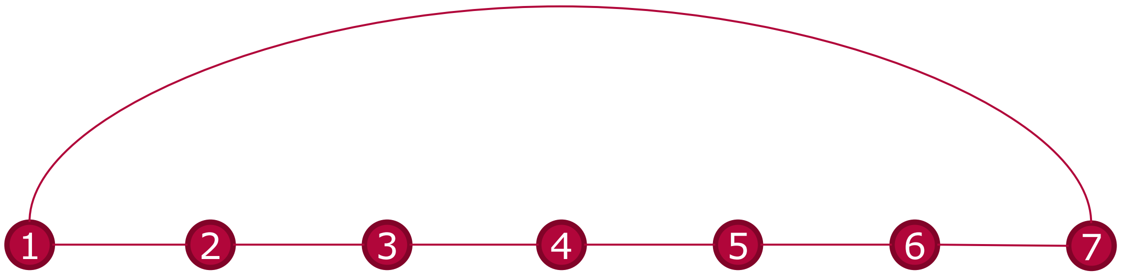


Growth Schedule

a) 1→2
b) 2→3 (3,1)
c) 3→4 (4,1)
d) 4→5 (5,1)
e) 5→6 (6,1)
f) 6→7 (7,1)

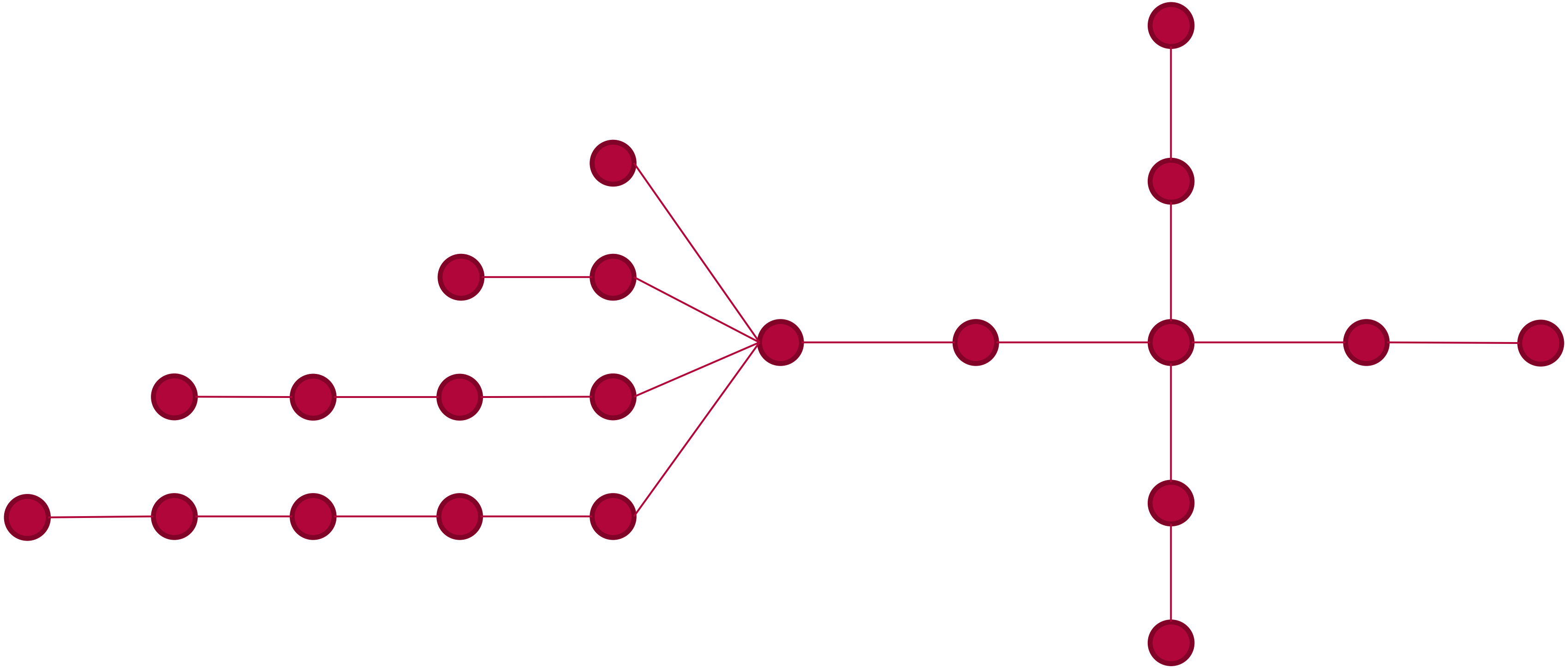
Growth Schedule for Tree Graphs

Theorem

The Tree *algorithm computes, in polynomial time, a growth schedule for any given tree graph G with $O(\log^2 n)$ slots and with $O(n)$ excess edges.*

- Decomposition strategy where vertices are removed in phases until a single node is present

Growth Schedule for Tree Graphs



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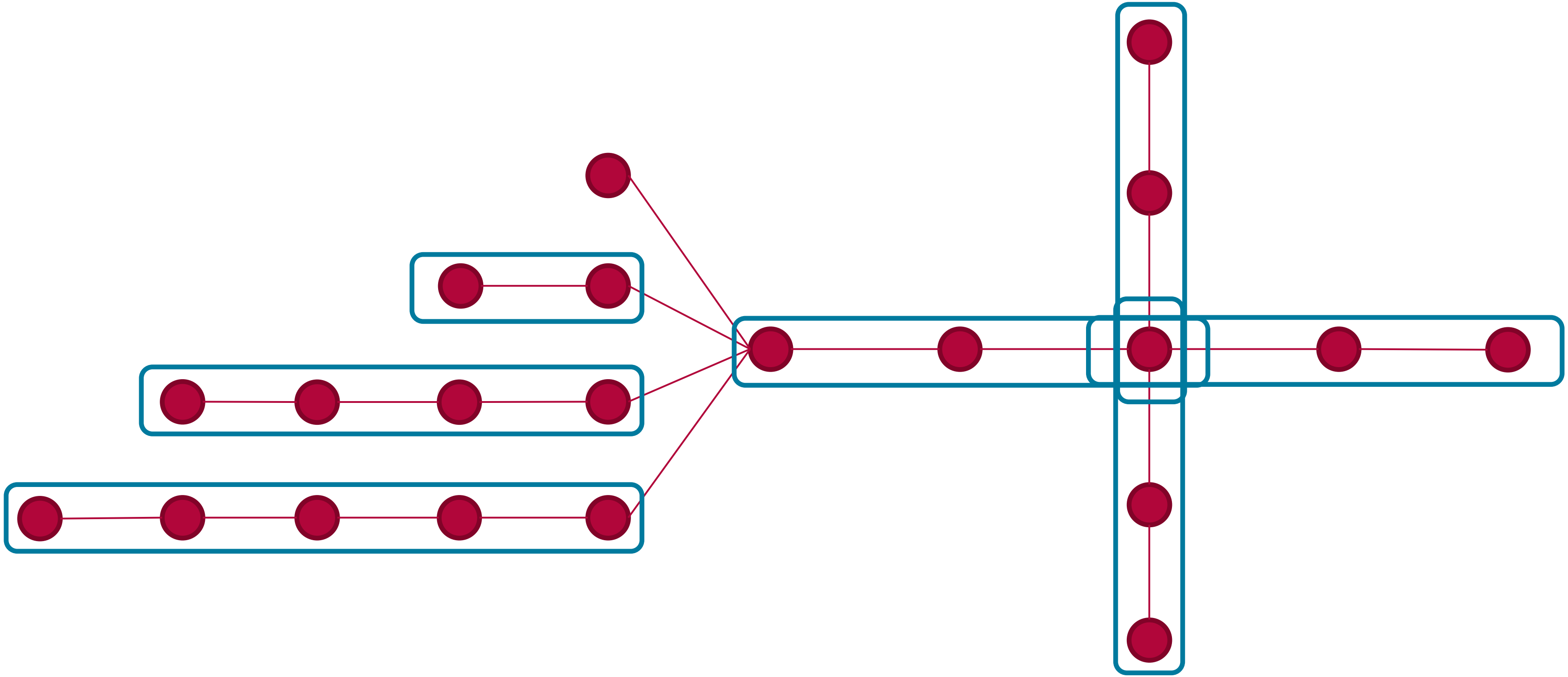

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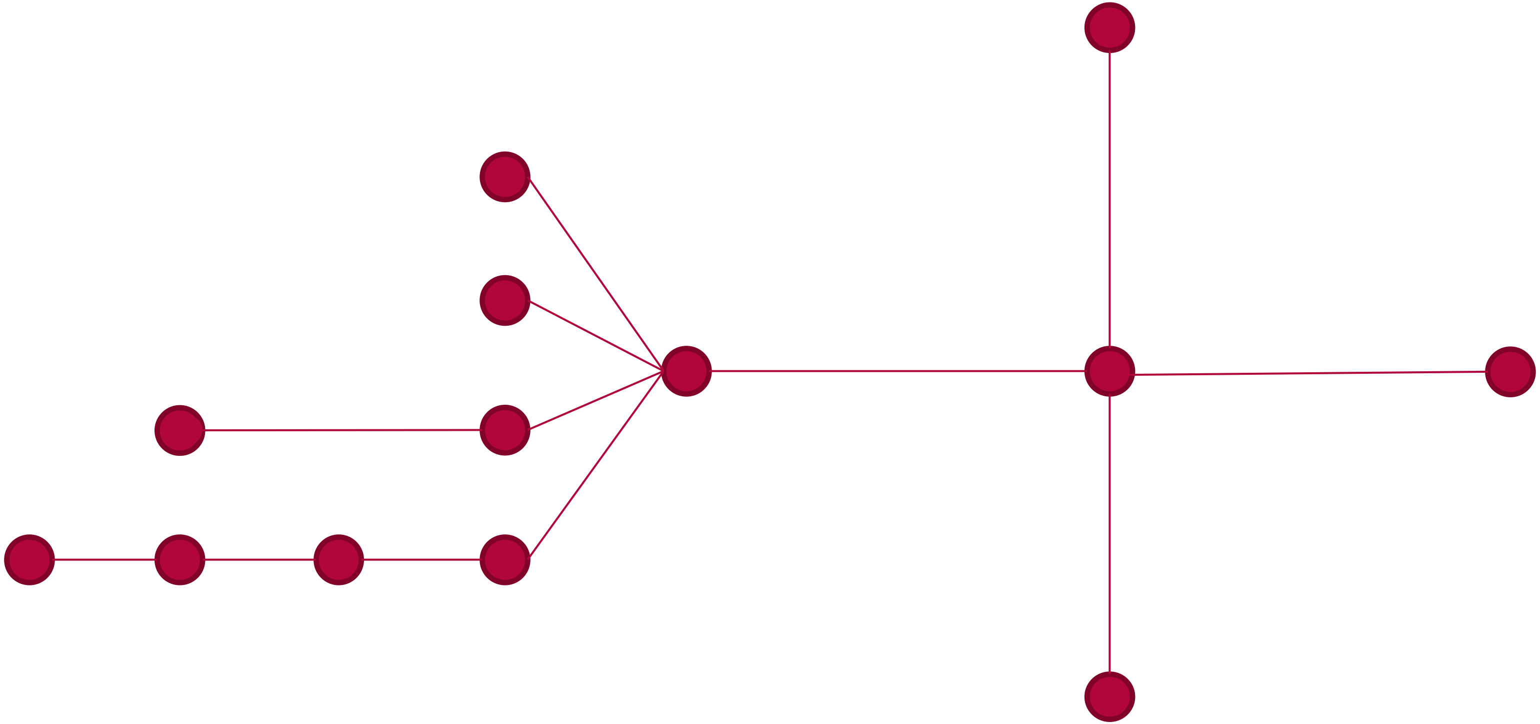

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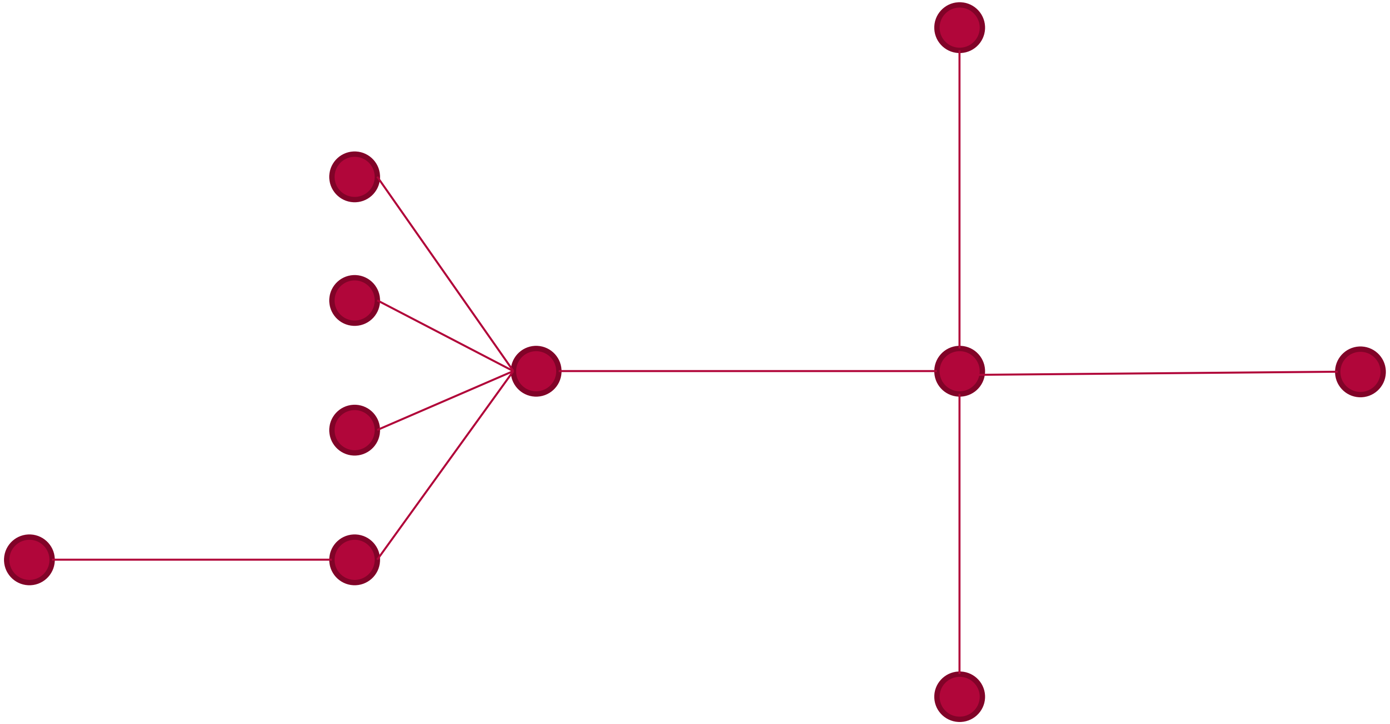

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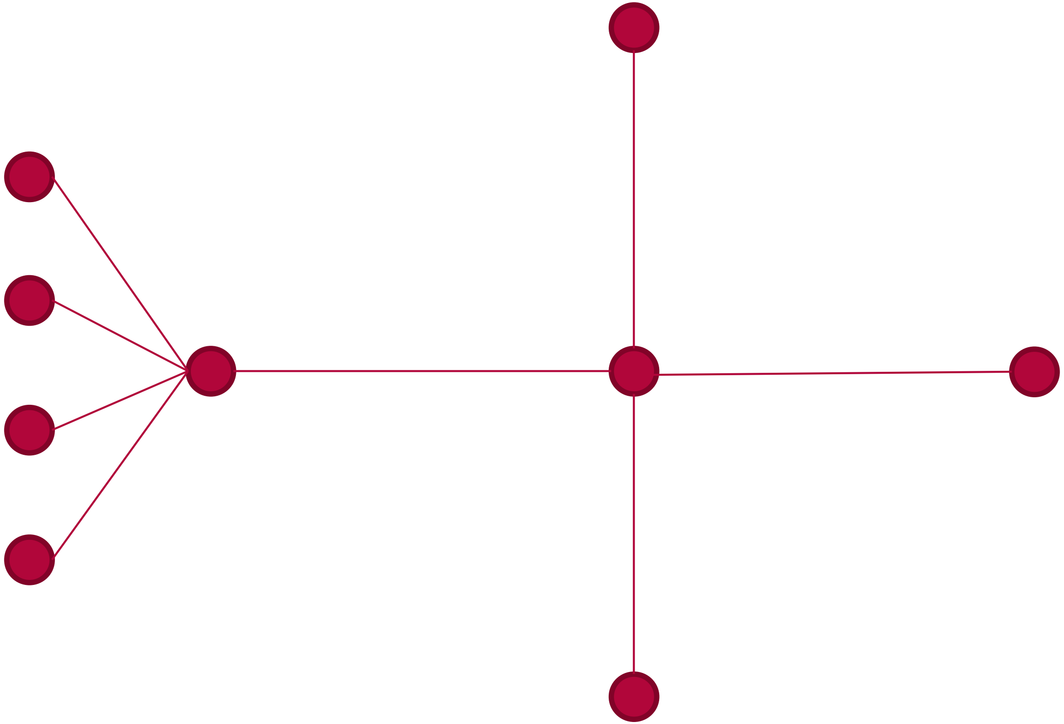

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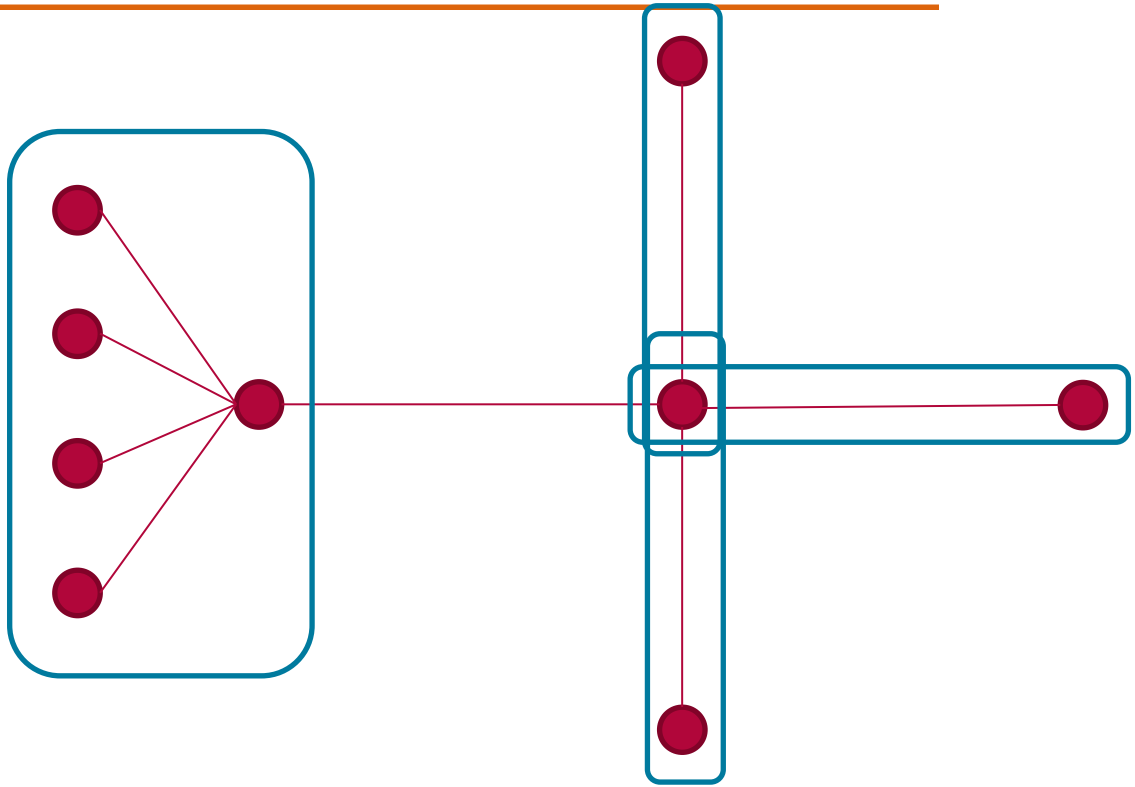

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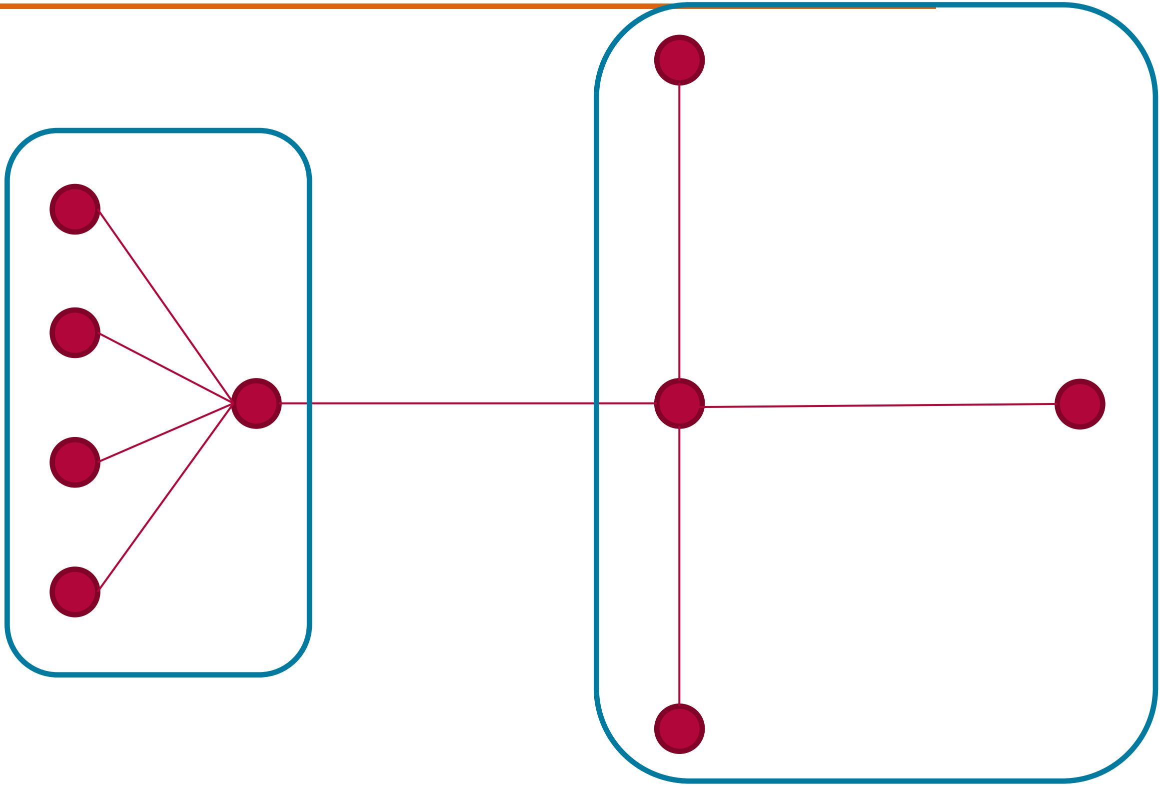

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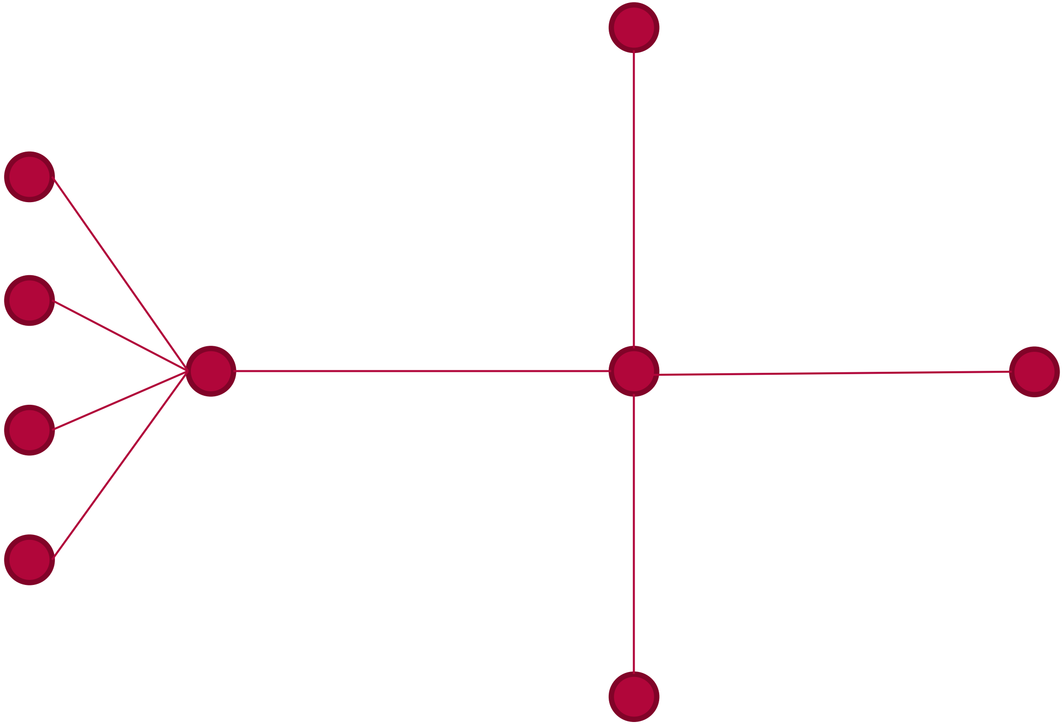

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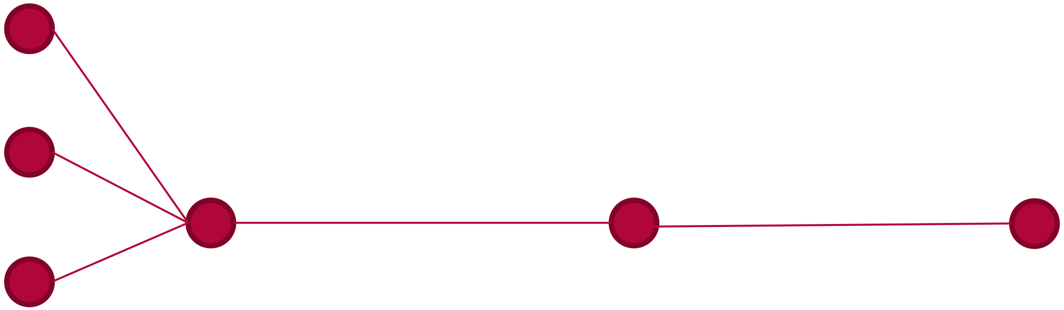

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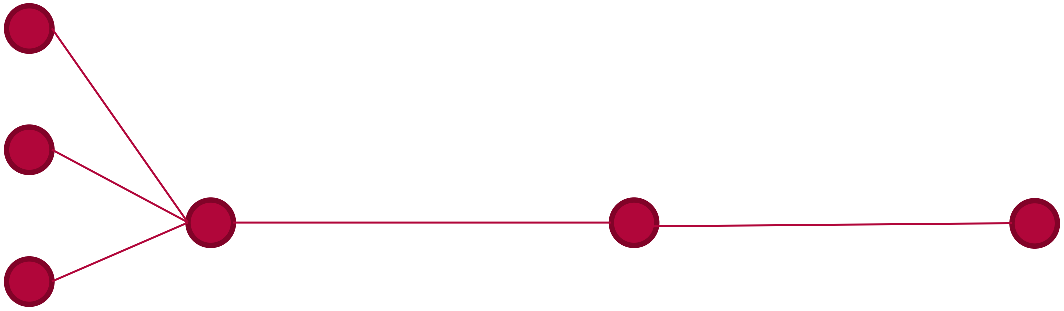

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The Tree algorithm computes, in polynomial time, a growth schedule for any given tree graph G with $O(\log^2 n)$ slots and with $O(n)$ excess edges.

- Decomposition strategy where vertices are removed in phases until a single node is present
- The phases can be reversed using $O(\log^2 n)$ slots and $O(n)$ excess edges

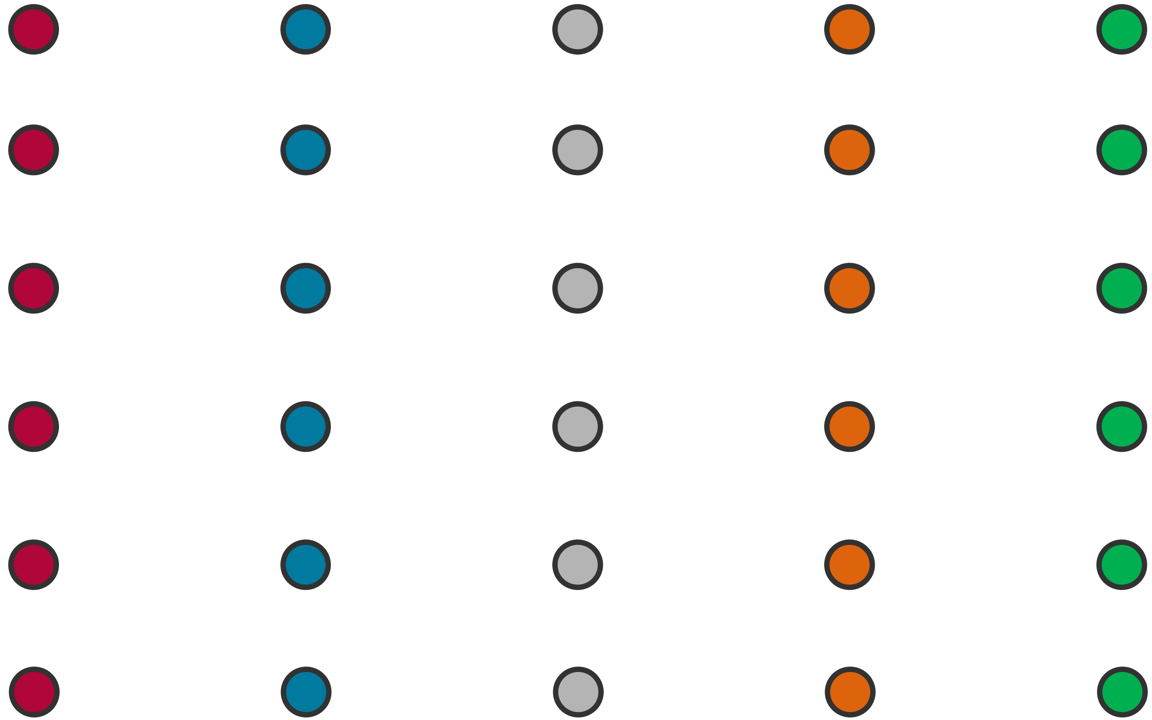
Growth Schedule for Planar Graphs

Theorem

The Planar algorithm computes, in polynomial time, a growth schedule for any given planar graph G with $O(\log n)$ slots and with $O(n \log n)$ excess edges.

- Compute a 5-coloring of the input planar graph
- Grow the vertices of each color class one by one using a star growth schedule for each color

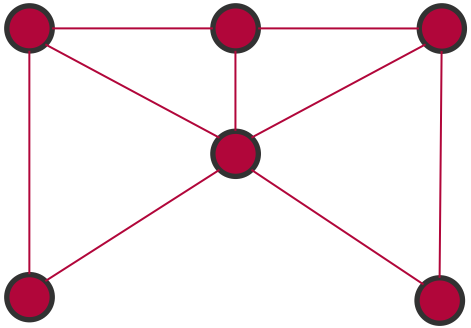
Growth Schedule for Planar Graphs



Zero Waste Growth Schedule Problem

Definition

Candidate Vertex: A vertex $v \in V$ can be the last generated vertex in a growth schedule of $\ell = 0$ for G if there exists a vertex $w \in V \setminus v$ such that $N[v] \subseteq N[w]$.



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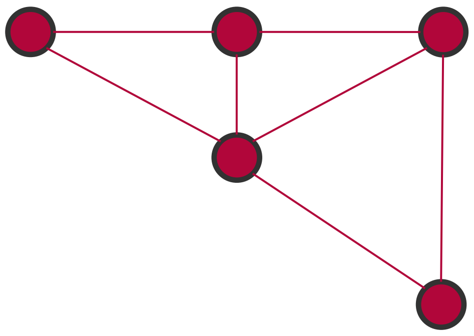

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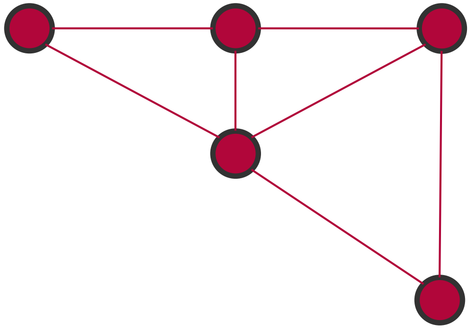

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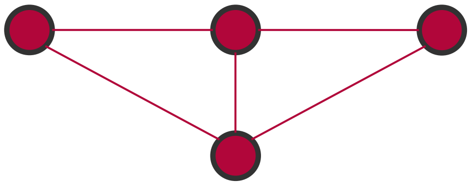

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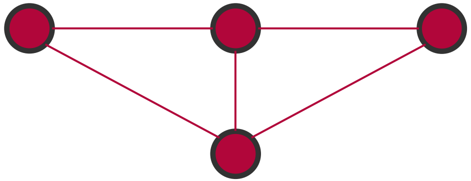

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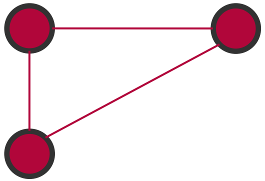

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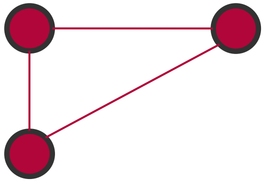

Zero Waste Growth Schedule Problem

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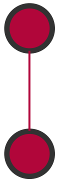

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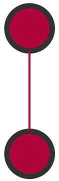

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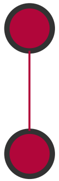

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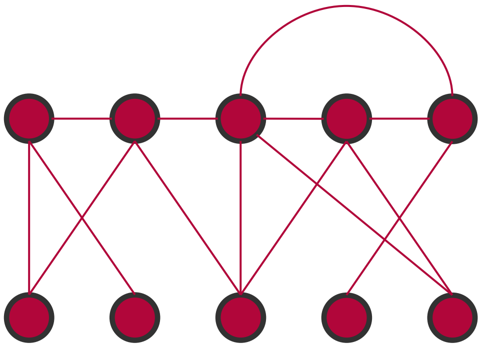
- A graph can be grown with ***l***=0 excess edges if and only if it has a candidate vertex ordering
- The Candidate vertex algorithm can decide in polynomial time, whether a given graph G has a growth schedule with n-1 slots and 0 excess edges.

Faster Growth Algorithm

Lemma

The fast growth algorithm computes in polynomial time a growth schedule σ for any graph $G = (V, E)$, where $|V| = 2^{\delta}$, with $\log n$ slots and $l = 0$ excess edges, if and only if such a σ exists for G.

- Find every candidate vertex and put them in set S
- Find a subset L of set S such that
  - L=n/2
  - L is an independent set
  - Perfect Matching



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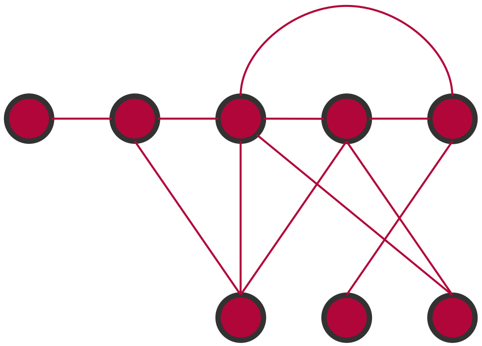

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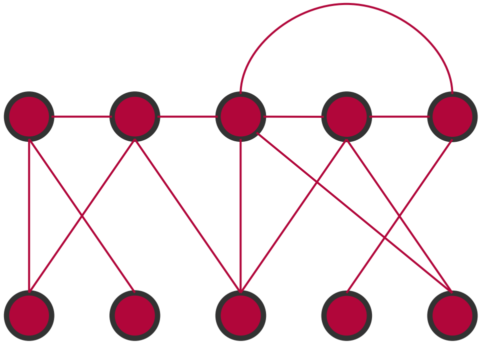

Negative Results

Theorem

The decision version of the zero-waste growth schedule problem is NP-complete.

Theorem

Let $\epsilon > 0$. If there exists a polynomial-time algorithm, which, for every graph G, computes a $n^{\frac{1}{3}-\epsilon}$-approximate growth schedule, then P=NP.

- Both reductions are from the coloring problem

Open Problems

- Distributed Control
- Minimum number of edges for k=n-1 slots
- Parameterized Complexity
- Changes to the model

Thank you!