On Geometric Shape Construction via Growth Operations

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Nada Almalki and Othon Michail

Department of Computer Science,
University of Liverpool, UK

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Introduction and Motivation

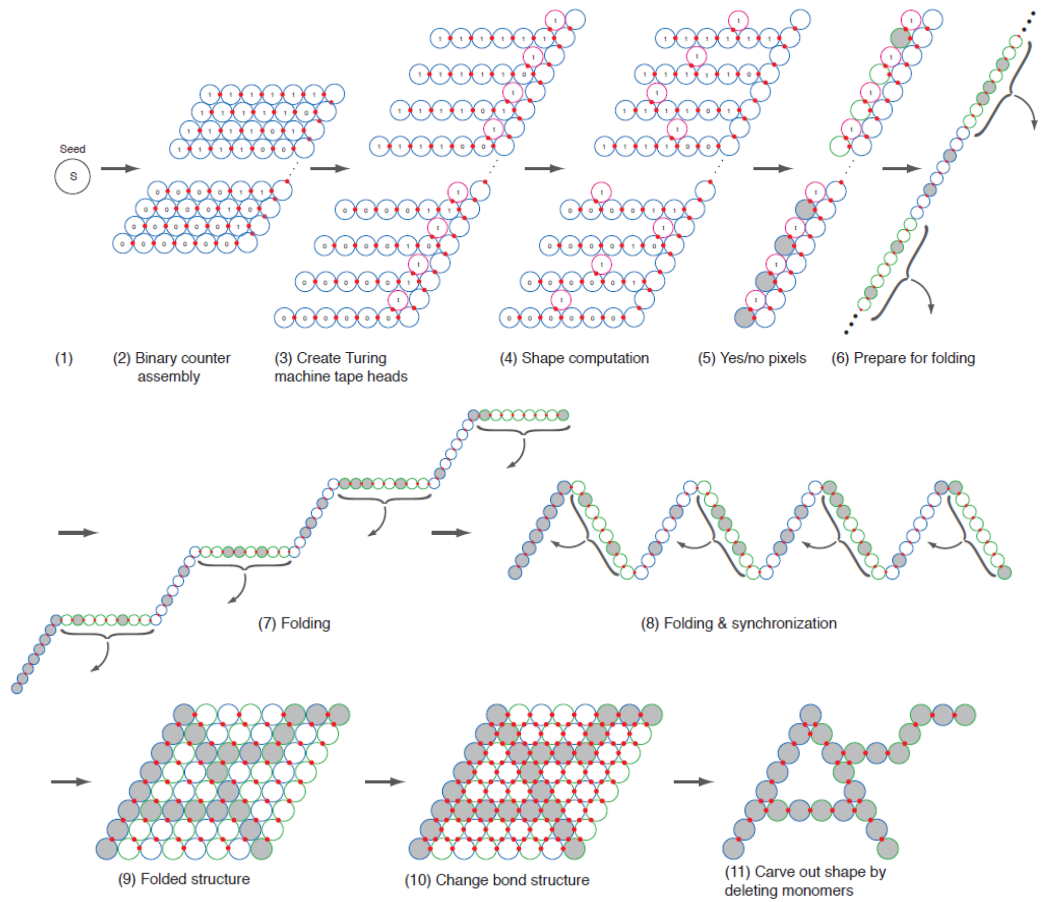
1. Woods *et al.* model

- How to generate connected 2D geometric shapes in polylogarithmic time
- Develop lines, squares and some arbitrary shapes

2. Mertzios *et al.* model

- Network-level abstraction of programmable matter systems
- Self-replication and local reconfiguration

[Mertzios *et al.*, ALGOSENSORS'22]



Source: [Woods *et al.*, ITCS'13]

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Contribution

We study three growth operations in a **centralized** **geometric** setting:

1. ***Full doubling:***
   - Characterize the structure of the class of shapes that are reachable from any S_I
2. ***RC doubling:***
   - There is a **linear-time** centralized algorithm that for any pair of shapes (S_I, S_F) decides if S_F can be constructed from S_I
   - If yes, returns an **$O(\log n)$**-time-step constructor
3. ***Doubling:***
   - Some shapes cannot be constructed in **sub-linear** time-steps
   - There are **two universal constructors** of any S_F from a singleton S_I
   - Both constructors can be computed by polynomial-time centralized algorithms for any shape S_F

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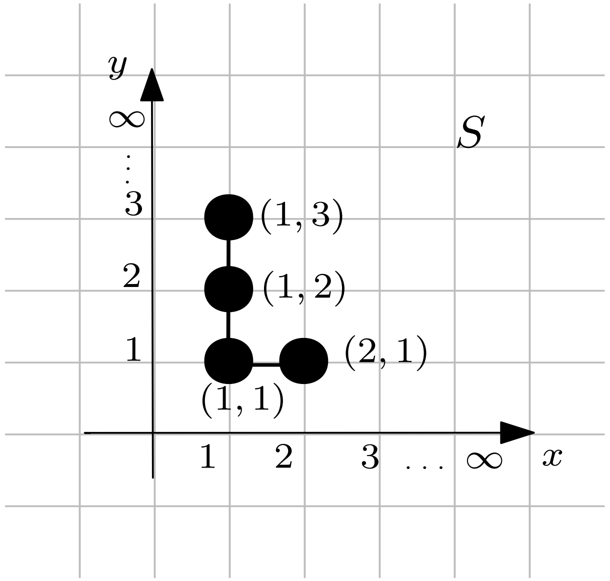
Our Model

- **Geometric** version of the abstract network-growth model of Mertzios *et al.*
- *n nodes* that form a connected shape *S*
- The considered model operates on a **2D square** grid
- Time consists of **discrete** time-steps
- Additional **properties** of growth operations:
  - Parallel operations
  - Single-direction growth operations



Note (two distinct notions):

- Time-steps of a growth operation **(*time-steps*)**
- The running time of a centralized algorithm **(*time*)**

Problems

1. Class characterization

What is the **class of shapes** that can be constructed efficiently from a given initial shape via a sequence of growth operations?

2. SHAPECONSTRUCTION

Is there a **polynomial-time** centralized algorithm that can decide if a given target shape S_F can be constructed from a given initial shape S_I and, whenever the answer is positive, return an **efficient constructor** of S_F from S_I?

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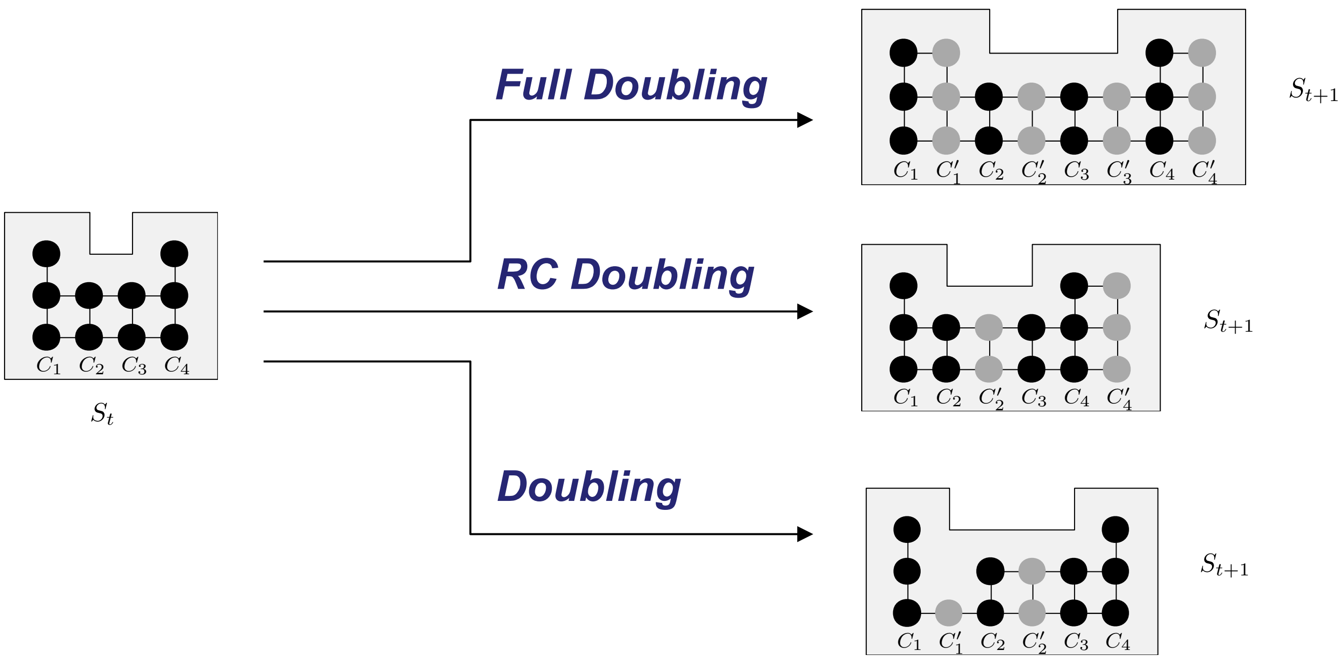
Growth Operations

A **growth operation *o*** is an operation that when applied on a shape instance S_t, for all time-steps $t \geq 0$, yields a new shape instance $\boldsymbol{S_{t+1}} = \boldsymbol{o(S_t)}$, such that $|\boldsymbol{S_{t+1}}| > |\boldsymbol{S_t}|$.



Full Doubling Operation

Reconfiguration Function

A reconfiguration function $F_{l,k}$ maps a shape to another shape in **two Phases:**

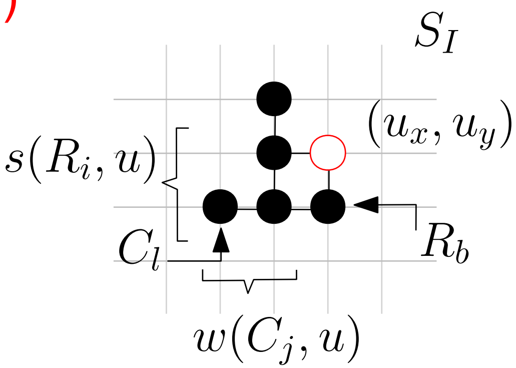
Phase 1

The coordinates of $|S|$ points of $F_{l,k}(S)$ are determined as a function of the coordinates of the points of S.

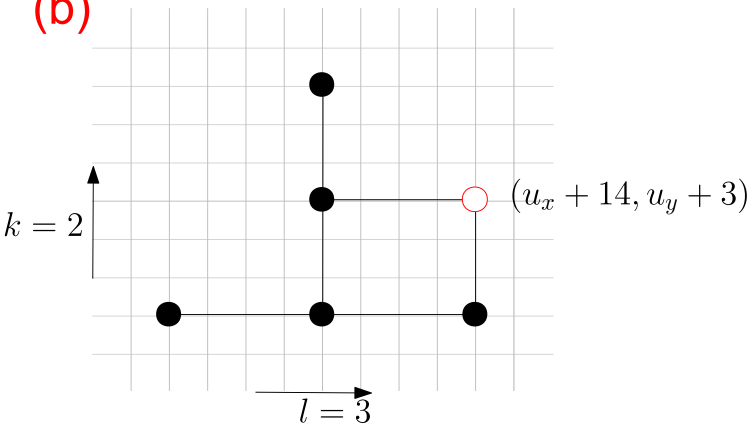
For each $u \in S$ the coordinates of $u' \in F_{l,k}(S)$ are given by

$(u_x + (2^l - 1)w(C_j, u), u_y + (2^k - 1)s(R_i, u))$.

(a)



(b)

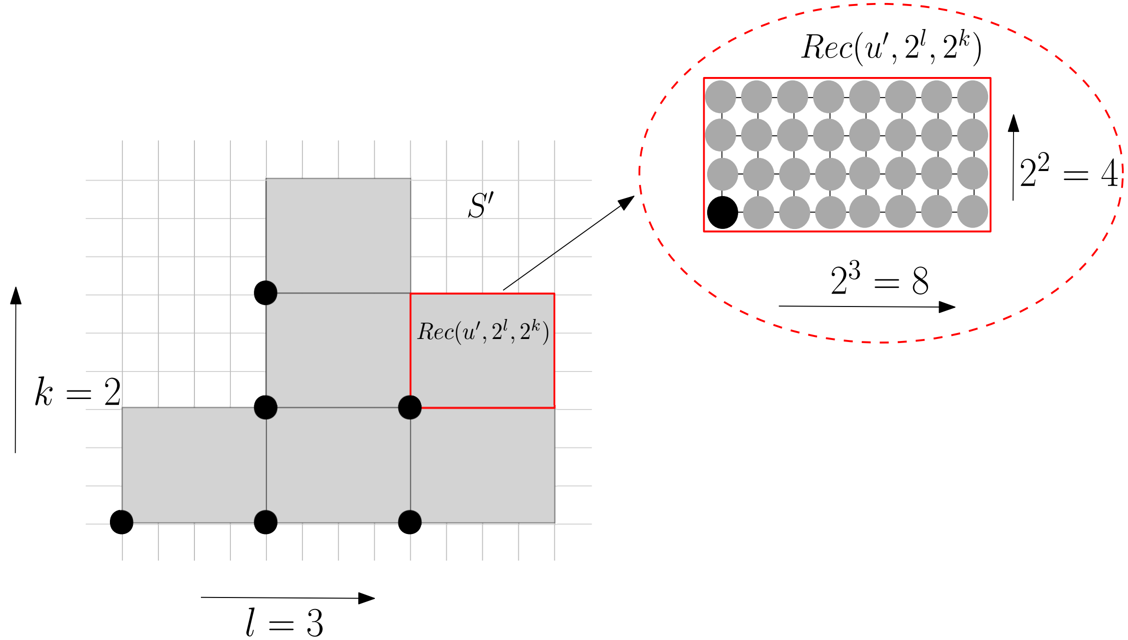


Full Doubling Operation

Reconfiguration Function

Phase 2

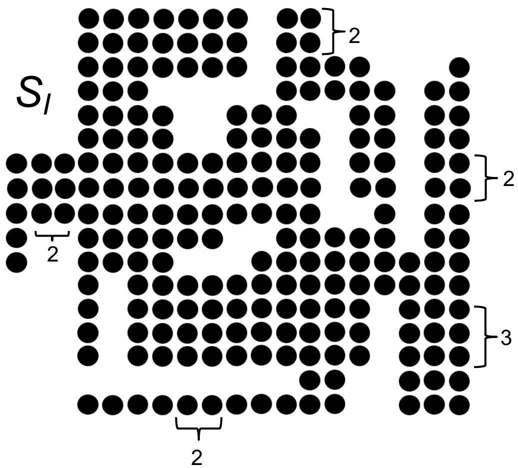
Generate the **Cartesian product** around u' such that, $Rec(u', 2^l, 2^k) = \{u'_x + 1, \ldots, u'_x + (2^l - 1)\} \times \{u'_y + 1, \ldots, u'_y + (2^k - 1)\}$ originating *at* u'.

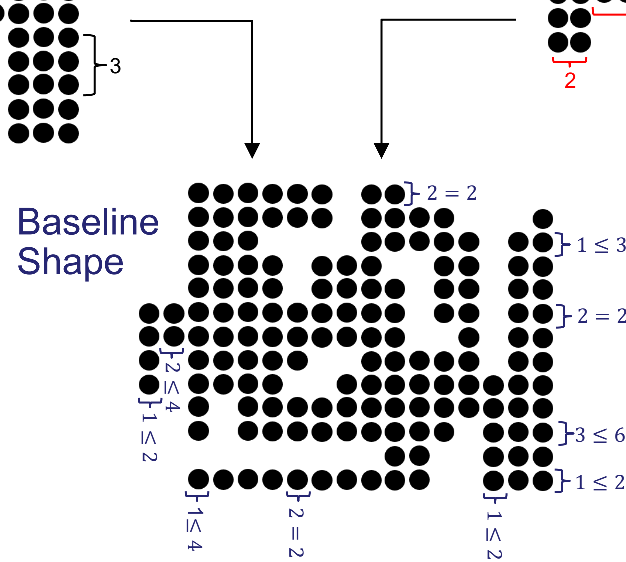


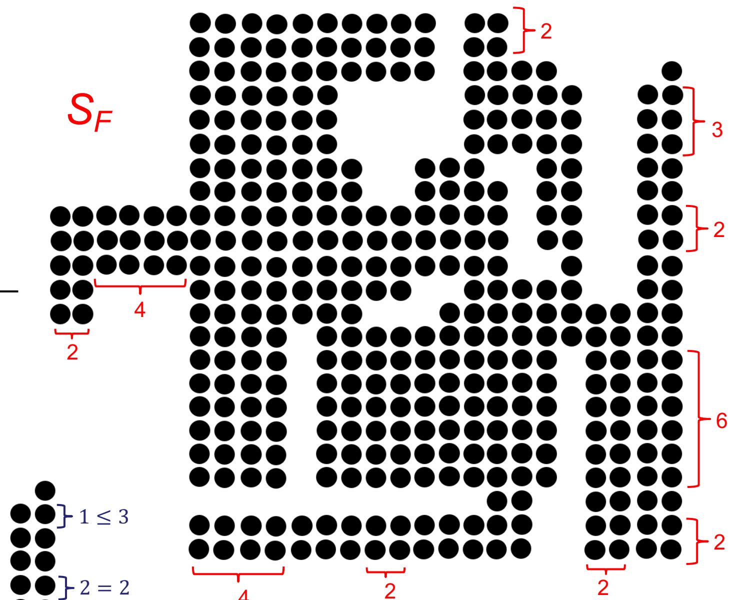
Theorem 1. Given any initial shape S_I and any sequence of l east and k north full doubling operations, the obtained shape is $S_F = F_{l,k}(S_I)$.

RC Doubling Operation

Theorem 2. A shape S_I can generate a shape S_F through a sequence of *RC* doubling operations iff $\boldsymbol{B(S_I) = B(S_F) = B}$ and for every column *C* and row *R* of *B* it holds that $\boldsymbol{M_{SF}(C) \geq M_{SI}(C)}$ and $\boldsymbol{M_{SF}(R) \geq M_{SI}(R)}$.







- $B(S_I) = B(S_F)$
- $M_{SF}(C) \geq M_{SI}(C)$
- $M_{SF}(R) \geq M_{SI}(R)$

RC Doubling Operation

Theorem 3. There is a **linear-time** algorithm for ***ShapeConstruction*** under *RC doubling* operations. In particular, given any pair of shapes (S_I , S_F), when ($S_I \rightsquigarrow S_F$) the algorithm returns a constructor σ of S_F from S_I of ***O*(log *n*)**-time-steps.

Algorithm 3: Constructor $S_I \overset{\sigma}{\rightsquigarrow} S_F$

Input : Decision, from Algorithm 2.
Output: Constructor σ.

while *Decision* **do**
    /* Find the constructor σ for columns. */
    for *every column* (C_j *of* S_F) **do**
        for *every column* (C_j *of* S_I) **do**
            if *the index of every column* C_j *of* S_F *and* S_I *are equal* **then**
                if *multiplicity of* $M_{S_F}(C_j)$ *are greater than* $M_{S_I}(C_j)$ **then**
                    compute the difference in $m(C_j)$ variable
                    append ($m(C_j)$ to σ)
compute the maximum of σ in max_{value} variable.
/* Count the steps k for doubling columns. */
if $k = \log max_{value}$ mod2 *does not equal* 0 **then**
    add one extra step to k variable to double the remaining columns that are not power of 2.
return $\sigma = \{m(C_1), m(C_2), \ldots, m(C_j)\}$.

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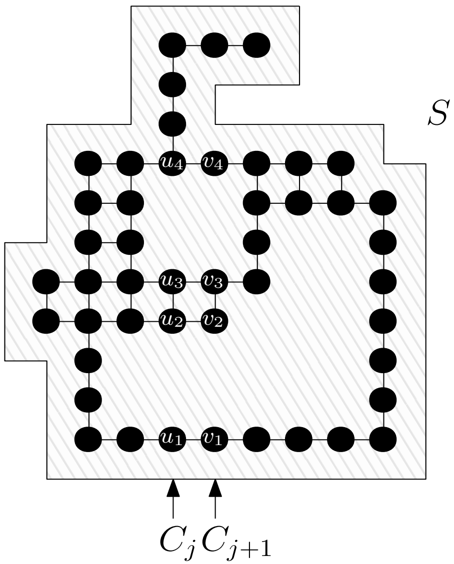
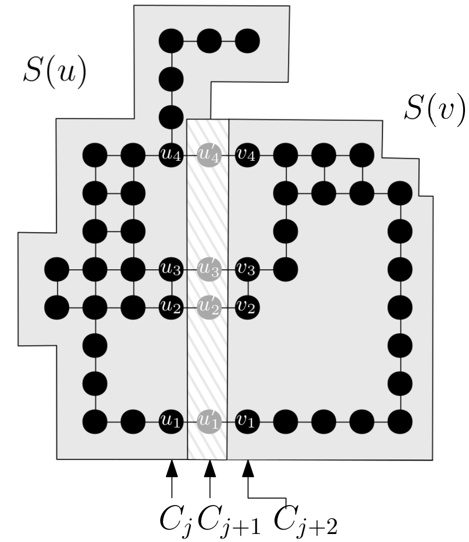
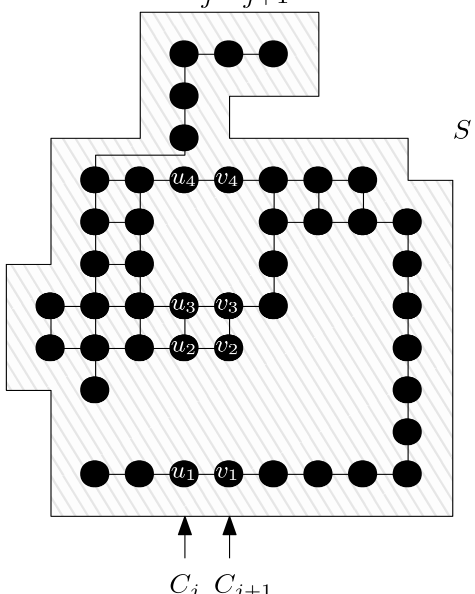
Doubling Operation



Rigidity-Preserving

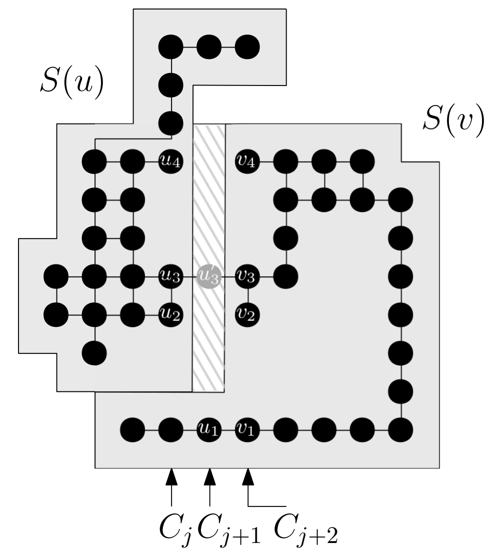
$I(C_j, u_1, \ldots, u_4)$





Rigidity-Breaking

$I(C_j(u_3))$



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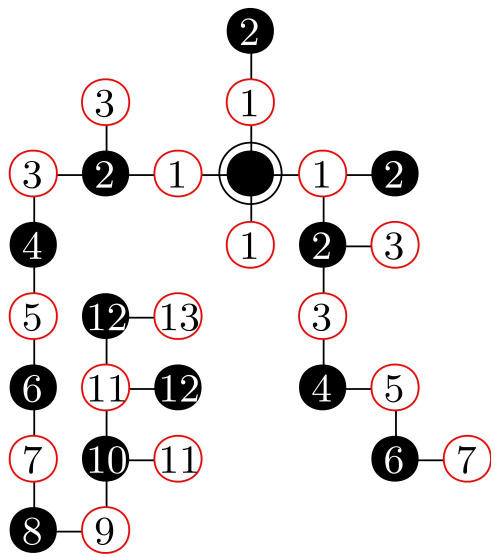
Doubling Operation

Proposition. For any shapes S_I and S_F, where $S_I \subseteq S_F$, there is a **linear** time-step constructor of S_F from S_I.



Proof

- Compute spanning tree ***T*** using **BFS**
- **Empty and adjacent** positions
- Fill all positions without **pushing existing nodes**

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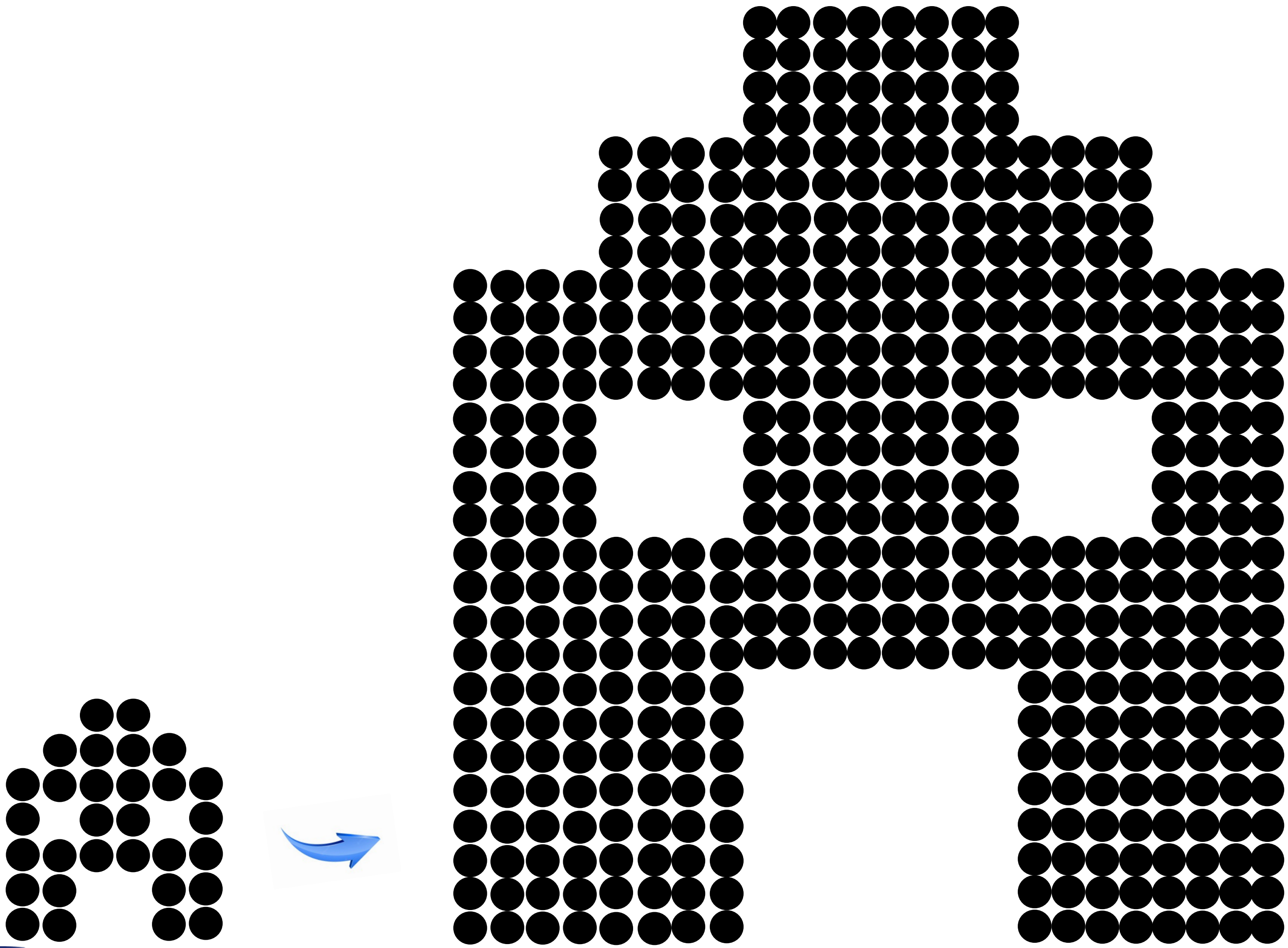
Doubling Operation

Theorem 4. Given any connected target shape S_F, there is an **[$O(|B(S_F)|) + O(\log |S_F|)$]**-time-step constructor of S_F from $S_I = \{u_0\}$ through doubling operations. Moreover, there is a polynomial-time algorithm computing such a constructor on every input S_F.



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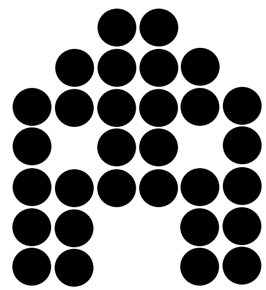

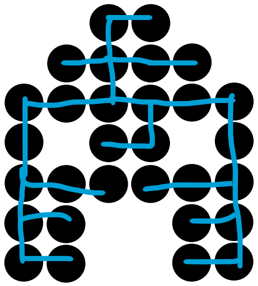

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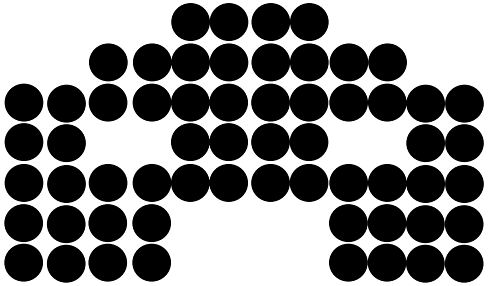

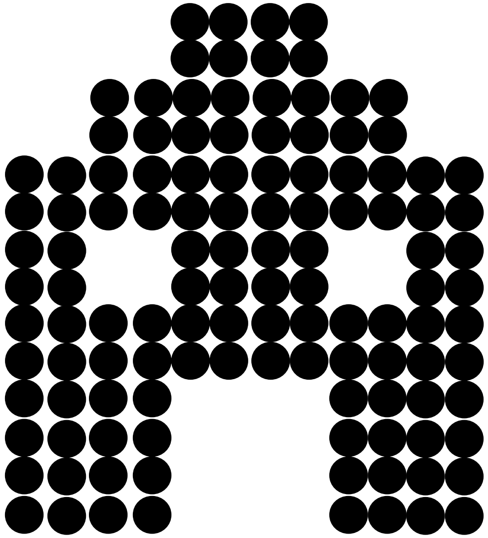

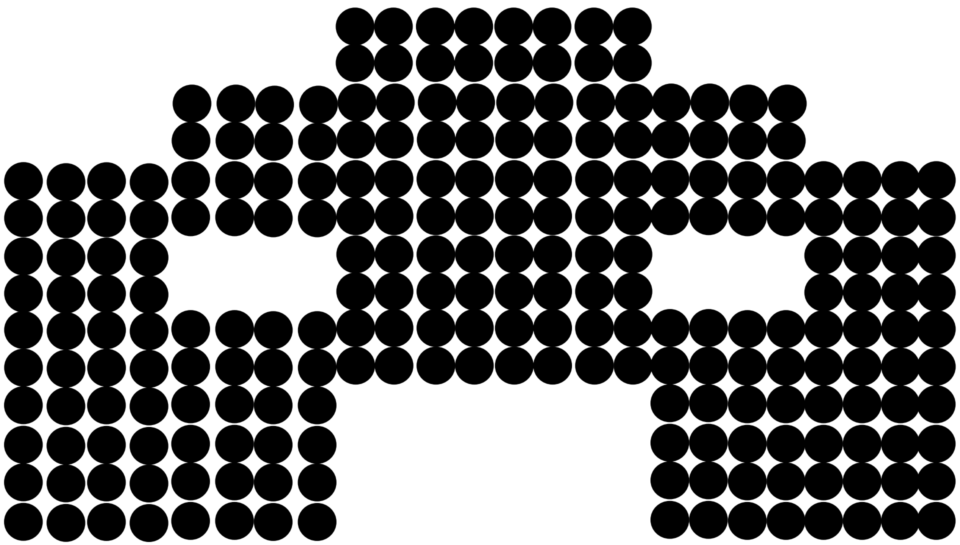

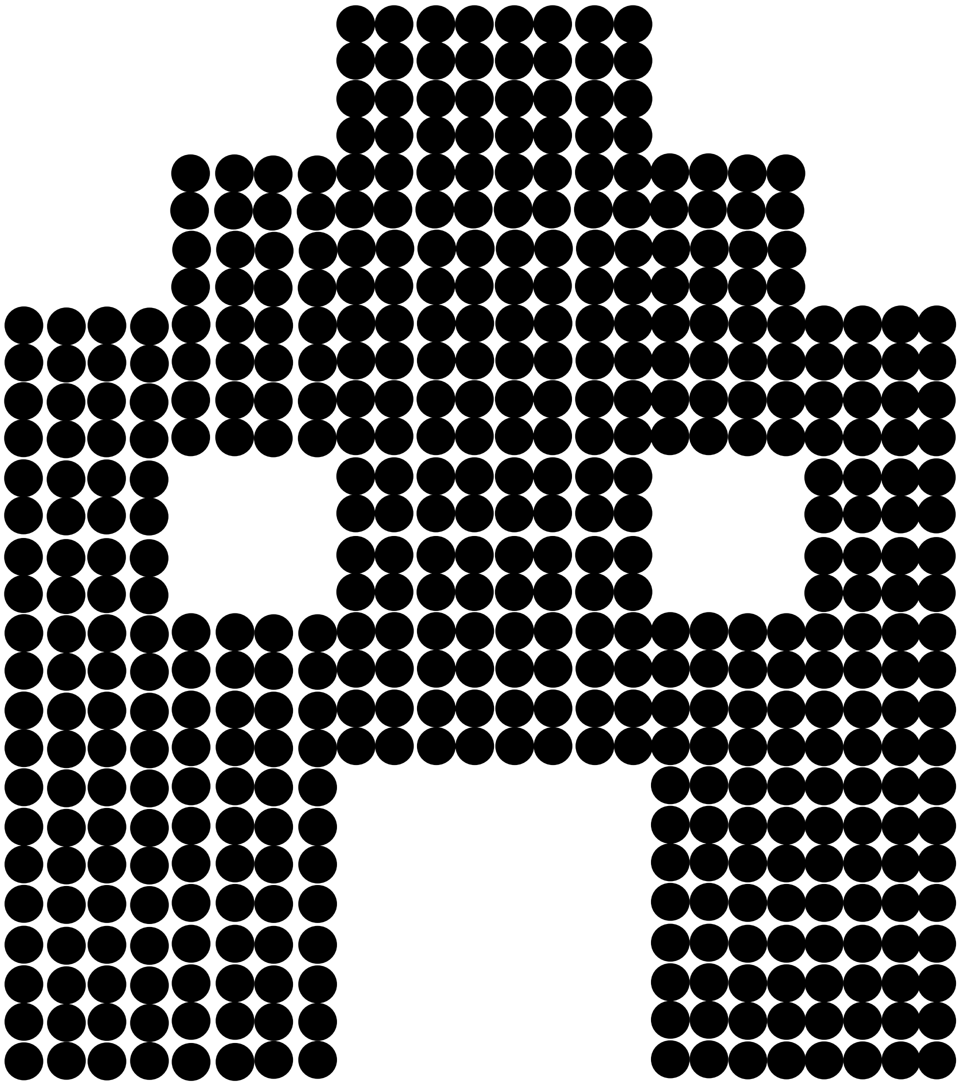

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Conclusion

- **Summary:**
  - Two problems were considered: **Class characterization** and **ShapeConstruction**
  - There is a **linear-time** centralized algorithm that for any pair of shapes (S_I, S_F) decides if S_F can be constructed from S_I, if yes, returns an ***O*(log *n*)-time-step constructor**
  - There are **two universal constructors** of any S_F from a singleton S_I

- **Future Research Directions:**
  - Obtain an **optimal** constructor
  - Explore our model in **3D** setting
  - Develop **distributed** version

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Any Questions?