Models Minimal Modulo Subset-Simulation for Expressive Propositional Modal Logics

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(Minimal) Model Generation

Useful for several tasks:

- hardware and software verification
- fault analysis
- commonsense reasoning
- query answering
- ...

Minimality criteria:

- domain minimality
- minimisation of a certain set of predicates
- minimal Herbrand models
- In this talk: models minimal modulo subset-simulation
IJCAR 2014

Minimal model procedures for all the sublogics of $S5$

- sound
- refutationally complete
- minimal model sound
- minimal model complete
- terminating
Done and to Do (Aims)

IJCAR 2014

Minimal model procedures for all the sublogics of \textbf{S5}

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Aims

Discuss how to generalise to more expressive modal logics

- multi-modal logics
- inclusion axioms
- universal modalities
Propositional Modal Logic

Syntax: \[ \phi ::= \bot \mid T \mid p_i \mid \neg \phi \mid \phi_1 \lor \phi_2 \mid \phi_1 \land \phi_2 \mid \Box \phi \mid \Diamond \phi \]

Kripke Semantics: An interpretation \( \mathcal{I} \) is a tuple \((W, R, V)\).
Propositional Modal Logic

Syntax: \[
\phi ::= \bot \mid \top \mid p_i \mid \neg \phi \mid \phi_1 \lor \phi_2 \mid \phi_1 \land \phi_2 \mid \Box \phi \mid \Diamond \phi
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Box semantics

\[ \Box p \]

\( V \) assigns a set of propositional symbols to each element of \( W \).
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## Frame Properties

<table>
<thead>
<tr>
<th></th>
<th>Axiom</th>
<th>Frame condition</th>
<th>First-order representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>K</td>
<td>□□p → p</td>
<td>reflexivity</td>
<td>∀xR(x, x)</td>
</tr>
<tr>
<td>T</td>
<td>□p → p</td>
<td>symmetry</td>
<td>∀x∀y(R(x, y) → R(y, x))</td>
</tr>
<tr>
<td>B</td>
<td>p → □◊p</td>
<td>seriality</td>
<td>∀x∃yR(x, y)</td>
</tr>
<tr>
<td>D</td>
<td>□p → ◇p</td>
<td>transitivity</td>
<td>∀x∀y∀z(R(x, y) ∧ R(y, z) → R(x, z))</td>
</tr>
<tr>
<td>4</td>
<td>□□p → □□□p</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>◇p → □◇p</td>
<td>Euclideanness</td>
<td>∀x∀y∀z(R(x, y) ∧ R(x, z) → R(y, z))</td>
</tr>
</tbody>
</table>

### Fifteen possible logics

K, KT, KB, . . ., K45, KD45, KB4 and KT5(= S5)
Subset-Simulation \( S_{\subseteq} \)

Relation between nodes of two models \( \mathcal{I} = (W, R, V) \) and \( \mathcal{I}' = (W', R', V') \) s.t. for any two worlds \( u \in W \) and \( u' \in W' \), if \( uSu' \) then the following hold.

- \( V(u) \subseteq V'(u') \), and
- if \( uRv \), then there exists a \( v' \in W' \) such that \( u'R'v' \) and \( vSv' \).

If for all \( u \in W \) there is at least one \( u' \in W' \) such that \( uSu' \), then we call \( S_{\subseteq} \) a full subset-simulation from \( \mathcal{I} \) to \( \mathcal{I}' \) (\( \mathcal{I} \leq_{\subseteq} \mathcal{I}' \)).
Subset-simulation is a preorder on models.

**Definition**

A model $\mathcal{I}$ of a modal formula $\phi$ is minimal modulo subset-simulation iff for any model $\mathcal{I}'$ of $\phi$, if $\mathcal{I}' \preceq \subseteq \mathcal{I}$, then $\mathcal{I} \preceq \subseteq \mathcal{I}'$.
Models Minimal Modulo Subset-Simulation

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Minimal models

Infinitely many minimal models can belong to a symmetry class.

F. Papacchini, R. A. Schmidt
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Infinitely many minimal models can belong to a symmetry class.
Minimal Model Soundness
A procedure is minimal model sound if it generates only models minimal modulo subset-simulation.

Minimal Model Completeness
A procedure is minimal model complete if it generates at least one model minimal modulo subset-simulation per symmetry class.
Properties of the Minimality Criterion

- loop free models are preferred
- syntax independent
- minimisation of the valuation function
- suitable for many non-classical logics
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Procedures for Computing Minimal Models

Combination of tableaux calculi and a minimality test.
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Tableaux calculi properties

- goal-oriented rules
- modularity
- termination
- minimal model completeness

Subset-simulation test

- closes unwanted branches of a tableau
- logic independent
- ensures minimal model soundness
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Generalisations

Multi-modal logics

\[(W, R, V) \Rightarrow (W, R_1, \ldots, R_n, V)\]

\[[R_1], ⟨R_1⟩, \ldots, [R_n], ⟨R_n⟩\]
Generalisations

Multi-modal logics

\((W, R, V) \Rightarrow (W, R_1, \ldots, R_n, V)\)

\([R_1], \langle R_1 \rangle, \ldots, [R_n], \langle R_n \rangle\)

Inclusion axioms

Syntax: \([R_i] \phi \rightarrow [R_1] \ldots [R_n] \phi\)

Semantics: \(R_1 \circ \ldots \circ R_n \subseteq R_i\)
Generalisations

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Universal modalities

\[[U] \phi\]
\(\phi\) holds in every world

\(\langle U \rangle \phi\)
\(\phi\) holds in some world
Challenges

- incorporate the semantics into the procedures
  - new rules
- preserve properties of the procedures
  - minimal model completeness
  - minimal model soundness
  - termination
New Rules

Multi-modal logics

Inclusion axioms

Universal modalities
New Rules

Multi-modal logics

• modification of existing rules

\[
\frac{(u, v) : R \quad u : \Box \phi}{v : \phi} \quad \Rightarrow \quad (\Box)_i \quad \frac{(u, v) : R_i \quad u : [R_i] \phi}{v : \phi}
\]

Inclusion axioms

Universal modalities
New Rules

Multi-modal logics

- modification of existing rules

\[
\begin{array}{c}
(u,v) : R \\
v : \phi \\
\end{array}
\quad \Rightarrow \quad 
\begin{array}{c}
(u,v) : R_i \\
v : [R_i] \phi \\
\end{array}
\]

Inclusion axioms

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[R_i] \phi \rightarrow [R_1] \ldots [R_n] \phi
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Universal modalities
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\frac{(u, v) : R}{v : \phi} \quad u : \Box \phi \quad \Rightarrow \quad (\Box)^i \quad \frac{(u, v) : R_i}{v : \phi} \quad u : [R_i] \phi
\]

Inclusion axioms

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[R_i] \phi \rightarrow [R_1] \ldots [R_n] \phi
\]

Universal modalities

\[
\langle U \rangle \phi \quad \frac{u : \langle U \rangle \phi}{v_{\langle U \rangle \phi} : \phi}
\]

where \( v_{\langle U \rangle \phi} \) is uniquely assigned to \( \langle U \rangle \phi \)

\[
[U] \phi \quad \frac{u : [U] \phi}{v : \phi}
\]

for any \( v \) appearing on the branch
Minimal model completeness

Adaptation of our previous proof

- take any minimal model $M$
- the tableau generates at least a model $M'$ s.t. $M' \subseteq M$
- minimality of $M \Rightarrow M$ and $M'$ same symmetry class
  $\Rightarrow$ minimal model completeness
Minimal Model Soundness and Completeness

Minimal model completeness

Adaptation of our previous proof

• take any minimal model $M$

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Minimal model soundness

Obtained by the application of the subset-simulation test.
Termination

Decision procedures exist $\Rightarrow$ blocking techniques exist
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Main Challenge

Blocking techniques that preserve minimal model completeness
Termination

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Most probable solution: variations of equality blocking
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Decision procedures exist $\Rightarrow$ blocking techniques exist

Main Challenge

Blocking techniques that preserve minimal model completeness

Most probable solution: variations of equality blocking
Benefits of Termination

- new decision procedures
- theoretical implications (e.g., finitely many symmetry classes)
- effective implementations
Where Are We Now?

- new rules
- minimal model soundness and completeness
- termination
  - multi-modal logics
  - inclusion axioms
  - universal modalities
Where Are We Now?

- new rules ✓
- minimal model soundness and completeness
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  - multi-modal logics
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Where Are We Now?

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Where Are We Now?

- new rules ✓
- minimal model soundness and completeness ✓
- termination
  - multi-modal logics – working on the proof
  - inclusion axioms
  - universal modalities – working on the proof
Where Are We Now?

- new rules ✓
- minimal model soundness and completeness ✓
- termination
  - multi-modal logics – working on the proof
  - inclusion axioms – more investigation needed
  - universal modalities – working on the proof
Conclusion and Further Work

- generalisations of the procedures are possible
- termination is the hardest challenge
  - almost solved for multi-modal logics and universal modalities
  - unsolved for inclusion axioms
Conclusion and Further Work

• generalisations of the procedures are possible

• termination is the hardest challenge
  – almost solved for multi-modal logics and universal modalities
  – unsolved for inclusion axioms

• there is no limit to generalisations!
  – converse relations
  – dynamic modal logics
  – other non-classical logics

• fragments of first-order logic

• implementation