A Tableau Calculus for Minimal Modal Model Generation

Fabio Papacchini and Dr.-Ing. Renate A. Schmidt

School of Computer Science
The University of Manchester

November 11, 2011
(Minimal) Model Generation

Useful for several tasks:

- hardware and software verification
- fault analysis
- commonsense reasoning
- ...
(Minimal) Model Generation

Useful for several tasks:

- hardware and software verification
- fault analysis
- commonsense reasoning
- ...

Even though non-classical logics are widely used in Computer Science, minimal model generation for such logics has not been deeply studied...
Minimality Criteria

Studying classical logics, we can deduce several minimality criteria:

\[ \langle \text{has-father} \rangle (p \lor q) \]

- Minimal Herbrand models
- Domain minimality
- Minimization of all predicates
- Minimization of a certain set of predicates
Minimality Criteria

Studying classical logics, we can deduce several minimality criteria:

\[ \langle \text{has\_father} \rangle (p \lor q) \]

- Minimal Herbrand models
- Domain minimality
- Minimization of all predicates
- Minimization of a certain set of predicates

Models generated by our calculus
Minimality Criteria

Studying classical logics, we can deduce several minimality criteria:

\[ \langle \text{has\_father} \rangle (p \lor q) \]

- Minimal Herbrand models
- Domain minimality
- Minimization of all predicates
- Minimization of a certain set of predicates
Minimality Criteria

Studying classical logics, we can deduce several minimality criteria:

\[ \langle \text{has\_father} \rangle (p \lor q) \]

- Minimal Herbrand models
- Domain minimality
- Minimization of all predicates
- Minimization of a certain set of predicates
Minimality Criteria

Studying classical logics, we can deduce several minimality criteria:

\[ \langle \text{has\_father} \rangle (p \lor q) \]

- Minimal Herbrand models
- Domain minimality
- Minimization of all predicates
- Minimization of a certain set of predicates

Minimization with respect to \( q \)
Minimality Criteria

Studying classical logics, we can deduce several minimality criteria:

\( \langle \text{has\_father} \rangle (p \lor q) \)

- Minimal Herbrand models
- Domain minimality
- Minimization of all predicates
- Minimization of a certain set of predicates

Minimization with respect to \( p \)
Minimality Criteria

Studying classical logics, we can deduce several minimality criteria:

\[ \langle \text{has\_father} \rangle (p \lor q) \]

- Minimal Herbrand models
- Domain minimality
- Minimization of all predicates
- Minimization of a certain set of predicates

Minimization with respect to \text{has\_father}
Aim

Set of multi-modal formulae
\((K_m, KT_m, KB_m, KTB_m)\)

Tableau method

All and only minimal modal Herbrand models
Tableau Language: Syntax

A tableau clause is defined as follows:

\[ TC ::= \top \]
\[ \quad | \bot \]
\[ \quad | u : \phi \]
\[ \quad | (u, v) : R_i \]
\[ \quad | (u, v) : \neg R_i \]
\[ \quad | TC \lor TC \]

- \( u : (p \lor \langle R_1 \rangle q) \)
- \( u : [R_1](p \lor q) \lor (u, v) : R_2 \)
- \( u : \langle R_2 \rangle \neg p \lor v : (p \land q) \)
Modal Herbrand universe \((W_U)\):

the set of all terms built from a supply of unary function symbols of the form \(f_{\langle R_i \rangle \phi_i}\) and \(f_{\langle R_i \rangle \sim \phi_i}\), and the terms appearing in \(N\).

\[
N = \{ u:_{\langle R_1 \rangle p}, v:_{\langle R_1 \rangle p} \land \neg [R_2] q, (u, v) : R_1 \}
\]

Constant symbols: \(u, v\)  
Unary function symbols: \(f_{\langle R_1 \rangle p}, f_{\langle R_2 \rangle \neg q}\)

\[
W_U = \{ u, v, f_{\langle R_1 \rangle p}(u), f_{\langle R_1 \rangle p}(v), f_{\langle R_2 \rangle \neg q}(u), f_{\langle R_2 \rangle \neg q}(v), \ldots \}
\]
Tableau Language: Semantics

Modal Herbrand universe \((W_\mathcal{U})\):

the set of all terms built from a supply of unary function symbols of the form \(f_{\langle R_i \rangle \phi_i}\) and \(f_{\langle R_i \rangle \sim \phi_i}\), and the terms appearing in \(N\).

\[
N = \{ \ u: \langle R_1 \rangle p, \ v: \langle R_1 \rangle p \land \neg [R_2] q, \ (u, v) : R_1 \} \]

Constant symbols:  \(u, v\)  
Unary function symbols:  \(f_{\langle R_1 \rangle p}, f_{\langle R_2 \rangle \neg q}\)

\[
W_\mathcal{U} = \{ \ u, v, f_{\langle R_1 \rangle p}(u), f_{\langle R_1 \rangle p}(v), f_{\langle R_2 \rangle \neg q}(u), f_{\langle R_2 \rangle \neg q}(v), \ldots \} \]
Tableau Language: Semantics

Modal Herbrand universe ($W_U$):

the set of all terms built from a supply of unary function symbols of the form $f_{\langle R_i \rangle} \phi_i$ and $f_{\langle R_i \rangle} \sim \phi_i$, and the terms appearing in $N$.

$$N = \{ u: \langle R_1 \rangle p, \; v: \langle R_1 \rangle p \land \neg [R_2] q, \; (u, v) : R_1 \}$$

Constant symbols: $u, v$  
Unary function symbols: $f_{\langle R_1 \rangle} p, f_{\langle R_2 \rangle} \neg q$

$$W_U = \{ u, \; v, \; f_{\langle R_1 \rangle} p(u), \; f_{\langle R_1 \rangle} p(v), \; f_{\langle R_2 \rangle} \neg q(u), \; f_{\langle R_2 \rangle} \neg q(v), \; \ldots \}$$
Modal Herbrand universe \((W_\mathcal{U})\):

the set of all terms built from a supply of unary function symbols of the form \(f_{\langle R_i \rangle \phi_i}\) and \(f_{\langle R_i \rangle \sim \phi_i}\), and the terms appearing in \(N\).

\[
N = \{ u: \langle R_1 \rangle p, v: \langle R_1 \rangle p \land \neg [R_2] q, (u, v) : R_1 \}
\]

Constant symbols: \(u, v\)  
Unary function symbols: \(f_{\langle R_1 \rangle p}, f_{\langle R_2 \rangle \neg q}\)

\[
W_\mathcal{U} = \{ u, v, f_{\langle R_1 \rangle p}(u), f_{\langle R_1 \rangle p}(v), f_{\langle R_2 \rangle \neg q}(u), f_{\langle R_2 \rangle \neg q}(v), \ldots \}
\]
Tableau Language: Semantics (cont’d)

- A modal Herbrand interpretation $I$ is a set of tableau atoms ($u : p_i$ and $(u, v) : R_i$)

- Truth in $I$:

  \[
  \begin{align*}
  I \not\models \bot & \quad I \models \top \\
  I \not\models u : \bot & \quad I \models u : \top \\
  I \models u : p_i & \quad \text{iff } u : p_i \in I \\
  I \models (u, v) : R_i & \quad \text{iff } (u, v) : R_i \in I \\
  I \models u : \neg \phi & \quad \text{iff } I \not\models u : \phi \\
  I \models (u, v) : \neg R_i & \quad \text{iff } I \not\models (u, v) : R_i \\
  I \models u : (\phi_1 \lor \phi_2) & \quad \text{iff } I \models u : \phi_1 \text{ or } I \models u : \phi_2 \\
  I \models \Delta_1 \lor \Delta_2 & \quad \text{iff } I \models \Delta_1 \text{ or } I \models \Delta_2 \\
  I \models u : [R_i]\phi & \quad \text{iff for every } v \text{ if } (u, v) : R_i \in I \text{ then } I \models v : \phi \\
  I \models u : \langle R_i\rangle \phi & \quad \text{iff } (u, f_{\langle R_i\rangle \phi}(u)) : R_i \in I \text{ and } I \models f_{\langle R_i\rangle \phi}(u) : \phi
  \end{align*}
  \]

- Semantics of a diamond formula is expressed in term of the functional symbol assigned to it

- $I \models u : (\phi_1 \lor \phi_2)$ iff $I \models u : \phi_1 \lor u : \phi_2$
Tableau Language: Semantics (cont’d)

- A modal Herbrand interpretation $I$ is a set of tableau atoms ($u : p_i$ and $(u, v) : R_i$)

- Truth in $I$:

\[
\begin{align*}
I \not\models \bot & \quad I \models \top \\
I \not\models u : \bot & \quad I \models u : \top \\
I \models u : p_i & \quad \text{iff } u : p_i \in I \\
I \models (u, v) : R_i & \quad \text{iff } (u, v) : R_i \in I \\
I \models u : \neg \phi & \quad \text{iff } I \not\models u : \phi \\
I \models (u, v) : \neg R_i & \quad \text{iff } I \not\models (u, v) : R_i \\
I \models u : (\phi_1 \lor \phi_2) & \quad \text{iff } I \models u : \phi_1 \text{ or } I \models u : \phi_2 \\
I \models \Delta_1 \lor \Delta_2 & \quad \text{iff } I \models \Delta_1 \text{ or } I \models \Delta_2 \\
I \models u : [R_i]\phi & \quad \text{iff for every } v \text{ if } (u, v) : R_i \in I \text{ then } I \models v : \phi \\
I \models u : \langle R_i \rangle \phi & \quad \text{iff } (u, f_{\langle R_i \rangle \phi}(u)) : R_i \in I \text{ and } I \models f_{\langle R_i \rangle \phi}(u) : \phi
\end{align*}
\]

- semantics of a diamond formula is expressed in term of the functional symbol assigned to it

\[
I \models u : (\phi_1 \lor \phi_2) \quad \text{iff } I \models u : \phi_1 \lor u : \phi_2
\]
Tableau Language: Semantics (cont’d)

- A modal Herbrand interpretation $I$ is a set of tableau atoms ($u : p_i$ and $(u, v) : R_i$)

- Truth in $I$:

  \[
  \begin{align*}
  I \not\models \bot & \quad I \models \top \\
  I \not\models u : \bot & \quad I \models u : \top \\
  I \models u : p_i & \quad \text{iff } u : p_i \in I \\
  I \models (u, v) : R_i & \quad \text{iff } (u, v) : R_i \in I \\
  I \models u : \neg \phi & \quad \text{iff } I \not\models u : \phi \\
  I \models (u, v) : \neg R_i & \quad \text{iff } I \not\models (u, v) : R_i \\
  I \models u : (\phi_1 \lor \phi_2) & \quad \text{iff } I \models u : \phi_1 \text{ or } I \models u : \phi_2 \\
  I \models \Delta_1 \lor \Delta_2 & \quad \text{iff } I \models \Delta_1 \text{ or } I \models \Delta_2 \\
  I \models u : [R_i] \phi & \quad \text{iff } \text{for every } v \text{ if } (u, v) : R_i \in I \text{ then } I \models v : \phi \\
  I \models u : \langle R_i \rangle \phi & \quad \text{iff } (u, f_{\langle R_i \rangle \phi}(u)) : R_i \in I \text{ and } I \models f_{\langle R_i \rangle \phi}(u) : \phi
  \end{align*}
  \]

- semantics of a diamond formula is expressed in term of the functional symbol assigned to it

- $I \models u : (\phi_1 \lor \phi_2) \iff I \models u : \phi_1 \lor u : \phi_2$
(Minimal) Modal Herbrand Model

- \( I \) is a modal Herbrand model of \( N \) if \( I \models N \)
- \( I \) is a minimal modal Herbrand model, iff for every other modal Herbrand model \( I' \), if \( I' \subseteq I \) then \( I = I' \)
(Minimal) Modal Herbrand Model

- $I$ is a *modal Herbrand model* of $N$ if $I \models N$
- $I$ is a *minimal modal Herbrand model*, iff for every other modal Herbrand model $I'$, if $I' \subseteq I$ then $I = I'$

\[
\begin{align*}
  u &: \Diamond (p \lor q) \\
  \begin{align*}
    &\{ f_{\Diamond (p \lor q)}(u) : p, (u, f_{\Diamond (p \lor q)}(u)) : R \} \\
    &\{ f_{\Diamond (p \lor q)}(u) : q, (u, f_{\Diamond (p \lor q)}(u)) : R \} \\
    &\{ f_{\Diamond (p \lor q)}(u) : p, f_{\Diamond (p \lor q)}(u) : q, (u, f_{\Diamond (p \lor q)}(u)) : R \}
  \end{align*}
\]
(Minimal) Modal Herbrand Model

- $I$ is a *modal Herbrand model* of $N$ if $I \models N$
- $I$ is a *minimal modal Herbrand model*, iff for every other modal Herbrand model $I'$, if $I' \subseteq I$ then $I = I'$

\[
 u : \Diamond (p \lor q)
\]

\[
\begin{align*}
\{ f_{\Diamond(p\lor q)}(u) : p, (u, f_{\Diamond(p\lor q)}(u)) : R \} \\
\{ f_{\Diamond(p\lor q)}(u) : q, (u, f_{\Diamond(p\lor q)}(u)) : R \} \\
\{ f_{\Diamond(p\lor q)}(u) : p, f_{\Diamond(p\lor q)}(u) : q, (u, f_{\Diamond(p\lor q)}(u)) : R \}
\end{align*}
\]
(Minimal) Modal Herbrand Model

- $I$ is a *modal Herbrand model* of $N$ if $I \models N$
- $I$ is a *minimal modal Herbrand model*, iff for every other modal Herbrand model $I'$, if $I' \subseteq I$ then $I = I'$

\[ u : \Diamond (p \lor q) \]

\[ \{ f_{\Diamond (p \lor q)}(u) : p, (u, f_{\Diamond (p \lor q)}(u)) : R \} \]
\[ \{ f_{\Diamond (p \lor q)}(u) : q, (u, f_{\Diamond (p \lor q)}(u)) : R \} \]
\[ \{ f_{\Diamond (p \lor q)}(u) : p, f_{\Diamond (p \lor q)}(u) : q, (u, f_{\Diamond (p \lor q)}(u)) : R \} \]
(Minimal) Modal Herbrand Model

- $I$ is a modal Herbrand model of $N$ if $I \models N$
- $I$ is a minimal modal Herbrand model, iff for every other modal Herbrand model $I'$, if $I' \subseteq I$ then $I = I'$

\[ u : \Diamond (p \lor q) \]

- $\{ f_{\Diamond (p \lor q)}(u) : p, \ (u, f_{\Diamond (p \lor q)}(u)) : R \}$
- $\{ f_{\Diamond (p \lor q)}(u) : q, \ (u, f_{\Diamond (p \lor q)}(u)) : R \}$
- $\{ f_{\Diamond (p \lor q)}(u) : p, \ f_{\Diamond (p \lor q)}(u) : q, \ (u, f_{\Diamond (p \lor q)}(u)) : R \}$
Minimal Modal Model Generation (3MG) Calculus

The input is obtained by clausal normal form transformation with box miniscoping $(\Box (\phi_1 \land \phi_2) \Rightarrow \Box \phi_1 \land \Box \phi_2)$.

Box miniscoping ensures that conjunctions appear only in the scope of diamonds.

The calculus is composed of:

- four expansion rules
- the model constraint propagation rule
- two rules for reflexivity and symmetry

The calculus use a depth-first left-to-right strategy
3MG: Expansion Rules

\[(\Diamond) \quad u : \Diamond(\phi_1 \land \ldots \land \phi_n) \quad \frac{}{(u,f_{\Diamond \phi}(u)) : R}
\]
\[f_{\Diamond \phi}(u) : \phi_1
\]
\[
\vdots
\]
\[f_{\Diamond \phi}(u) : \phi_n
\]

where $\phi = \phi_1 \land \ldots \land \phi_n$ and $f_{\langle R_i \rangle \phi}$ is the function symbol uniquely associated with $\langle R_i \rangle \phi$

\[(\lor)_E \quad u : (\phi_1 \lor \ldots \lor u : \phi_n) \lor \Delta \quad \frac{}{u : \phi_1 \lor \ldots \lor u : \phi_n \lor \Delta}
\]
3MG: Expansion Rules (cont’d)

\[ (CS) \quad \frac{P_1 \lor \ldots \lor P_n}{\begin{array}{c} P_1 \\ \text{neg}(P_i) \\ P_2 \lor \ldots \lor P_n \end{array}} \]

where \( \text{neg}(P_i) \) stands for \( \text{neg}(P_2), \ldots, \text{neg}(P_n) \)

\[
\text{neg}(P) = \begin{cases} 
(u, f_{\Diamond \phi}(u)) : \neg R & \text{if } P = u : \Diamond \phi \\
 u : \neg p_i & \text{if } P = u : p_i \\
(u, v) : \neg R & \text{if } P = (u, v) : R
\end{cases}
\]

- applicable only to disjunctions of positive literals \((u : p_i, u : \langle R_i \rangle \phi, (u, v) : R_i)\)
- ensures that no model is generated more than once
- ensures that the first model is a minimal model
- results in a reduction of the search space
3MG: Expansion Rules (cont’d)

\[(CS) \quad \frac{P_1 \lor \ldots \lor P_n}{P_1 \lor P_2 \lor \ldots \lor P_n \lor \neg(P_i)}\]

where \(\neg(P_i)\) stands for \(\neg(P_2), \ldots, \neg(P_n)\)

\[\neg(P) = \begin{cases} 
(u, f_{\Diamond \phi}(u)) : \neg R & \text{if } P = u : \Diamond \phi \\
u : \neg p_i & \text{if } P = u : p_i \\
(u, v) : \neg R & \text{if } P = (u, v) : R
\end{cases}\]

- applicable only to disjunctions of positive literals \((u : p_i, u : \langle R_i \rangle \phi, (u, v) : R_i)\)
- ensures that no model is generated more than once
- ensures that the first model is a minimal model
- results in a reduction of the search space
3MG: Expansion Rules - \((SBR)\) rule

- the most complex rule of the calculus
- aims to expand a clause with negative literals \((u : \neg p_i, u : [R_i]\phi, (u, v) : \neg R_i)\) iff it is necessary
- aims to minimize splitting
- can be thought as the composition of the common closure rules and the box expansion rule
3MG: Expansion Rules - \((SBR)\) rule

- the most complex rule of the calculus
- aims to expand a clause with negative literals \((u : \neg p_i, u : [R_i]\phi, (u, v) : \neg R_i)\) iff it is necessary
- aims to minimize splitting
- can be thought as the composition of the common closure rules and the box expansion rule

\[
\begin{align*}
  u : \neg p & \lor u : q \text{ is expanded iff } u : p \text{ is on the branch} \\
  u : \Box p & \lor u : q \text{ is expanded iff there exists } (u, v) : R \text{ on the branch}
\end{align*}
\]
3MG: Expansion Rules - \((SBR)\) rule

- the most complex rule of the calculus
- aims to expand a clause with negative literals \((u : \neg p_i, u : [R_i] \phi, (u, v) : \neg R_i)\) iff it is necessary
- aims to minimize splitting
- can be thought as the composition of the common closure rules and the box expansion rule

\[
\begin{align*}
  u : \neg p \lor u : q \text{ is expanded iff } u : p \text{ is on the branch} \\
  u : \Box p \lor u : q \text{ is expanded iff there exists } (u, v) : R \text{ on the branch}
\end{align*}
\]

\[
\begin{array}{c}
  \frac{u : p \quad u : \neg p}{\bot} \\
  \frac{(u, v) : R \quad (u, v) : \neg R}{\bot} \\
  \frac{u : \Box \phi \quad (u, v) : R}{v : \phi}
\end{array}
\]
3MG: Expansion Rules - \((SBR)\) rule

- the most complex rule of the calculus
- aims to expand a clause with negative literals \((u : \neg p_i, u : [R_i] \phi, (u, v) : \neg R_i)\) iff it is necessary
- aims to minimize splitting
- can be thought as the composition of the common closure rules and the box expansion rule

\[
\begin{align*}
    u : \neg p \lor u : q & \text{ is expanded iff } u : p \text{ is on the branch} \\
    u : \Box p \lor u : q & \text{ is expanded iff there exists } (u, v) : R \text{ on the branch}
\end{align*}
\]

\[
\begin{array}{c c c c c c}
\hline
u : p & u : \neg p & (u, v) : R & (u, v) : \neg R & u : \Box \phi & (u, v) : R \\
\hline
\bot & \bot & \bot & \bot & v : \phi
\end{array}
\]
3MG: Expansion Rules - \( (SBR) \) rule

- the most complex rule of the calculus
- aims to expand a clause with negative literals \((u : \neg p_i, u : [R_i] \phi, (u, v) : \neg R_i)\) iff it is necessary
- aims to minimize splitting
- can be thought as the composition of the common closure rules and the box expansion rule

\[
\begin{align*}
  u_1 : p_1 & \quad \ldots \quad u_n : p_n \\
  (v_1, w_1) : R & \quad \ldots \quad (v_m, w_m) : R \\
  (s_1, t_1) : R & \quad \ldots \quad (s_j, t_j) : R \\
  u_1 : \neg p_1 \lor \ldots \lor u_n : \neg p_n \lor v_1 : \square \phi_1 \lor \ldots \lor v_m : \square \phi_m \\
  \lor (s_1, t_1) : \neg R \lor \ldots \lor (s_j, t_j) : \neg R \lor \Delta^+ \\
  w_1 : \phi_1 \lor \ldots \lor w_m : \phi_m \lor \Delta^+
\end{align*}
\]
Model Extraction

Once the calculus generates an open and fully expanded branch, the set of tableau atoms appearing on such branch is a modal Herbrand model for the original input.

\[\begin{align*}
    u & : \Diamond (p \lor q) \\
    (u, f_{\Diamond (p \lor q)}(u)) & : R \\
    f_{\Diamond (p \lor q)}(u) & : (p \lor q) \\
    f_{\Diamond (p \lor q)}(u) & : p \lor f_{\Diamond (p \lor q)}(u) : q \\
    f_{\Diamond (p \lor q)}(u) & : p \quad (CS) \\
    f_{\Diamond (p \lor q)}(u) & : \neg q \quad (CS) \\
    f_{\Diamond (p \lor q)}(u) & : q \quad (CS)
\end{align*}\]
Model Extraction

Once the calculus generates an open and fully expanded branch, the set of tableau atoms appearing on such branch is a modal Herbrand model for the original input.

\[
\begin{align*}
{u} : \diamond (p \lor q) & \quad \text{Input} \\
(u, {f} \diamond (p \lor q)(u)) : R & \quad (\diamond) \\
{f} \diamond (p \lor q)(u) : (p \lor q) & \quad (\diamond) \\
{f} \diamond (p \lor q)(u) : p \lor {f} \diamond (p \lor q)(u) : q & \quad (\lor)_E \\
{f} \diamond (p \lor q)(u) : p & \quad (CS) \\
{f} \diamond (p \lor q)(u) : \neg q & \quad (CS) \\
{f} \diamond (p \lor q)(u) : q & \quad (CS) \\
I_1 = \{ {f} \diamond (p \lor q)(u) : p, (u, {f} \diamond (p \lor q)(u)) : R \} 
\end{align*}
\]
Model Extraction

Once the calculus generates an open and fully expanded branch, the set of tableau atoms appearing on such branch is a modal Herbrand model for the original input.

\[
\begin{align*}
\text{Input} & : \diamond(p \lor q) \\
(u, f_{\diamond(p \lor q)}(u)) & : R \\
f_{\diamond(p \lor q)}(u) & : (p \lor q) \\
f_{\diamond(p \lor q)}(u) & : p \lor f_{\diamond(p \lor q)}(u) : q \\
& \quad (\lor)_E \\
f_{\diamond(p \lor q)}(u) & : p \quad (CS) \\
f_{\diamond(p \lor q)}(u) & : \neg q \quad (CS) \\
f_{\diamond(p \lor q)}(u) & : q \quad (CS) \\
\end{align*}
\]

\[
I_1 = \{ f_{\diamond(p \lor q)}(u) : p, \ (u, f_{\diamond(p \lor q)}(u)) : R \} \\
I_2 = \{ f_{\diamond(p \lor q)}(u) : q, \ (u, f_{\diamond(p \lor q)}(u)) : R \} 
\]
If \( I = \{u_1 : p_1, \ldots, u_n : p_n, (v_1, w_1) : R, \ldots, (v_m, w_m) : R\} \) is the (minimal) modal Herbrand model extracted from a branch \( \mathcal{B} \), then

\[
\begin{align*}
  u_1 : \neg p_1 \lor \ldots \lor u_n : \neg p_n \lor (v_1, w_1) : \neg R \lor \ldots \lor (v_m, w_m) : \neg R
\end{align*}
\]

is added to all the branches to the right of \( \mathcal{B} \).

\[
\begin{align*}
  u : \Diamond (p \lor q) \\
  (u,f_\Diamond (p\lor q)(u)) : R \\
  f_\Diamond (p\lor q)(u) : (p \lor q) \\
  f_\Diamond (p\lor q)(u) : p \lor f_\Diamond (p\lor q)(u) : q
\end{align*}
\]

\( I_1 = \{f_\Diamond (p\lor q)(u) : p, (u,f_\Diamond (p\lor q)(u)) : R\} \)
If \( I = \{u_1 : p_1, \ldots, u_n : p_n, (v_1, w_1) : R, \ldots, (v_m, w_m) : R\} \) is the (minimal) modal Herbrand model extracted from a branch \( B \), then

\[
\begin{align*}
    u_1 : & \neg p_1 \lor \ldots \lor u_n : \neg p_n \lor (v_1, w_1) : \neg R \lor \ldots \lor (v_m, w_m) : \neg R
\end{align*}
\]

is added to all the branches to the right of \( B \).
3MG calculus, an example

Derivation for:
\{ u : \diamond (p \lor q), \ u : \Box p \}
3MG calculus, an example

\[ u : \Diamond (p \lor q) \quad \text{Input} \]
\[ u : \Box p \quad \text{Input} \]
\[ (u, f_{\Diamond (p \lor q)}(u)) : R \quad (\Diamond) \]
\[ f_{\Diamond (p \lor q)}(u) : (p \lor q) \quad (\Diamond) \]
\[ f_{\Diamond (p \lor q)}(u) : p \quad (SBR) \]
\[ f_{\Diamond (p \lor q)}(u) : p \lor f_{\Diamond (p \lor q)}(u) : q \quad (\lor)_E \]
\[ \frac{f_{\Diamond (p \lor q)}(u) : p \quad (CS)}{f_{\Diamond (p \lor q)}(u) : q \quad (CS)} \]
\[ f_{\Diamond (p \lor q)}(u) : \neg q \quad (CS) \quad \frac{f_{\Diamond (p \lor q)}(u) : q \quad (CS)}{MC} \]
\[ \bot \quad (SBR) \]

\[ I_1 = \{ f_{\Diamond (p \lor q)}(u) : p, \ (u, f_{\Diamond (p \lor q)}(u)) : R \} \]
\[ MC = f_{\Diamond (p \lor q)}(u) : \neg p \lor (u, f_{\Diamond (p \lor q)}(u)) : \neg R \]
Reflexivity and Symmetry

\[(B)^i \quad \frac{(u, v) : R_i}{(v, u) : R_i} \quad \text{if } R_i \text{ is symmetric}\]

\[(T)^i \quad \frac{(u, u) : R_i}{(u, u) : R_i} \quad \text{if } R_i \text{ is reflexive and } u \text{ appears in a tableau formula of the form } u : \phi, (u, v) : R_j \text{ or } (v, u) : R_j \text{ on the current branch}\]
Reflexivity and Symmetry

\[(B)^i \frac{(u, v) : R_i}{(v, u) : R_i}\] if \(R_i\) is symmetric

\[(T)^i \frac{(u, u) : R_i}{(u, u) : R_i}\] if \(R_i\) is reflexive and \(u\) appears in a tableau formula of the form \(u : \phi, (u, v) : R_j\) or \((v, u) : R_j\) on the current branch

\[
\begin{align*}
  u &: (p \lor \Diamond q) & \text{Input} \\
  u &: p \lor u : \Diamond q & (\lor)_E \\
  (u, u) &: R & \text{Std. } (T) \\
  \end{align*}
\]

\[
\begin{align*}
  u &: p & (CS) \\
  (u, f_{\Diamond q}(u)) &: \neg R & (CS) \\
  (f_{\Diamond q}(u), f_{\Diamond q}(u)) &: R & \text{Std. } (T) \\
\end{align*}
\]

\[
\begin{align*}
  u &: \Diamond q & (CS) \\
  (u, f_{\Diamond q}(u)) &: R & (\Diamond) \\
  f_{\Diamond q}(u) &: q & (\Diamond) \\
  (f_{\Diamond q}(u), f_{\Diamond q}(u)) &: R & \text{Std. } (T) \\
\end{align*}
\]

\[MC = u : \neg p \lor (u, u) : \neg R \lor (f_{\Diamond q}(u), f_{\Diamond q}(u)) : \neg R\]
Reflexivity and Symmetry

\[(B)^i \frac{(u, v) : R_i}{(v, u) : R_i} \quad \text{if } R_i \text{ is symmetric}\]

\[(T)^i \frac{(u, u) : R_i}{(u, u) : R_i} \quad \text{if } R_i \text{ is reflexive and } u \text{ appears in a tableau formula of the form } u : \phi, (u, v) : R_j \text{ or } (v, u) : R_j \text{ on the current branch}\]

\[\begin{align*}
\text{Input} & \quad u : (p \lor \diamond q) & (\lor)_E \\
& \quad u : p \lor u : \diamond q \\
& \quad (u, u) : R & (T)^i \\
\text{Input} & \quad u : \diamond q & (CS) \\
& \quad (u, f \diamond q(u)) : \neg R & (CS) \\
\text{Input} & \quad u : \diamond q & (CS) \\
& \quad (u, f \diamond q(u)) : R & (\diamond) \\
& \quad f \diamond q(u) : q & (\diamond) \\
& \quad (f \diamond q(u), f \diamond q(u)) : R & (T)^i \\
\end{align*}\]

\[MC = u : \neg p \lor (u, u) : \neg R\]
Minimal Model Soundness and Completeness

**Minimal model soundness:** only minimal models are generated

**Minimal model completeness:** all minimal models are generated

Our proof is based on showing a bisimulation between the 3MG calculus and the PUHR approach of Bry and Yahya for first-order logic.

This indirect proof has the advantage of:

- giving us useful insight about the relation of the two calculi
- giving us the opportunity to study how structural transformation affects model generation for first-order clauses
- allowing us to compare our calculus with techniques for fragments of first-order logic (e.g. Georgieva - Hustadt - Schmidt, Niemelä)
Minimal Model Soundness and Completeness

**Minimal model soundness**: only minimal models are generated

**Minimal model completeness**: all minimal models are generated

Our proof is based on showing a bisimulation between the 3MG calculus and the PUHR approach of Bry and Yahya for first-order logic.

This indirect proof has the advantage of:

- giving us useful insight about the relation of the two calculi
- giving us the opportunity to study how structural transformation affects model generation for first-order clauses
- allowing us to compare our calculus with techniques for fragments of first-order logic (e.g. Georgieva - Hustadt - Schmidt, Niemelä)
Minimal Model Soundness and Completeness (cont’d)

Modal Formulae

PUHR

3MG

The 3MG calculus is minimal model sound and complete.

F. Papacchini, R. A. Schmidt

Minimal Modal Model Generation

November 11, 2011 18 / 21
The 3MG calculus is minimal model sound and complete.

F. Papacchini, R. A. Schmidt
Minimal Modal Model Generation
November 11, 2011 18 / 21
Minimal Model Soundness and Completeness (cont’d)

The 3MG calculus is minimal model sound and complete.

FOL formulae \rightarrow \text{New Translation} \rightarrow \text{Modal Formulae} \rightarrow 3MG

\text{PUHR} \rightarrow \text{3MG}
Minimal Model Soundness and Completeness (cont’d)

The 3MG calculus is minimal model sound and complete.

FOL formulae → New Translation → Modal Formulae

Modified in PUHR → SBR → 3MG
Minimal Model Soundness and Completeness (cont’d)

The 3MG calculus is minimal model sound and complete.
Minimal Model Soundness and Completeness (cont’d)

The 3MG calculus is minimal model sound and complete.
The 3MG calculus is minimal model sound and complete.

F. Papacchini, R. A. Schmidt
Minimal Modal Model Generation
November 11, 2011 18 / 21
The 3MG calculus is minimal model sound and complete.

F. Papacchini, R. A. Schmidt

Minimal Modal Model Generation

November 11, 2011 18 / 21
The 3MG calculus is minimal model sound and complete.
The 3MG calculus is minimal model sound and complete. F. Papacchini, R. A. Schmidt
The 3MG calculus is minimal model sound and complete.
Evaluation of $(CS)$ with $neg$

- for standard tableaux
- in our implementation complement splitting is applied only to diamond formulae
Evaluation of \((CS)\) with \textit{neg}

- for standard tableaux
- in our implementation complement splitting is applied only to diamond formulae

The \textit{neg} function is applicable due to the uniquely assigned functional symbols
Conclusion and Further Work

The 3MG calculus

- is the only calculus generating minimal modal Herbrand models
- works directly on modal formulae
- is minimal model sound and complete
- each model is generated exactly once
- terminates

Several extensions are possible:

- avoiding the clausal normal form transformation
- extending to more expressive logics (e.g. dynamic modal logics)
Conclusion and Further Work

The 3MG calculus

- is the only calculus generating minimal modal Herbrand models
- works directly on modal formulae
- is minimal model sound and complete
- each model is generated exactly once
- terminates

Several extensions are possible:

- avoiding the clausal normal form transformation
- extending to more expressive logics (e.g. dynamic modal logics)
Thank You!