Review of Second Part of Module

Calculus, Complex Numbers, Statistics

In this lecture, as with the earlier instance . . .

We review, with some examples

1. The key notions with which you should feel comfortable.
2. Give some instances of "typical" questions that might arise.

Areas dealt with subsequently

a. Basic calculus and its uses in CS: optimisation.
b. Complex Numbers: representation methods and applications.
c. Statistics and CS as an experimental discipline.

Basic Calculus – Origins, Derivatives, and Optima

Recap

Motivating aim is to study the "*behaviour of* **functions**" "in terms of" **lines**.

Examples: the "gradient" of the "line touching a curve" at a given point; the notions of "increasing", "decreasing" and "**turning points**"; the link to idea of (local) "minimal" and "maximal" values of a function.

For these (you will be given):

- The general expressions for "*the line between 2 points*" **and** "*the line of gradient m through a point* $< p, q >$".
- The "Table of 8 Simple Rules" for obtaining derivatives of "standard" functions.

You will need to recall:

How to **find** the **turning points** of a function, $f(x)$.

How to determine whether these are (local) **minima** or **maxima**.

Some typical example style questions

Lines and Gradients

Questions

A car travels from the base of hill in a straight line to the hill's summit. If the car starts from a map coordinate $< 25, 40 >$ and the hill's peak is at $< 40, 100 >$ what is the **gradient** of the hill?
When the car has reached a height of 60 metres on the hill, how far has it travelled horizontally, i.e. what x-coordinate is its location?

Answers

Subtract the y-coordinates $(100 - 40)$ and divide by the result of subtracting the x-coordinates $(40 - 25)$. The result is $60/15 = 4$.
The line has gradient 4 so the point x in travelling from 25 must be consistent with $(60 - 40)/(x - 25)$ being equal to this gradient. Hence $4(x - 25) = 20$, i.e. $4x = 120$ and, thus $x = 30$.

Turning Points and Local Optima

Finding Turning Points

The **turning points** of $f(x)$ are those values, x, where the "gradient of the line touching the point $< x, f(x) >$" is 0, i.e. the line is *horizontal* (**parallel** with the x-axis).
The gradient of the line touching the point $< x, f(x) >$ is described by the **function** $f'(x)$: the *first derivative* of $f(x)$.
So the "**turning points** of $f(x)$" are "*those values x for which $f'(x) = 0$*".

Maxima and minima

To check whether a turning point, t, is a (local) minimum or maximum:
Find the **function** $f''(x)$ the *second* derivative of $f(x)$ by constructing the (**first**) derivative of $f'(x)$.
Find the value of $f''(t)$.
If $f''(t) < 0$ then t is a (local) maximum;
if $f''(t) > 0$ then t is a (local) minimum.

Example Question

A **Law Department** discovers the existence of $+$ grades and uses $\{G, G+, F, F+, E, E+, D, D+, C, C+, B, B+, A\}$ to grade student essays.
A grade corresponds to a "raw" mark of
$G = 0,\ F = 1,\ \ldots,\ B = 5,\ A = 6$
with $+$ grades scoring 0.5 more than the grade these qualify,
eg $G+ = 0.5,\ D+ = 3.5,\ B+ = 5.5$.
The "raw mark", x, is converted to a "final" mark $Final(x)$ by applying the formula:

$$Final(x) \ = \ 2x^3 \ - \ 18x^2 \ + \ 30x \ + \ 27$$

What **grade** will attract the **minimum** final mark?

Warning: In principle you could attempt this by "brute-force" calculation. Do **not** do this. You will waste a lot of time.

Answer

First step

Find the derivative of the function $Final(x)$:

$$Final'(x) \;=\; 6x^2 \;-\; 36x \;+\; 30$$

Second step: turning points

When is $Final'(x) = 0$? $Final'(x) \equiv 6(x^2 - 6x + 5)$, so $Final'(x) = 0$ when $x^2 - 6x + 5 = 0$, i.e. when $x = 5$ and $x = 1$ – using $Final'(x) = 6(x - 5)(x - 1)$.

Final step: Minimum or Maximum?

Find $Final''(x)$ this is $Final''(x) = 12x - 36$.
$Final''(1) = 12 - 36 = -24 < 0$: $x = 1$ is a **maximum** (grade: F)
$Final''(5) = 60 - 36 = 24 > 0$: $x = 5$ is a **minimum** (grade: B)
So the *minimum* final mark is given by grade B.

Complex Numbers

Main Points to Remember

A **complex number**, z, is defined by two **real numbers**: x the so-called **Real** part of z (denoted $\Re(z) = x$) and y the so-called **Imaginary** part of z (denoted $\Im(z) = y$).

The entity $\imath$ is treated as an object having the property $\imath^2 = -1$.

The complex number z gives rise to its **complex conjugate** ($\overline{z}$) for which $\overline{z} = \Re(z) - \Im(z)$, ie if $z = x + \imath y$ then $\overline{z} = x - \imath y$.

The **modulus** of z (denoted $|z|$) is a measure of the "size" of z and is the (positive) square root of $x^2 + y^2$.

Addition and subtraction of $x + \imath y$ with $u + \imath v$ follows **exactly** the same rules used to add/subtract the (Real) 2-vectors $< x, y >$ and $< u, v >$.

Multiplication and **Division** are more involved. Make sure you understand these (although the process required will be given – at a high level – on an accompanying formula sheet).

Equivalent Representations

Explicit: $z = x + \imath y$.

Coordinate system (Argand diagram): z is a point in 2-dimensional space: "Horizontal" axis being the "Real part" and "Vertical axis" the "Imaginary part".
Remember that, in this scheme, $z =< x, y >$ is a pair of **Real values**.

Polar coordinates: z is described by a pair (r, θ): $0 \leq \theta < 2\pi$ an angle in **radians**, r the length of the line from the origin $< 0, 0 >$ to the point $< \Re(z), \Im(z) >$. Here θ (denoted $\texttt{arg}z$) is the angle measured counter-clockwise from the x-axis to the the line (2-vector) $< (0, 0), (\Re(z), \Im(z)) >$.

Euler form: $z = r \exp(\imath\theta)$: $\Re(z) = r \cos\theta$, $\Im(z) = r \sin\theta$.

Example style questions

Mapping to and from polar coordinates and "target scores"

These have a similar form to the "dartboard example" on Worksheet 4, e.g. for a particular collection of complex numbers $\{< r_1, \theta_1 >, \ < r_2, \theta_2 >, \ldots, < r_t, \theta_t >\}$ what would the **total score** achieved be? Context could be dartboard, archery target, etc.

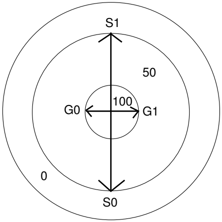
Composition sequences

A "simple" (iterative) algorithm to generate a sequence of complex numbers is given (an approach used by some automated music composition methods). Given a starting value (ie the complex number z_1 when the loop involved begins), determine what the output (complex) number would be for some subsequent (specified) value of the loop counter.

Two Examples

Robot Archery

A **robot archer** is trained to shoot arrows at a target.



The diameter of the line $< G0, G1 >$ is 1 metre. The diameter of the line $< S0, S1 >$ is 2 metres. The entire target has diameter 3 metres. Only arrows that landing within the circles $< G0, G1 >$ and $< S0, S1 >$ score points (100 and 50 respectively).

A robot generates a **random complex number**, in **polar coordinates**. and hits the target areas $\{< 48, \pi >, \ < 75, \pi/2 >, \ < 120, 3\pi/4 >\}$

What is the **total score** achieved?

Sequences of Complex Numbers

Algorithmic Composition Example

An automated composition method generates its score using the algorithm

```
Z := x + ıy;
C := p + ıq; //* <x, y> ∈ ℝ^2 and <p, q> ∈ ℝ^2
Counter := 0;
while Counter < MAX_LENGTH_SCORE
   Output note corresponding to (current) value of Z;
   Z := Z^2 + C;
   Counter := Counter + 1;
end while
```

If this algorithm is run with starting input of $Z = 2 + \imath$ and $C = 1$, what will be the value of Z when the **while** loop is entered with *Counter* equal to 2?

Computer Science as Experimental Study – Statistics

Points to remember

a. The notions of **probability distribution** and **expectation** (wrt a distribution).

b. The ideas underlying **standard deviation**, **hypothesis testing**, **confidence levels**.

c. How the q-test is applied to determine significance.

What you don't need to memorize

Formulae for "computing the expected value of a random variable within a population S and distribution D".

Formulae for computing the standard deviation.

The "significance values" (for the q-test) that indicate $(5\%, 1\%, 0.1\%)$.

All of these will be given on an accompanying handout.

Some typical questions

Exam misconduct

The marks obtained by a collection of 10 students in an exam are,

$$\{\,66,\ 79,\ 43,\ 87,\ 25,\ 51,\ 66,\ 43,\ 89,\ 96\}$$

After (correctly) computing that the **standard deviation** in this collection is 23.5, some students complain that the mark of 96 is so far in excess of the average that the student obtaining it must have used some "underhand means" of boosting their performance. What support does the *q*-test provide for such claims?

To answer this . . .

a. Compute the average mark: this turns out to be 64.5.

b. Find "how many standard deviations" a mark of 96 differs from this average: $\frac{96-64.5}{23.5} \sim 1.34$.

c. Conclusion: the *q*-test provides no support for the allegation.