COMP116 – Work Sheet One – Solutions

Associated Module Learning Outcome

Basic understanding of the range of techniques used to analyse and reason about computational settings

Numbers and Polynomial Properties

The topics on this work sheet deal with properties of *types* of *number* and basic polynomial properties (Q1). More detailed review of this behaviour is the theme of (Q2).

Question 1

I. Consider the following situations involving **numerical data**. What **types** of number are most suitable to use for describing possible outcomes?

Give brief, informal reasons for your answers.

a. The number of correctly answered questions on an MCQ test (Multiple Choice Questions).

b. The goal difference of the middle placed team in the Scottish Lowland League football division at the end of a season.

c. The pay of *soi-disant* 'senior' managers relative to that of Workers and Academics in universities.

d. The total number of points awarded to the UK entry in the Eurovision Song Contest in a particular year.

e. The time a vehicle takes to reach a speed of $80Km/h$ after starting.

f. The majority achieved when legislation is *successfully* passed in the House of Commons, i.e. without requiring the Speaker to separate outcomes by use of a casting vote.

Answers

a. $\mathbb{W}$: not $\mathbb{N}$ because 0 is possible; similarly not $\mathbb{Z}$ (scores are non-negative), nor $\mathbb{Q}$, $\mathbb{R}$.

b. $\mathbb{Z}$: $\mathbb{W}$ or $\mathbb{N}$ presumes the middle placed team always has non-negative $For - Against$ but this may not be the case.

c. $\mathbb{Q}$: The amount is a **ratio** whether expressed in pounds or pence paid in each category. Although there is a literal interpretation of senior management salaries being “irrational” there is no numerical interpretation of this form.

d. $\mathbb{W}$: It is possible to score zero points, however, as the rules stand at present, it is not possible to have points deducted so $\mathbb{Z}$ is too broad a class.

e. $\mathbb{R}$: this is the only choice the allows for the fact that the exact unit of time has not been specified, e.g. microseconds, milliseconds, minutes, hours, etc.

f. $\mathbb{N}$: At least one more vote in favour must have been cast, so ruling out $\mathbb{W}$ and $\mathbb{Z}$.

II. You have seen on COMP109 and COMP124 that computers represent data as collections of **bits** (binary 0 and 1). What implications do you think this convention has for computing applications requiring **Real** numbers ($\mathbb{R}$)?
Answer: Only a finite amount of space is available (64, 128 bits etc.). While sufficient exactly to encode all rational values within a given range, it is not possible to represent every element of $\mathbb{R}$ between, say 0 and 1, to an **arbitrary degree of precision**. Computer architectures, in effect, realise a “rational approximation” to $\mathbb{R}$.

III. The following questions deal with some simple manipulation of polynomial expressions.

a. Suppose $p(x)$ has degree k and $q(x)$ has degree r. What is the degree of $p(x) \cdot q(x)$?

b. If all coefficients of $p(x)$ are *greater than* zero and all coefficients of $q(x)$ are greater than zero how many coefficients of $p(x) \cdot q(x)$ are non-zero?

c. If the condition on the coefficients of $p(x)$ and $q(x)$ is changed from *greater than zero* to *not equal to zero* is your answer to (b) unchanged? [**Hint:** Consider the two degree 1 polynomials $x - 1$ and $x + 1$ all of whose coefficients are non-zero.]

d. Suppose that

$$\begin{aligned} p(x) &= 7x^5 + 3x^4 + 9x^3 + 5x^2 + 3x + 1 \\ q(x) &= 8x^5 + 2x^3 + x^2 + x + 9 \end{aligned}$$

Describe one way of interpreting $p(10)$, $q(10)$ and $p(10) \cdot q(10)$. What does this suggest to you as a reason why being able to compute the product of two polynomials quickly may be **computationally** useful?

Answers

a. Degree $p(x) \cdot q(x)$ is the result of adding the two degrees, i.e. $k + r$. The coefficient c_k of x^k in $p(x)$ and the coefficient d_r of x^r in $q(x)$ will be multiplied when construction $p(x) \cdot q(x)$, so giving $(c_k d_r)$ as the coefficient of x^{k+r}. There cannot be any larger power of x in $p(x) \cdot q(x)$ since this would imply that the degree of $p(x) > k$ or degree of $q(x) > r$. Also neither c_k nor d_r are zero.

b. All $k + r + 1$ coefficients are greater than 0.

c. Considering $p(x) = x - 1$ and $q(x) = x + 1$ the outcome $p(x) \cdot q(x)$ is $x^2 - 1$: a degree 2 polynomial but with only two non-zero coefficents, $c_0 = -1$, $c_1 = 0$ and $c_2 = 1$.

d. $p(10) = 739531$, $q(10) = 802119$. The result of $p(10) \cdot q(10)$ is the arithmetic outcome $739531 * 802119$. There is a close link between "computing the coefficients of the result of multiplying two polynomials" and "compute the outcome of multiplying two numbers" (in a given base). Fast methods for the former can be adapted to given fast methods for the latter.

Question 2 - Manipulating Polynomials
An extremely important operation involving polynomials (and, as we shall later in the course, functions in general) is that of finding their **roots**. If the degree of $p(x)$ is "small" (4 or less) then there are various direct methods than can be used. For example for so-called quadratics $p(x) = ax^2 + bx + c$ the two roots are found by computing,

$$\left\{ \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right\}$$

(which will always find answers, provided that $b^2 \geq 4ac$).

Similar formulae can be written for the three roots of a cubic: $ax^3 + bx^2 + cx + d$ (see pages 44-45 of the recommended course textbook) and even for the four roots of a quartic: $ax^4 + bx^3 + cx^2 + d$ (you do not want to go there, this solution is notationally horrendous).

No such direct "closed-form" can be used when the degree is five or more. This is more than "no such solution has been *discovered*" it means "no such solution is *possible*".

In applications, however, one often finds polynomial expressions for which the degree is much larger than five and (as will be seen when considering the important CS area called "Optimization" and, at the end of the course, when discussing the notion of "eigenvalue") finding roots of such polynomials is required.

Suppose that $p(x)$ is a polynomial whose degree is k with $k \geq 3$. Suppose, further, that we can find **two distinct** Real values, a and b, for which $p(a) < 0$ (i.e. evaluating $p(x)$ when $x = a$ produces a *negative* outcome) and $p(b) > 0$, (i.e. evaluating $p(x)$ when $x = b$ produces a *positive* outcome).

a. What does this allow us to deduce about **one** root of $p(x)$?

b. Describe (using pseudo-code) how the fact that $p(a) < 0$ and $p(b) > 0$ can be used in an **algorithm** to find either an **exact** root of $p(x)$ (that is a value c for which $p(c) = 0$) or, at worst an arbitrarily good **approximation** to an exact root (that is, a value c for which, although we may have $p(c) \neq 0$ we can make $|p(c)|$ as "small as desired", e.g. $< 10^{-5}$ or $< 10^{-30}$ etc.)

c. Having found (or found a "good enough approximation to") one root, c, of a degree k polynomial, $p(x)$, how could we use this information to find other (Real) roots of $p(x)$? [**Hint:** Look at pages 33-37 of the recommended course textbook.]

d. There are (at least) **three** difficulties that might arise when trying to find other roots of $p(x)$. Describe (in informal terms) **two** such difficulties.

e. In principle, the thinking underlying part (a) could be applied to find zeros (as they are called) of arbitrary Real valued functions, e.g. $f(x) = \sin^2(x) - 2\cos(x)$. Why might it be the case that such an approach would fail despite having found a and b with $f(a) < 0$ and $f(b) > 0$?
[**Hint:** Consider the function $f(x) = x - 1/x$ with choices $a = 0.5$ and $b = -0.5$ as well as the choice $a = 0.5$ and $b = 1.5$.]

Answers

a. The functional behaviour of polynomials indicates that should $p(a) < 0$ and $p(b) > 0$ then **at some point** x between a and b $p(x) = 0$. Graphically we can justify this claim by observing that $p(a) < 0$ means that $(a, p(a))$ is in the negative half of the y-axis and $p(b) > 0$ that $(b, p(b))$ is in the positive half. Moving "from a to b" the polynomial changes in value from "strictly negative" to "strictly positive" and so at som stage must be neither, i.e. equal zero.

b. Use a "binary search" approach: given (a, b) and $p(a) < 0$, $p(b) > 0$ compute $m = (a + b)/2$ and $p(m)$. If $p(m) < 0$ repreat the process but with the pair (m, b); if $p(m) > 0$ repeat the process but using (a, m). Obviously if $p(m) = 0$ then we have found a root m. Even if the only roots are irrational going through the same method repeatedly gets a more and more accurate estimate, i.e. this "converges".

c. If c is one root of the degree k polynomial $p(x)$ then the other $k - 1$ are found by computing the roots of the polynomial $p(x)/(x - c)$. This will be a polyomial of degree $k - 1$.

d. The method relies on two starting values. Although 0 gives one such value depending on c_0 we then have to identify another value with opposite sign, i.e. $c_0 < 0$ need b with $p(b) > 0$, $c_0 > 0$ need a with $p(a) < 0$. Such a choice may be difficult. Secondly, how quickly the method runs depends on the initial (a, b) if these are "badly" chosen the method may take longer to reach a conclusion. Finally there may be no other roots one can find. For example consider $p(x) = x^3 - 1$: using $(0, 2)$ we get $p(0) = -1$, $p(2) = 7$ and $m = (0 + 2)/2 = 1$ with $p(1) = 0$, what, however, is $(x^3 - 1)/(x - 1)$? It's the polynomial $x^2 + x + 1$ which has no "real" roots.

e. In the example $f(x) = x - 1/x$ choosing $m = (-0.5 + 0.5)/2$ gives $m = 0$ so that evaluating $m - 1/m$ is problematic. On the other hand $m = (0.5 + 1.5)/2$ gives $m = 1$ with $1 - 1/1 = 0$. One choice leads to a zero the other causes an error.