COMP116 – Work Sheet Two – Solutions

Associated Module Learning Outcome

> Ability to solve problems involving the outcome of matrix-vector products as might arise in standard transformations

Vectors & Matrices

Q1: There are a number of ways of defining the "size" of an n-vector. These include, often regarded as the standard approach, what is called the **Euclidean** distance (also known as the L_2-norm) which is denoted $\|\mathbf{x}\|$ and, as we shall use, $\|\mathbf{x}\|_2$. For the n-vector, $\mathbf{x} =< x_1, x_2, \ldots, x_n >$, as described in the lectures (and on page 67 of the course textbook)

$$\|< x_1, x_2, \ldots, x_n >\|_2 \;=\; \sqrt{\sum_{k=1}^{n} |x_k|^2}$$

The *positive* square root being used. Although, strictly speaking the $|x_i|$ qualification is not needed as the operation of squaring renders this redundant, i.e $(-x_i)^2 = (x_i)^2 = |x_i|^2$, it is, however, useful to include this as it helps with describing generalizations.

Another widely used, in CS contexts, definition of "vector size" is the so-called *Manhattan distance* (also called the L_1-norm) which is denoted $\|\mathbf{x}\|_1$. This often arises in robot-motion control contexts in which the robot cannot turn about through arbitrary angles and is only able to move in one of four ways: continue in the direction being travelled; turn around and move in the opposite direction that had been used; turn 90° right; turn 90° left.

The Manhattan distance of the n-vector $\mathbf{x} =< x_1, x_2, \ldots, x_n >$ is

$$\|< x_1, x_2, \ldots, x_n >\|_1 \;=\; \sum_{k=1}^{n} |x_k|$$

One way of thinking of this is as moving in an n-dimensional grid with movement limited to being able to reach an immediately adjacent, i.e. connected by a "grid line" point from the current point.

The first collection of questions concerns differences between these two measures of distance.

a. Suppose that we have 4-vectors, $\mathbf{x} =< -2, 3, -5, 8 >$ and $\mathbf{y} =< 5, -3, 7, 10 >$. What is the value of the following quantities?

1. $\|\mathbf{x}\|_2$.
2. $\|\mathbf{x}\|_1$.
3. $\|\mathbf{y}\|_2$.
4. $\|\mathbf{y}\|_1$.

Answers:

1. $\|\mathbf{x}\|_2 =$

$$\sqrt{|-2|^2 + |3|^2 + |-5|^2 + |8|^2} \;=\; \sqrt{4 + 9 + 25 + 64} \;=\; \sqrt{102}$$

2. $\|\mathbf{x}\|_1 =$

$$|-2| + |3| + |-5| + |8| \;=\; 2 + 3 + 5 + 8 \;=\; 18$$

3. $\|\mathbf{y}\|_2 =$

$$\sqrt{|5|^2 + |-3|^2 + |7|^2 + |10|^2} \;=\; \sqrt{25 + 9 + 49 + 100} \;=\; \sqrt{183}$$

4. $\|\mathbf{y}\|_1 =$

$$|5| + |-3| + |7| + |10| \;=\; 5 + 3 + 7 + 10 \;=\; 25$$

b. Similarly what are the values of the following quantities?

1. $\|\mathbf{x}\|_2 + \|\mathbf{y}\|_2$.
2. $\|\mathbf{x}\|_1 + \|\mathbf{y}\|_1$.
3. $\|\mathbf{x} + \mathbf{y}\|_2$.
4. $\|\mathbf{x} + \mathbf{y}\|_1$.

Answers

1. $\|\mathbf{x}\|_2 + \|\mathbf{y}\|_2 = \sqrt{102} + \sqrt{183}$
2. $\|\mathbf{x}\|_1 + \|\mathbf{y}\|_1 = 18 + 25 = 43$
3. $\|\mathbf{x} + \mathbf{y}\|_2 = \|< 3, 0, 2, 18 >\|_2 =$

$$\sqrt{9 + 0 + 4 + 324} = \sqrt{337}$$

4. $\|\mathbf{x} + \mathbf{y}\|_1 = \|< 3, 0, 2, 18 >\|_1 \;=\; 23$

c. What relationship is suggested by your answer to (b1) compared with your answer to (b3). Similarly when comparing your answer to (b2) with that of (b4)?
Answer: $\sqrt{102} + \sqrt{183} \sim 23.63$ but $\sqrt{337} \sim 18.36$. In $\|\cdots\|_2$ traversing $\mathbf{x}$ then $\mathbf{y}$ is a longer distance than going directly from the starting point of $\mathbf{x}$ to where the endpoint after adding $\mathbf{y}$ would result. Similarly, with $\|\cdots\|_1$: $43 \; > 23$.

d. Is the behaviour you notice in answering (c) indicative of a more general relationship between $\|\mathbf{x}\|_2 + \|\mathbf{y}\|_2$ and $\|\mathbf{x}+\mathbf{y}\|_2$. Similarly between $\|\mathbf{x}\|_1 + \|\mathbf{y}\|_1$ and $\|\mathbf{x}+\mathbf{y}\|_1$.
Answer: These relationships describe the so-called **Triangle Inequality** which (in standard arithmetic) indicates that what is called Euclidean length will satisfy $Length(\mathbf{x}+\mathbf{y}) \leq Length(\mathbf{x}) + Length(\mathbf{y})$. Equality will hold if and only if $\mathbf{x}$ and $\mathbf{y}$ are "on the same line".

e. What does your answer to (d) allow you to deduce (if anything) about combining **three** or more 4-vectors, i.e. $\|\mathbf{x}\|_2$, $\|\mathbf{y}\|_2$, $\|\mathbf{z}\|_2$ and $\|\mathbf{x}+\mathbf{y}+\mathbf{z}\|_2$? Is the same true if the Manhattan distance ($\|\mathbf{x}\|_1$) is used?
Answer: In general for n-vectors we have

$$Length\left(\sum_{i=1}^{k} \mathbf{x}_i\right) \quad \leq \quad \sum_{i=1}^{k} Length(\mathbf{x}_i)$$

f. Suppose one tries to compute $\|\mathbf{x}\|_1$ for the 3-vector $\mathbf{x} =< x_1, x_2, x_3 >$ using a **matrix-vector** product $\mathbf{M} \cdot \mathbf{x}^\top$ with $\mathbf{M}$ the 1×3 matrix

$$\mathbf{M} \;=\; (1,\ 1,\ 1)$$

What important feature of the Manhattan distance is such an attempt **failing** to consider?
Answer: This approach fails to recognise that $|x|$ must be used, whereas $(1,\ 1,\ 1)$ just implements simple addition.
If $\mathbf{M}$ was replaced by the 8×3 matrix

$$\mathbf{P} \;=\; \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

How might $\mathbf{P} \cdot \mathbf{x}^\top$ (which is an 8×1-matrix) be used to find $\|\mathbf{x}\|_1$?

Answer: By selecting the maximum component in the outcome.

2-D Graphics Transformations and Matrices

Q2 This question deals with a very basic 2-D vector-graphic game that involves moving a circular puck around a playing area 100 units wide and 100 units high. The general configuration is shown in Figure 1.

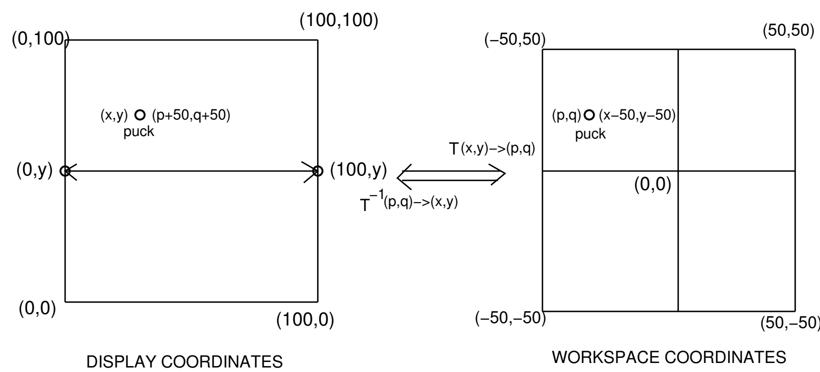


Figure 1: Simple Game Environment

Standard reference systems for graphical images will often use x (Horizonal/width) values between 0 and some maximum W and, similarly y (Vertical/height) values between 0 and some maximum H.

This can, however, prove rather awkward as a framework for realizing effects (for example if integers less than 0 could result). In such cases an animation effect might be thought of as carried out in three stages:

S1. Map the coordinate (x, y) to be manipulated within the display system to a coordinate (p, q) in another coordinate reference scheme.

S2. Compute the coordinates (p', q') resulting by applying some transformation, $\mathbf{M}$ to (p, q).

S3. Map the coordinate (p', q') back to the corresponding coordinate (x', y') in the display system.

In Figure 1 effects in the Display System are realized by mapping to the coordinate system in which x values range between -50 and 50; similarly y values between -50 and 50.

a. What are the **two** 3×3-matrices, $\mathbf{M}_{d-to-w}$ and $\mathbf{M}_{w-to-d}$ that should be used to translate the homogenous coordinate $< x, y, 1 >$ in the Display setting to the homogenous coordinate $< x - 50, y - 50, 1 >$ in Workspace system and *vice-versa*? That is for which

$$\begin{array}{rcl} \mathbf{M}_{d-to-w} \cdot < x, y, 1 >^\top & = & < x - 50, y - 50, 1 >^\top \\ \mathbf{M}_{w-to-d} \cdot < p, q, 1 >^\top & = & < p + 50, q + 50, 1 >^\top \end{array}$$

Answer:

$$\mathbf{M}_{d-to-w} = \begin{pmatrix} 1 & 0 & -50 \\ 0 & 1 & -50 \\ 0 & 0 & 1 \end{pmatrix} \quad ; \quad \mathbf{M}_{w-to-d} = \begin{pmatrix} 1 & 0 & 50 \\ 0 & 1 & 50 \\ 0 & 0 & 1 \end{pmatrix}$$

b. Consider the point $(0, y)$ (Display coordinates). Describe the animation effect (on the Display) of applying

$$\mathbf{M}_{w-to-d} \cdot \mathbf{F}_1 \cdot \mathbf{M}_{d-to-w} \cdot < 0, y, 1 >^\top$$

where $\mathbf{F}_1$ is the 3×3 matrix,

$$\mathbf{F}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 100 \\ 0 & 0 & 1 \end{pmatrix}$$

(for instance the effect could be displayed by drawing a line on the display joining (x, y) to the new position).

Answer:
Applying $\mathbf{M}_{d-to-w}$ to the **display** coordinate, $(0, y, 1)$ will produce the **workspace** coordinate $(-50, y-50, 1)$. The result of applying $\mathbf{F}_1$ to $(-50, y-50, 1)$ is the workspace coordinate $(y - 50, 50, 1)$ which then is translated back to the display coordinate (through $\mathbf{M}_{w-to-d}$) to give $(y, 100, 1)$. Displaying this as a line will connect $(0, y, 1)$ to $(y, 100, 1)$, i.e. the "old" y values becomes the "new" x coordinate and the new y-value becomes the maximum possible (whereas the previous X value in $(0, y, 1)$ had been the minimal possible x value, i.e. the line drawn would go from the extreme left of the display to the top of the drawing area).

c. Now suppose we introduce 2 new 3×3 matrices $\mathbf{F}_2$ and $\mathbf{F}_3$ with

$$\mathbf{F}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ; \quad \mathbf{F}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -100 \\ 0 & 0 & 1 \end{pmatrix}$$

Let

$$\begin{aligned}
\mathbf{T}_1 &= \mathbf{M}_{w-to-d} \cdot \mathbf{F}_1 \cdot \mathbf{M}_{d-to-w} \\
\mathbf{T}_2 &= \mathbf{M}_{w-to-d} \cdot \mathbf{F}_2 \cdot \mathbf{M}_{d-to-w} \\
\mathbf{T}_3 &= \mathbf{M}_{w-to-d} \cdot \mathbf{F}_3 \cdot \mathbf{M}_{d-to-w}
\end{aligned}$$

1. Describe the effect on $(0, y, 1)$ displayed after animating

$$\begin{array}{ll}
\mathbf{T}_1 \cdot < 0, y, 1 >^\top & \text{then} \\
\mathbf{T}_2 \cdot \mathbf{T}_1 \cdot < 0, y, 1 >^\top & \text{then} \\
\mathbf{T}_3 \cdot \mathbf{T}_2 \cdot \mathbf{T}_1 \cdot < 0, y, 1 >^\top & \text{then} \\
\mathbf{T}_2 \cdot \mathbf{T}_3 \cdot \mathbf{T}_2 \cdot \mathbf{T}_1 \cdot < 0, y, 1 >^\top &
\end{array}$$

   Answer:
   We have already seen that $\mathbf{T}_1$ takes the point $(0, y, 1)$ to $(y, 100, 1)$. Applying $\mathbf{T}_2$ to $(y, 100, 1)$ first takes this display coordinate and maps it to the workspace coordinate $(y - 50, 50, 1)$. The effect of $\mathbf{F}_2$ on $(y - 50, 50, 1)$ is to produce the workspace coordinate $(50, y - 50, 1)$ which (after mapping map to the display coordinates) becomes $(100, y, 1)$. This is the point at the same height as the original coordinate, $(0, y, 1)$ but now at the extreme right of the display. In total $\mathbf{T}_1$ followed by $\mathbf{T}_2$ traces the perimeter of (half) a quadrilateral extending from the starting point, $(0, y, 1)$ to a point $(y, 100, 1)$ and then to the “opposite wall” at $(100, y, 1)$. When $\mathbf{T}_3$ is applied to $(100, y, 1)$ (in the display system) we get $(50, y - 50, 1)$ in the workspace coordinate scheme, a coordinate which translates to $(y - 50, -50, 1)$ (worskpace) then to $(y, 0, 1)$ (display). This point is directly opposite $(y, 100, 1)$ so tracing the line produced gives a third side of the four-sided figure. Finally applying $\mathbf{T}_2$ to $(y, 0, 1)$ gives $(y - 50, -50, 1)$ (workspace) then $(-50, y - 50, 1)$ (workspace from $\mathbf{F}_2$) and $(0, y, 1)$ (display), i.e. the start point.

   Overall the effect is to trace out a four-sided figure with corners $(0, y)$, $(y, 100)$, $(100, y)$ and $(y, 0)$ and connecting lines drawn in that order.

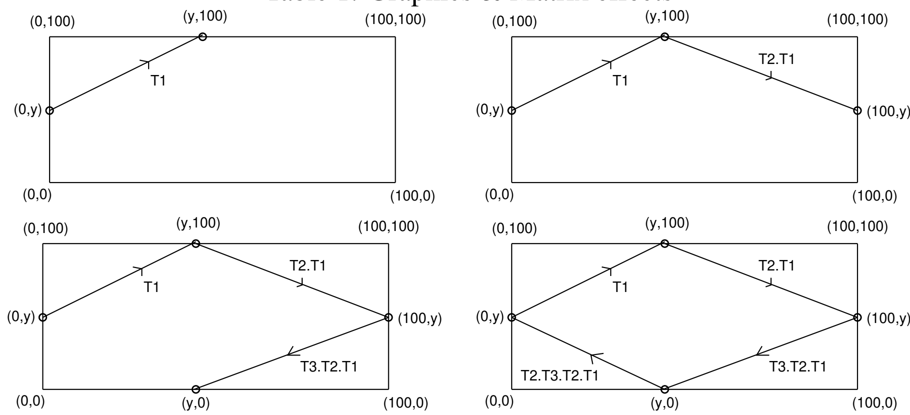
   The whole process is illustrated in Table 1.

2. If the ordering was

$$\begin{array}{ll}
\mathbf{T}_2 \cdot < 0, y, 1 >^\top & \text{then} \\
\mathbf{T}_3 \cdot \mathbf{T}_2 \cdot < 0, y, 1 >^\top & \text{then} \\
\mathbf{T}_2 \cdot \mathbf{T}_3 \cdot \mathbf{T}_2 \cdot < 0, y, 1 >^\top & \text{then} \\
\mathbf{T}_1 \cdot \mathbf{T}_2 \cdot \mathbf{T}_3 \cdot \mathbf{T}_2 \cdot < 0, y, 1 >^\top &
\end{array}$$

   What would be the resulting effect?
   Answer: The perimeter would be traced out anticlockwise (instead of clockwise). See Figure 2.

Table 1: Graphics & Matrix effects



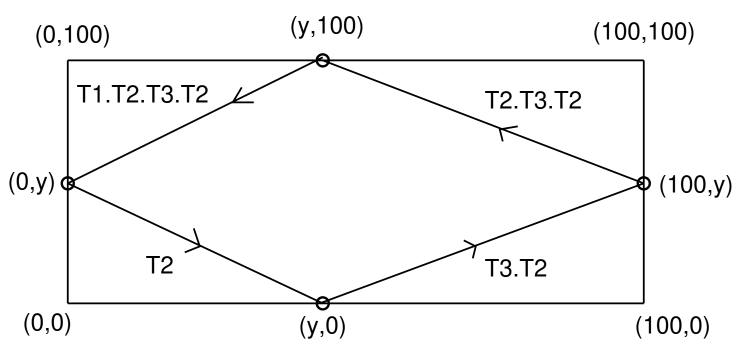


Figure 2: Applying transformations in Reverse order