COMP116 – Work Sheet One

Associated Module Learning Outcome

Basic understanding of the range of techniques used to analyse and reason about computational settings

Numbers and Polynomial Properties

The topics on this work sheet deal with properties of *types* of *number* and basic polynomial properties (Q1). More detailed review of this behaviour is the theme of (Q2).

Question 1

I. Consider the following situations involving **numerical data**. What **types** of number are most suitable to use for describing possible outcomes?

   Give brief, informal reasons for your answers.

   a. The number of correctly answered questions on an MCQ test (Multiple Choice Questions).
   b. The goal difference of the middle placed team in the Scottish Lowland League football division at the end of a season.
   c. The pay of *soi-disant* 'senior' managers relative to that of Workers and Academics in universities.
   d. The total number of points awarded to the UK entry in the Eurovision Song Contest in a particular year.
   e. The time a vehicle takes to reach a speed of $80Km/h$ after starting.
   f. The majority achieved when legislation is *successfully* passed in the House of Commons, i.e. without requiring the Speaker to separate outcomes by use of a casting vote.

II. You have seen on COMP109 and COMP124 that computers represent data as collections of **bits** (binary 0 and 1). What implications do you think this convention has for computing applications requiring **Real** numbers ($\mathbb{R}$)?

III. The following questions deal with some simple manipulation of polynomial expressions.

a. Suppose $p(x)$ has degree k and $q(x)$ has degree r. What is the degree of $p(x) \cdot q(x)$?

b. If all coefficients of $p(x)$ are *greater than* zero and all coefficients of $q(x)$ are greater than zero how many coefficients of $p(x) \cdot q(x)$ are non-zero?

c. If the condition on the coefficients of $p(x)$ and $q(x)$ is changed from *greater than zero* to *not equal to zero* is your answer to (b) unchanged? [**Hint:** Consider the two degree 1 polynomials $x - 1$ and $x + 1$ all of whose coefficients are non-zero.]

d. Suppose that

$$\begin{aligned} p(x) &= 7x^5 + 3x^4 + 9x^3 + 5x^2 + 3x + 1 \\ q(x) &= 8x^5 + 2x^3 + x^2 + x + 9 \end{aligned}$$

Describe one way of interpreting $p(10)$, $q(10)$ and $p(10) \cdot q(10)$. What does this suggest to you as a reason why being able to compute the product of two polynomials quickly may be **computationally** useful?

Question 2 - Manipulating Polynomials
An extremely important operation involving polynomials (and, as we shall later in the course, functions in general) is that of finding their **roots**. If the degree of $p(x)$ is "small" (4 or less) then there are various direct methods than can be used. For example for so-called quadratics $p(x) = ax^2 + bx + c$ the two roots are found by computing,

$$\left\{ \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right\}$$

(which will always find answers, provided that $b^2 \geq 4ac$).

Similar formulae can be written for the three roots of a cubic: $ax^3 + bx^2 + cx + d$ (see pages 44-45 of the recommended course textbook) and even for the four roots of a quartic: $ax^4 + bx^3 + cx^2 + d$ (you do not want to go there, this solution is notationally horrendous).

No such direct "closed-form" can be used when the degree is five or more. This is more than "no such solution has been *discovered*" it means "no such solution is *possible*".

In applications, however, one often finds polynomial expressions for which the degree is much larger than five and (as will be seen when considering the important CS area called "Optimization" and, at the end of the course, when discussing the notion of "eigenvalue") finding roots of such polynomials is required.

Suppose that $p(x)$ is a polynomial whose degree is k with $k \geq 3$. Suppose, further, that we can find **two distinct** Real values, a and b, for which $p(a) < 0$ (i.e. evaluating $p(x)$ when $x = a$ produces a *negative* outcome) and $p(b) > 0$, (i.e. evaluating $p(x)$ when $x = b$ produces a *positive* outcome).

a. What does this allow us to deduce about **one** root of $p(x)$?

b. Describe (using pseudo-code) how the fact that $p(a) < 0$ and $p(b) > 0$ can be used in an **algorithm** to find either an **exact** root of $p(x)$ (that is a value c for which $p(c) = 0$) or, at worst an arbitrarily good **approximation** to an exact root (that is, a value c for which, although we may have $p(c) \neq 0$ we can make $|p(c)|$ as "small as desired", e.g. $< 10^{-5}$ or $< 10^{-30}$ etc.)

c. Having found (or found a "good enough approximation to") one root, c, of a degree k polynomial, $p(x)$, how could we use this information to find other (Real) roots of $p(x)$? [**Hint:** Look at pages 33-37 of the recommended course textbook.]

d. There are (at least) **three** difficulties that might arise when trying to find other roots of $p(x)$. Describe (in informal terms) **two** such difficulties.

e. In principle, the thinking underlying part (a) could be applied to find zeros (as they are called) of arbitrary Real valued functions, e.g. $f(x) = \sin^2(x) - 2\cos(x)$. Why might it be the case that such an approach would fail despite having found a and b with $f(a) < 0$ and $f(b) > 0$?
[**Hint:** Consider the function $f(x) = x - 1/x$ with choices $a = 0.5$ and $b = -0.5$ as well as the choice $a = 0.5$ and $b = 1.5$.]

Two (of the many such that have been discovered) approaches to finding roots of polynomials one of which (Halley's Method) can, under suitable conditions, be adapted to find zeros of arbitrary functions; the other (Laguerre's Method) is specifically intended for polynomials, are discussed in Section 4.6, pages 152-161 of the course textbook.

The most efficient known polynomial root-finding techniques are the Jenkins-Traub algorithms from 1972 and 1975. These are technically quite complex.