On the Complexity of Rationalizing Behavior

Jose Apesteguia and Miguel A. Ballester

Universitat Pompeu Fabra and Universitat Autònoma de Barcelona

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INTRODUCTION

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- **Classic result**: Only *rational* choice can be rationalized as the maximization process of an ordering.

- But what if rationality does not hold?
  - To consider a wider notion of rationalization, by relaxing the way in which the choice function is explained.
  - Rationalization by multiple rationales (Kalai, Rubinstein, and Spiegler 2002; KRS): behavior is rationalized through a collection of linear orders. For every choice problem there is a linear order that rationalizes it.
  - It is as if the DM had in mind a partition of the set of choice problems, and applies one rationale to each element of the partition.
Definition (CC, CF)

Given a set of elements $X$ and a domain $\mathcal{D} \subseteq \mathcal{U}$, a map $c : \mathcal{D} \rightarrow \mathcal{U}$ is a choice correspondence if for every $A \in \mathcal{D}$, $c(A) \subseteq A$. If for every $A \in \mathcal{D}$, $c(A)$ is a singleton, we say that $c$ is a choice function.
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Definition (RMR)

A $K$-tuple of complete preorders $(\succ_k)_{k=1,...,K}$ on $X$ is a rationalization by multiple rationales (RMR) of choice correspondence $c$ if for every $A \in \mathcal{D}$, the set of elements $c(A)$ is $\succ_k$-maximal in $A$ for some $k$. 
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\[
\begin{array}{c|c}
\succ_1 & \succ_2 \\
\hline
1 & 2 \\
2 & 1 \\
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\end{array}
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There are multiple books of rationales that can rationalize a given choice behavior. KRS propose to focus on those that use the minimal number of rationales.
OUR AIMS

Drawing on the tools of theoretical computer science, we study the question of how complex it is to find the preference relations that rationalize choice behavior. Unless stated, results apply both to CC and CF.
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- Our basic result shows that in the general case, finding a minimal book is a difficult computational problem.
- Now, the question arises whether it is the conjunction of (i) unstructured choice behavior and (ii) unrestricted choice domain that leads to the computational hardness of the problem of rationalization.
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- Restriction of choice behavior. The choice correspondence satisfies the well-known consistency property known as the *weak axiom of revealed preference* (WARP). In other words, the minimal number of rationales is 1 with certainty. The problem is polynomial.
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This is especially useful since there is a wealth of algorithms for graph problems that may be used to solve the problem of rationalization of certain choice structures.
Rationalization of any $c$ by Linear Orders in $\mathcal{D}$ (RLO-$\mathcal{D}$): Given a choice function $c$ on $\mathcal{D}$, can we find $k \leq K$ linear orders that constitute a rationalization by multiple rationales of $c$?
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**Theorem**

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Sketch of Proof of Theorem
We use the proof-by-reduction technique to prove that the problem is NP-complete. That is, we show that it contains a known NP-complete problem as a special case.
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Rationalization of any $c$ by Linear Orders in $D$ (RLO-$D$):
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Partition into Cliques (PIC): Given a graph $G = (V, E)$, can the vertices of $G$ be partitioned into $k \leq K$ disjoint sets $V_1, V_2, \ldots, V_k$ such that for $1 \leq i \leq k$ the subgraph induced by $V_i$ is a complete graph?
**c-Maximal Sets:** A subset $S \in \mathcal{D}$ is said to be $c$-maximal if for all $T \in \mathcal{D}$, with $S \subset T$, it is the case that $c(S) \neq c(T)$. Denote the family of $c$-maximal sets under the choice domain $\mathcal{D}$ by $M^\mathcal{D}_c$. 

**Weak Axiom of Revealed Preference (WARP):** Let $A, B \in \mathcal{D}$ and assume $x, y \in A \cap B$; if $x = c(A)$ then $y \neq c(B)$. 

**Theorem:** Let the choice function $c$ be a rational procedure on $\mathcal{D}$. Then $|M^\mathcal{D}_c| \leq |X| - 1$ and the problem of finding the linear order $\succ$ that rationalizes $c$ is polynomial.
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RESTRICTION OF CHOICE DOMAIN

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This result remains an open question for the case of Choice Correspondences (RCP-$\mathcal{U}$).
Definition
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**Cycle:** The collection $\{A_t\}_{t=1}^n \in M^D_c$, $n \geq 2$, is a cycle if $A_1 = A_n$ and for every $i \in \{1, \ldots, n-1\}$, $A_i \rightarrow A_{i+1}$. 
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**Partition into DAGs** A partition of \( M^D_c \) \( \{V_p\}_{p=1}^P \) is said to be a Partition into DAGs if every class \( V_p \) is a DAG, i.e., it admits no cycle. It is said to be minimal if any other Partition into DAGs has at least \( P \) classes.
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- If $\{\succeq_p\}_{p=1,\ldots,P}$ is a minimal RMR, then there is a minimal Partition into DAGs $\{V_p\}_{p=1,\ldots,P}$ of $M^D_c$ where all the choice problems explained by any rationale are grouped together.

- If $\{V_p\}_{p=1,\ldots,P}$ is a minimal Partition into DAGs of $M^D_c$, then there is a minimal RMR $\{\preceq_p\}_{p=1,\ldots,P}$ where all the choice problems in the same equivalence class are explained by the same rationale.
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Conclusions

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- In the choice correspondences case, it may well be the case that the difficulty in finding a minimal book is triggered by choice behavior per se.

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