COMP108
Algorithmic Foundations

Mathematical Induction

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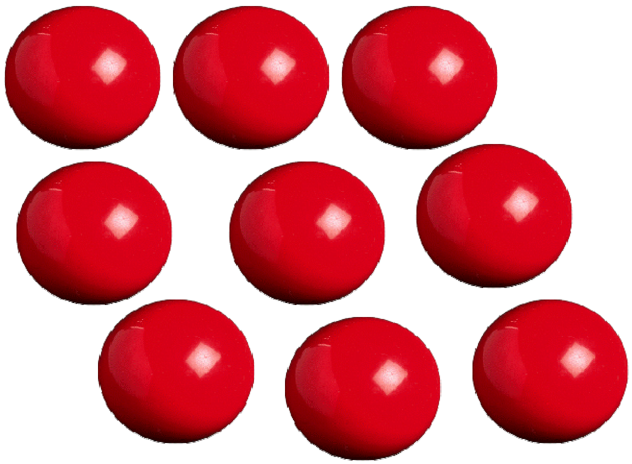
Which Ball is Heavier?

balance scale



9 balls look identically the same
but 1 is heavier than the rest

How to find the heavier one
by weighing 2 times only?

Learning outcomes

- Understand the concept of Induction
- Able to prove by Induction

Analysis of Algorithms

After designing an algorithm, we analyze it.

- Proof of correctness: show that the algorithm gives the desired result
- Time complexity analysis: find out how fast the algorithm runs
- Space complexity analysis: find out how much memory space the algorithm requires
- Look for improvement: can we improve the algorithm to run faster or use less memory? is it best possible?

A typical analysis technique

Induction

- technique to prove that a property holds for all natural numbers (or for all members of an infinite sequence)

$\forall$ **for all**

E.g., To prove 1+2+…+n = n(n+1)/2 $\forall$ +ve integers n

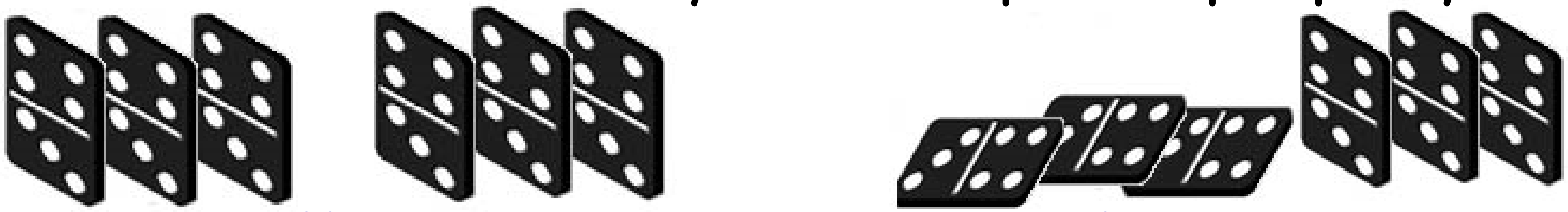
n	LHS	RHS	LHS = RHS?
1	1	1*2/2 = 1	
2	1+2 = 3	2*3/2 = 3	
3	1+2+3 = 6	3*4/2 = 6	

However, this isn't a proof and we cannot enumerate over all possible numbers.

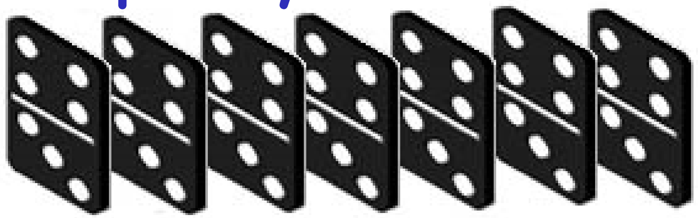
$\Rightarrow$ Induction

Intuition – Long Row of Dominoes

- How can we be sure each domino will fall?
- Enough to ensure the 1^{st} domino will fall?
  - No. Two dominoes somewhere may not be spaced properly



- Enough to ensure all are spaced properly?
  - No. We need the 1^{st} to fall



- **Both** conditions required:
  - 1^{st} will fall; & after the k^{th} fall, $k+1^{st}$ will also fall
  - then even infinitely long, all will fall

Induction

To prove that a property holds for every positive integer n

Two steps

- **Base case:** Prove that the property holds for n = 1
- **Induction step:** Prove that **if** the property holds for n = k (for some positive integer k), **then** the property holds for n = k + 1
- Conclusion: The property holds for every positive integer n

Example

To prove: $1+2+3+\ldots+n = \frac{n(n+1)}{2}$ ∀+ve integers n

- **Base case:** When **n=1**, LHS=1, RHS= $\frac{1\times 2}{2}$ = 1.
  So, the property holds for n=1.
  Left hand side
- **Induction hypothesis:**
  Assume that the property holds when n=k for some integer k≥1.
  - i.e., assume that $1+2+3+\ldots+k = \frac{k(k+1)}{2}$
- **Induction step:** When n=k+1,
  LHS becomes 1 + 2 + 3 + … + k + (k+1)
  RHS becomes $\frac{(k+1)((k+1)+1)}{2}$

∴ we want to prove

$$1+2+3+\ldots+k+(k+1) = \frac{(k+1)(k+2)}{2}$$

Induction hypothesis

$$1 + 2 + \ldots + k = \frac{k(k+1)}{2}$$

Target: to prove

$$1 + 2 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Induction Step: When n=k+1

- LHS = 1+2+...+k+(k+1)

$$= \frac{k(k+1)}{2} + (k+1) \quad \leftarrow \textbf{by hypothesis}$$

$$= (k+1)(\frac{k}{2} + 1)$$

$$= \frac{(k+1)(k+2)}{2}$$

= RHS

- So, property also holds for n=k+1
- Conclusion: property holds for all +ve integers n

Conclusion

We have proved

1. property holds for n=1
2. if property holds for n=k, then also holds for n=k+1

In other words,

- holds for n=1 implies holds for n=2 (**induction step**)
- holds for n=2 implies holds for n=3 (**induction step**)
- holds for n=3 implies holds for n=4 (**induction step**)
- and so on ......

By principle of induction: holds for all +ve integers n

Example 2

To prove **n^3+2n is divisible by 3** $\forall$ integers $n \geq 1$

n	n^3+2n	divisible by 3?
1	1+2 = 3	
2	8+4 = 12	
3	27+6 = 33	
4	64+8 = 72	

Prove it by induction…

Example 2

To prove n^3+2n is divisible by 3 $\forall$ integers $n \geq 1$

- **Base case:** When $n=1$, $n^3+2n=1+2=3$, divisible by 3. So property holds for $n=1$.
- **Induction hypothesis:** Assume property holds for $n=k$, for some +ve int k, i.e., **assume** k^3+2k is divisible by 3
- **Induction step:** When $n=k+1$, LHS becomes $(k+1)^3 + 2(k+1)$

Target: to prove $(k+1)^3+2(k+1)$ is divisible by 3

Induction hypothesis
k^3+2k is divisible by 3

Target: to prove
$(k+1)^3+2(k+1)$ is divisible by 3

- **Induction step:** When $n=k+1$,
  - $(k+1)^3+2(k+1) = (k^2+2k+1)(k+1) + (2k+2)$
    $= (k^3+3k^2+3k+1) + (2k+2)$
    $= (k^3+2k) + 3(k^2+k+1)$

by hypothesis, divisible by 3 — divisible by 3

sum is divisible by 3

- Property holds for $n=k+1$
- **By principle of induction: property holds $\forall$ integers $n \geq 1$**

Example 3

Factorial:
$n! = n(n-1)(n-2) \ldots *2*1$

To prove $2^n < n!$ $\forall$ +ve integers $n \geq 4$.

n	2^n	n!	LHS < RHS?
1	2	1	
2	4	2	
3	8	6	
4	16	24	
5	32	120	
6	64	720	

Prove it by induction…

Example 3

$n! = n(n-1)(n-2) \ldots *2*1$

To prove $2^n < n!$ $\forall$ +ve integers $n \geq 4$.

- **Base case:** When n=4,
  LHS = 2^4 = 16, RHS = 4! = 4*3*2*1 = 24,
  LHS < RHS
  So, property holds for n=4
- **Induction hypothesis:** Assume property holds for n=k for some integer $k \geq 4$, i.e., assume $2^k < k!$
- **Induction step:** When n=k+1, LHS becomes 2^{k+1}, RHS becomes (k+1)!

Target: to prove $2^{k+1} < (k+1)!$

Induction hypothesis
$2^k < k!$

Target: to prove
$2^{k+1} < (k+1)!$

- **Induction step:** When n=k+1,
  - LHS = $2^{k+1} = 2 * 2^k < 2 * k!$ ← by hypothesis, $2^k < k!$
  - RHS = $(k+1)! = (k+1) * k! > 2 * k! >$ LHS ← because $k+1 > 2$
  - So, property holds for n=k+1
- **By principle of induction:** property holds ∀ +ve integers $n \geq 4$

Example 3

Why base case is n=4?

When n=1, $2^1=2$, $1!=1$

When n=2, $2^2=4$, $2!=2$

When n=3, $2^3=8$, $3!=6$

Property does not hold for n=1, 2, 3

Challenges …

Exercise

To prove $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ $\forall$ +ve int n ≥ 1.

n	LHS	RHS	LHS = RHS?
1	1	1*2*3/6 = 1	
2	1+4 = 5	2*3*5/6 = 5	
3	1+4+9 = 14	3*4*7/6 = 14	
4	1+4+9+16 = 30	4*5*9/6 =30	
5	1+4+9+16+25 = 55	5*6*11/6 = 55	

Prove it by induction…

Exercise

To prove $$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

- **Base case:** when n=1, LHS = 1^2 = **1**, RHS = $\frac{1 \times 2 \times 3}{6}$ =**1**=LHS
- **Induction hypothesis:** Assume property holds for n=k
  - i.e., assume that $1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$
- **Induction step:** When n=k+1,

  LHS becomes $1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2$

  RHS becomes $\frac{(k+1)(k+2)(2k+3)}{6}$

  Target is to prove

  $$1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

  work on LHS & work on RHS and show that they reach the same expression

Induction hypothesis: $1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$

Target: to prove $1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Induction Step: When n = k+1

LHS $= \underbrace{1^2 + 2^2 + 3^2 + ... + k^2}_{} + (k+1)^2$

← **by hypothesis**

Exercise 2

Prove that $1+3+5+\ldots+(2n-1) = n^2$ $\forall$ +ve integers ≥ 1

(sum of the ***first n odd integers*** equals to n^2)

n	LHS	RHS	LHS = RHS?
1	1	$1^2 = 1$	🙂
2	1+3 = 4	$2^2 = 4$	🙂
3	1+3+5 = 9	$3^2 = 9$	🙂
4	1+3+5+7 = 16	$4^2 = 16$	🙂
5	1+3+5+7+9 = 25	$5^2 = 25$	🙂

Prove it by induction…

Exercise 2

Prove that $1+3+5+\ldots+(2n-1) = n^2$ $\forall$ +ve integers ≥ 1

- **Base case:** When n=1,
  LHS=2*1-1=**1**, RHS=1^2=**1**
- **Induction hypothesis:**
  Assume property holds for some integer k,
  i.e., assume $1+3+5+\ldots+(2k-1)=k^2$
- **Induction step:** When n=k+1,
  LHS becomes $1+3+5+\ldots+(2(k+1)-1)$
  RHS becomes $(k+1)^2$

Target: to prove
$1+3+5+\ldots+(2k-1)+(2(k+1)-1))=(k+1)^2$

Induction hypothesis: $1+3+5+\ldots+(2k-1) = k^2$

Target: to prove $1+3+5+\ldots+(2k-1)+(2(k+1)-1))=(k+1)^2$

- **Induction step:** When $n=k+1$,

  LHS $= 1+3+5+\ldots+(2k-1)+(2(k+1)-1)$

  RHS $= (k+1)^2 = k^2+2k+1 =$ LHS

Therefore, property holds for $n=k+1$

By principle of induction, property holds for all +ve integers

More advanced ...

Note

The **induction step** means that **if** property holds for some integer k, **then** it also holds for k+1.

It does **NOT** mean that the property must hold for k nor for k+1.

Therefore, we **MUST** prove that property holds for some starting integer n_0, which is the **base case**.

Missing the base case will make the proof fail.

What's wrong with this?

Claim: For all n, n=n+1

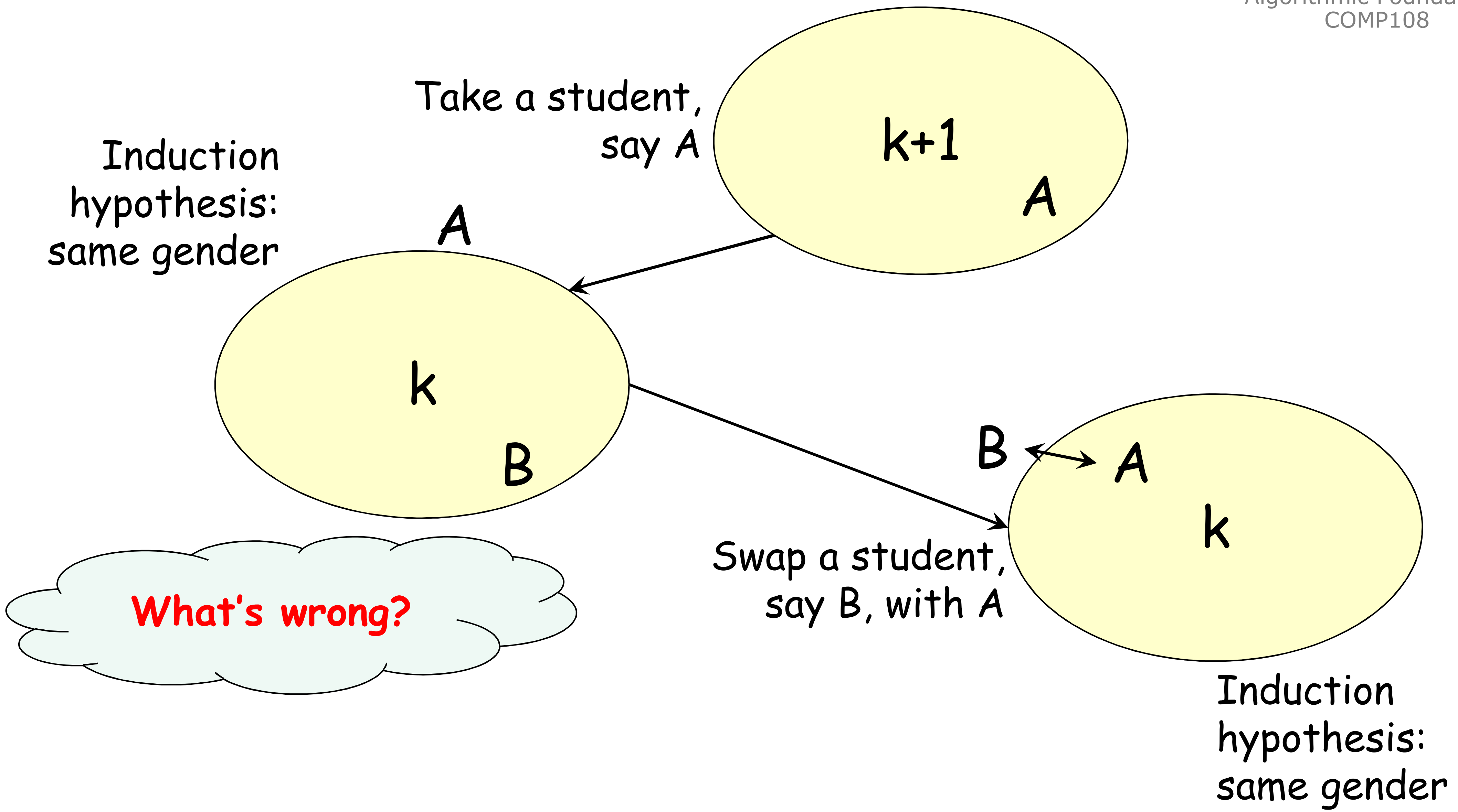
- Assume the property holds for n=k, i.e., assume k = k+1
- Induction Step:
  - Add 1 to both sides of the induction hypothesis
  - We get: **k+1 = (k+1)+1**, i.e., k+1 = k+2
- The property holds for n=k+1

BUT, we know this isn't true, what's wrong?

What about this?

Claim: All comp108 students are of the same gender

- **Base case:** Consider any group of **ONE** comp108 student. Same gender, of course.
- **Induction hypothesis:** Assume that any group of **k** comp108 students are of same gender
- **Induction step:** Consider any group of **k+1** comp108 students…



So, A, B & other (k-1) students
are of the same gender

Recall: Finding minimum

```
input: a[1], a[2], ..., a[n]
i = 1
M = a[1]
while (i <= n) do
begin
  if a[i] < M then        **
    M = a[i]              **
  i = i + 1
end
output M
```

Consider at the end of the statement **

Base case: When i=1, M is min(a[1])

Induction hypothesis: Assume the property holds when i=k for some k≥1.

Induction step: When i=k+1,

- **If a[k+1] < min(a[1],…,a[k]),** M is set to a[k+1], i.e., min(a[1],…,a[k+1]),
- **Else**, a[k+1] is not min, M is unchanged & M equals min(a[1],…,a[k+1])

Property: After each iteration of statement **, the value of M is min(a[1], …, a[i])