COMP108
Algorithmic Foundations

Greedy methods

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http://www.csc.liv.ac.uk/~pwong/teaching/comp108/201617

Coin Change Problem

Suppose we have 3 types of coins

10p

20p

50p

Minimum number of coins to make £0.8, £1.0, £1.4 ?

Greedy method

Learning outcomes

- Understand what greedy method is
- Able to apply Kruskal's algorithm to find minimum spanning tree
- Able to apply Dijkstra's algorithm to find single-source shortest-paths
- Able to apply greedy algorithm to find solution for Knapsack problem

Greedy methods

How to be greedy?

- At every step, make the best move you can make
- Keep going until you're done

Advantages

- Don't need to pay much effort at each step
- Usually finds a solution very **quickly**
- The solution found is usually **not bad**

Possible problem

- The solution found may **NOT** be the best one

Greedy methods - examples

Minimum spanning tree

- Kruskal's algorithm

Single-source shortest-paths

- Dijkstra's algorithm

Both algorithms find one of the BEST solutions

Knapsack problem

- greedy algorithm does NOT find the BEST solution

Kruskal’s algorithm …

Minimum Spanning tree (MST)

Given an undirected connected graph G

- The edges are labelled by weight

Spanning tree of G

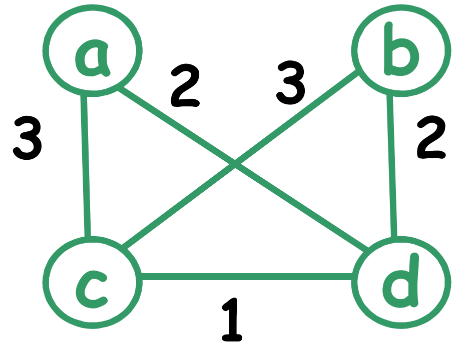
- a tree containing all vertices in G

Minimum spanning tree of G

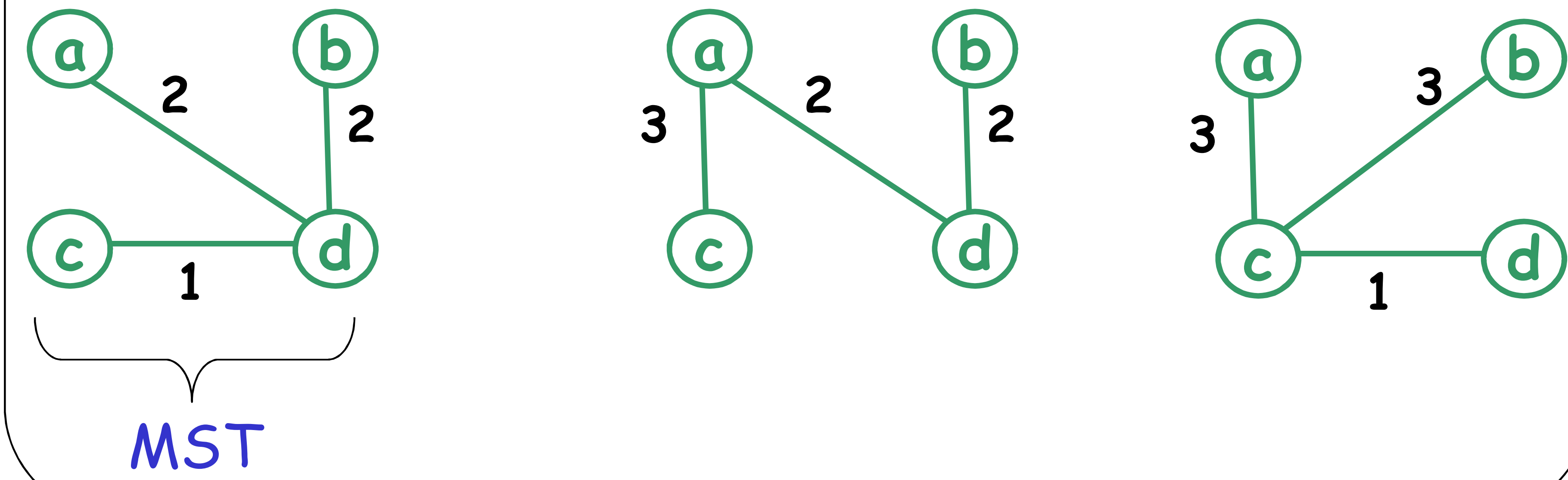
- a spanning tree of G with minimum weight

Examples

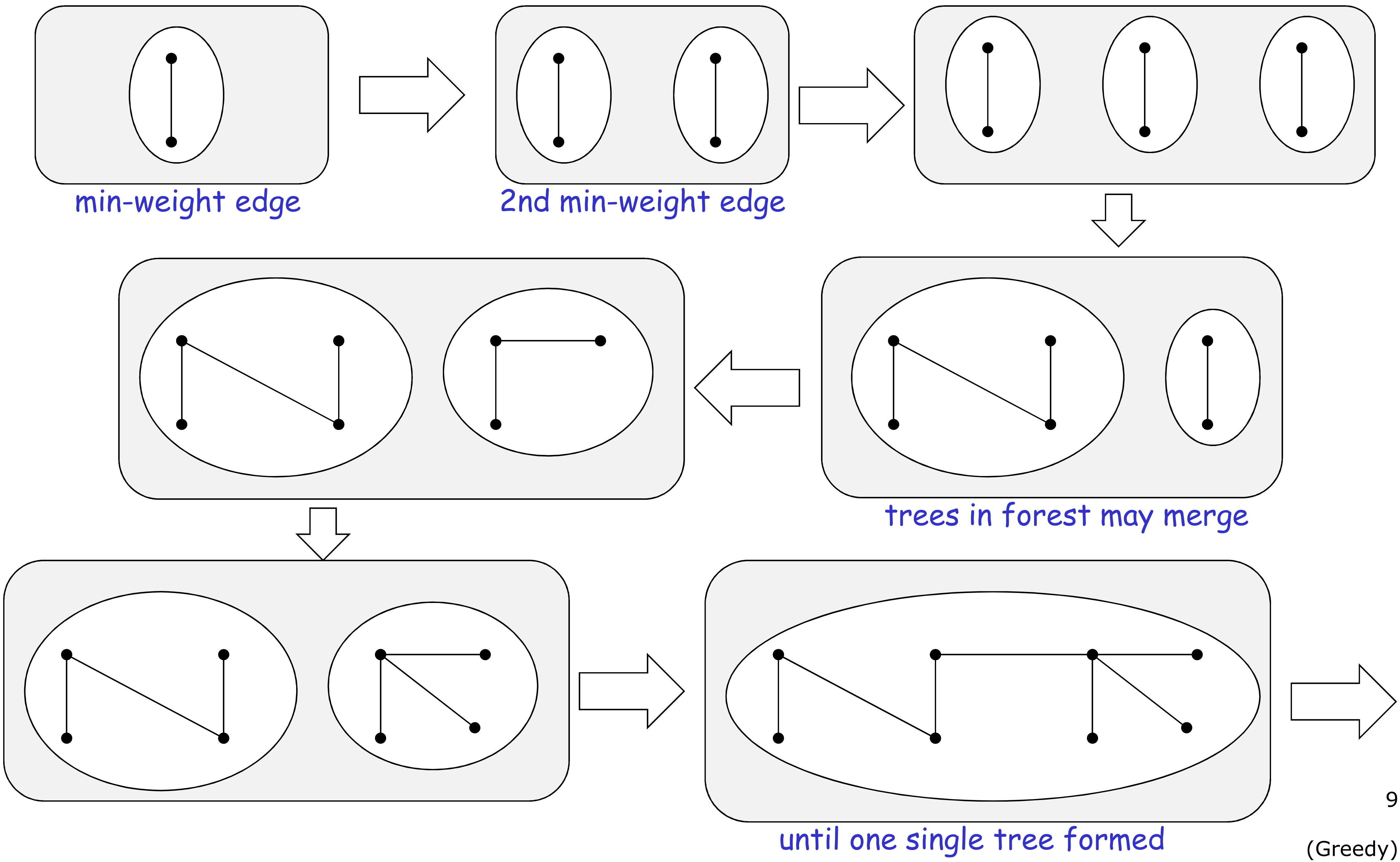
Graph G
(edge label is weight)



Spanning trees of G

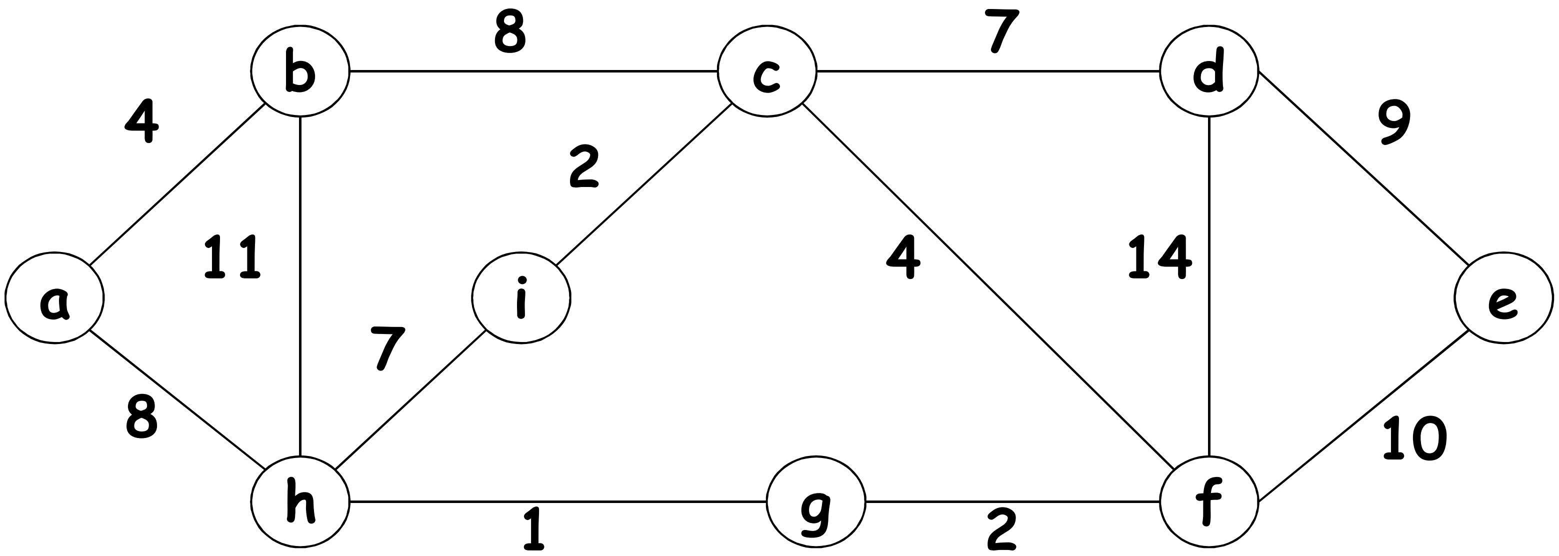


Idea of Kruskal's algorithm - MST



(Greedy)

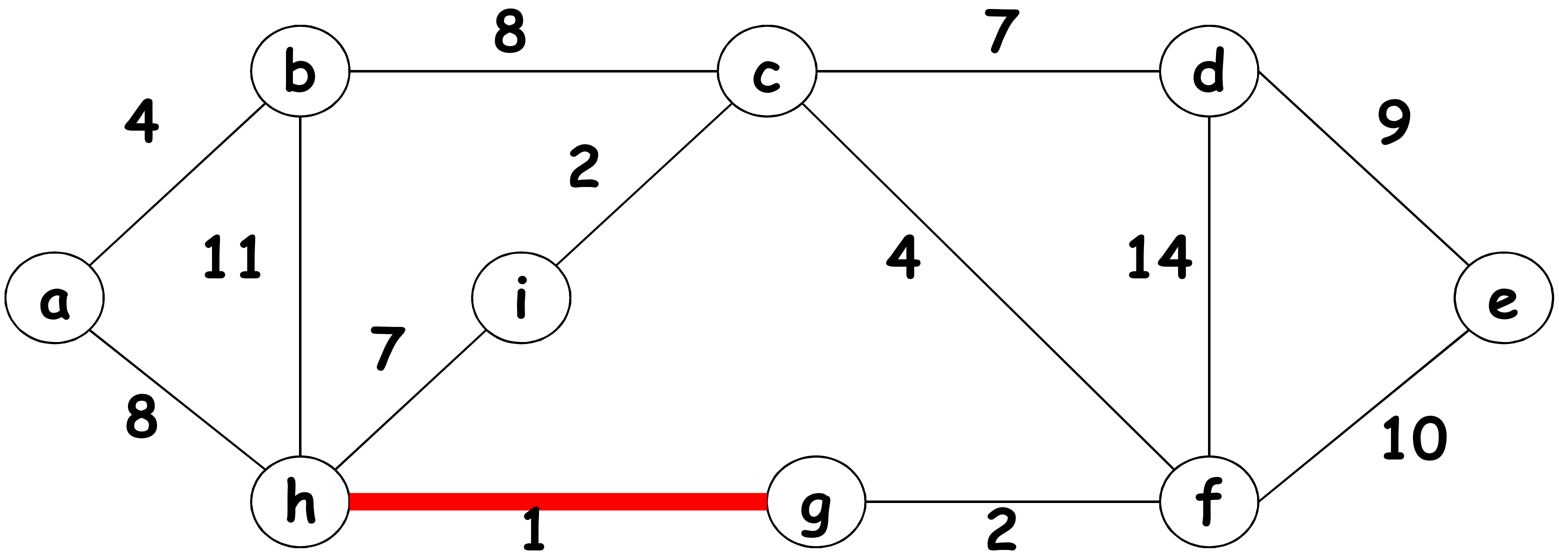
Kruskal's algorithm - MST



Arrange edges from smallest to largest weight

(h,g)	1
(i,c)	2
(g,f)	2
(a,b)	4
(c,f)	4
(c,d)	7
(h,i)	7
(b,c)	8
(a,h)	8
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

Kruskal's algorithm - MST

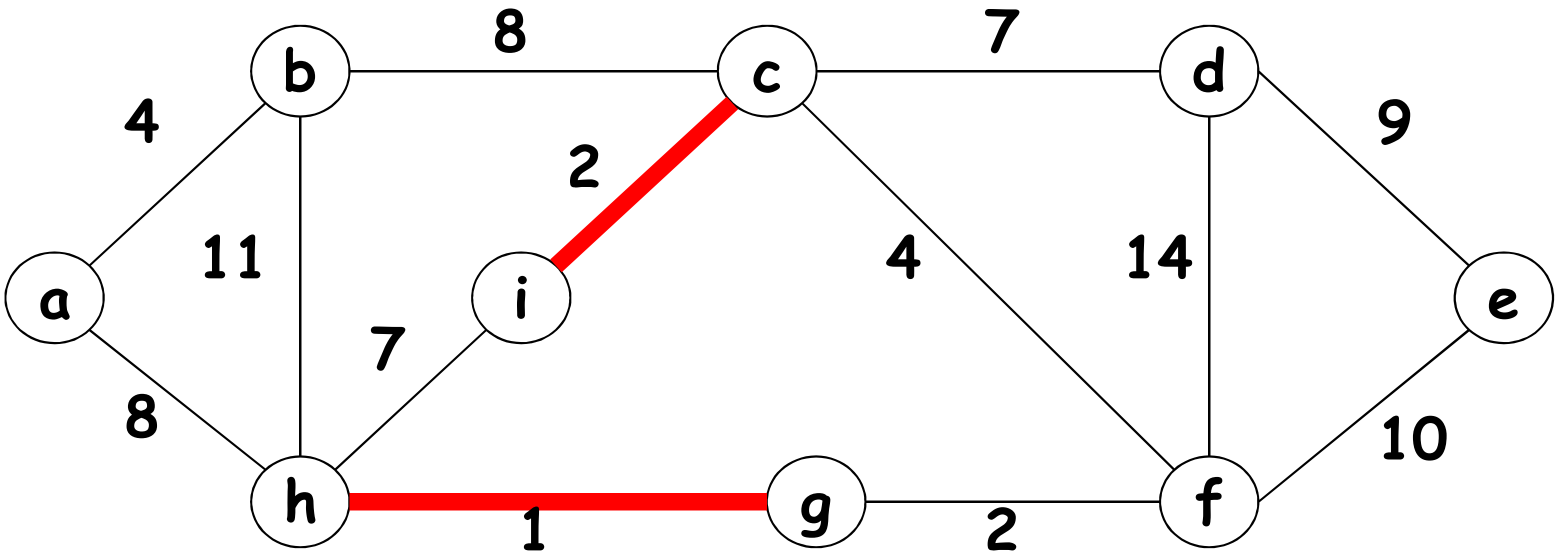


(h,g)	*1*
(i,c)	2
(g,f)	2
(a,b)	4
(c,f)	4
(c,d)	7
(h,i)	7
(b,c)	8
(a,h)	8
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

Choose the minimum weight edge

italic: chosen

Kruskal's algorithm - MST

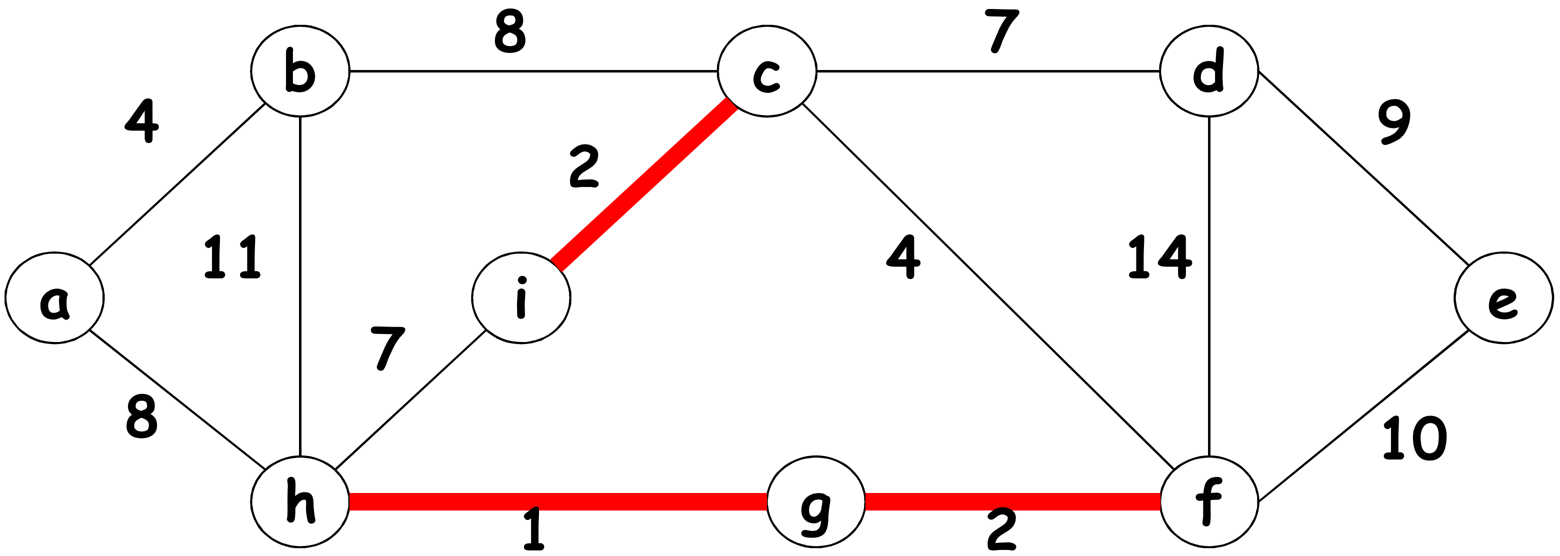


Choose the next minimum weight edge

(h,g)	*1*
(i,c)	*2*
(g,f)	2
(a,b)	4
(c,f)	4
(c,d)	7
(h,i)	7
(b,c)	8
(a,h)	8
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

italic: chosen

Kruskal's algorithm - MST

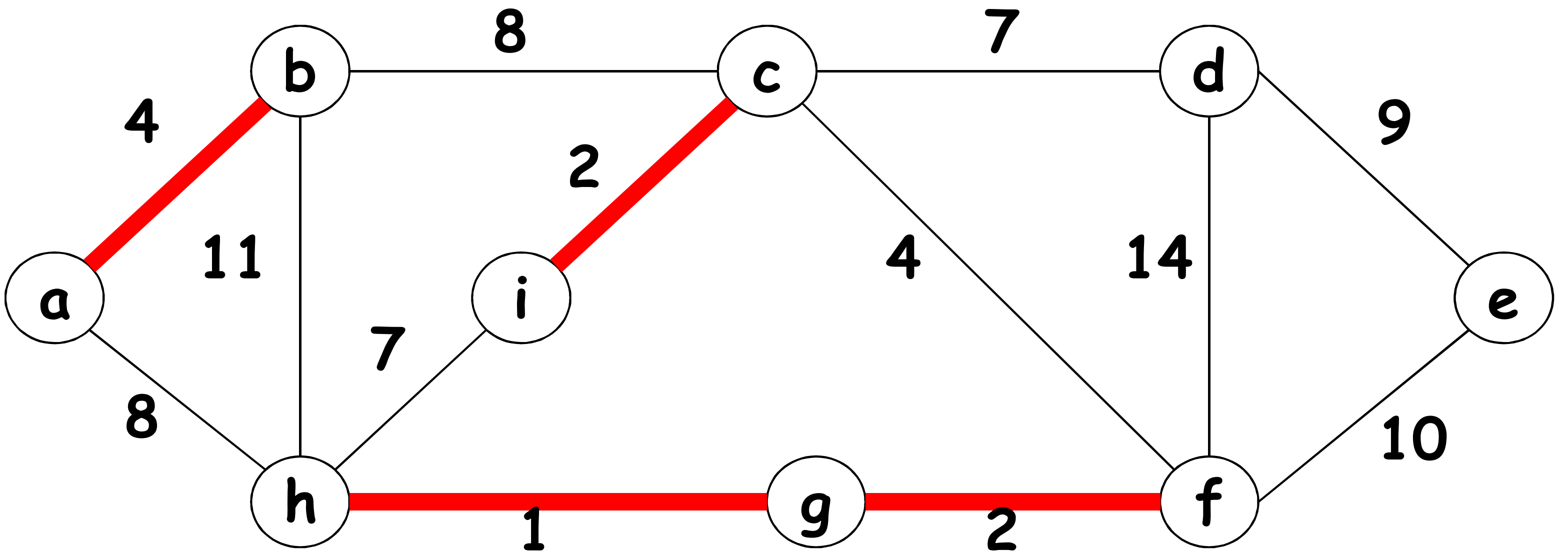


(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	4
(c,f)	4
(c,d)	7
(h,i)	7
(b,c)	8
(a,h)	8
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

italic: chosen

Continue as long as no cycle forms

Kruskal's algorithm - MST

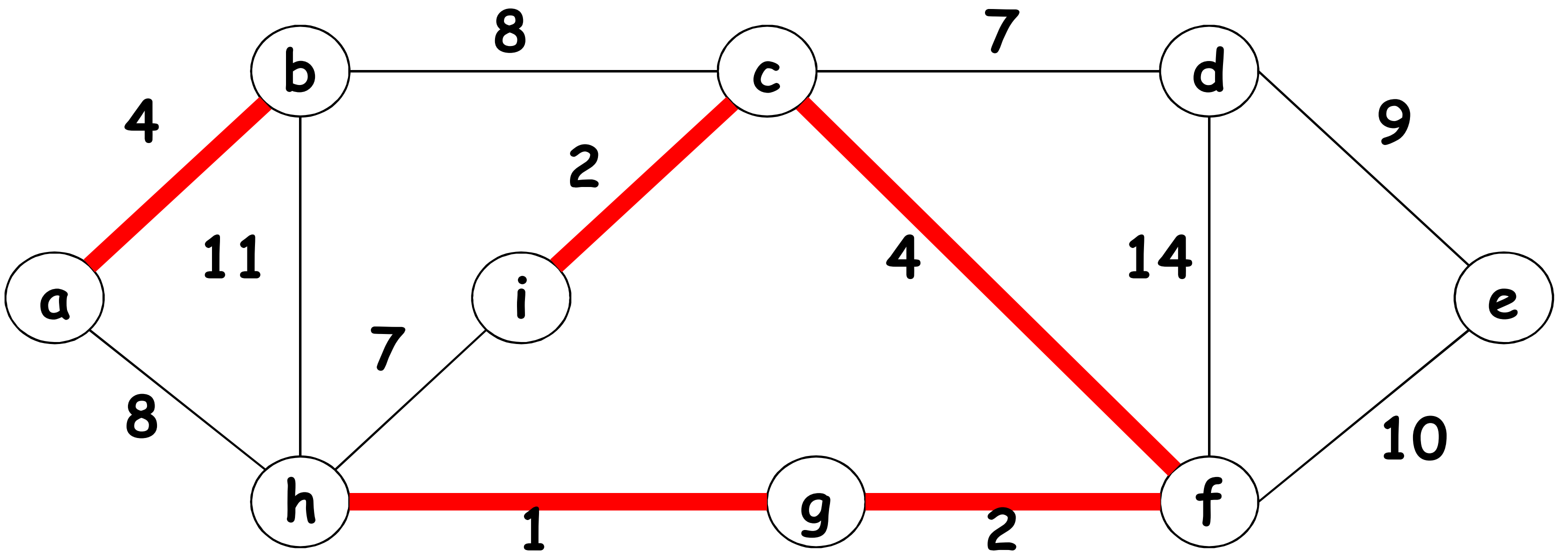


Continue as long as no cycle forms

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	4
(c,d)	7
(h,i)	7
(b,c)	8
(a,h)	8
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

italic: chosen

Kruskal's algorithm - MST

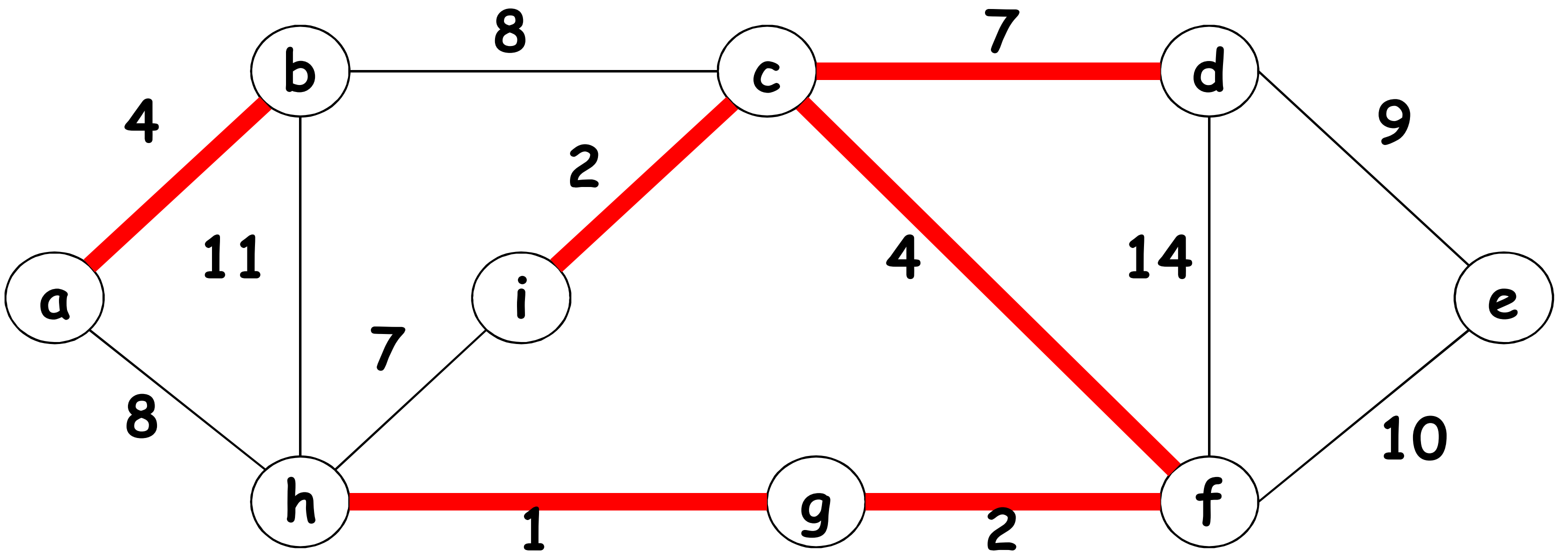


Continue as long as no cycle forms

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	7
(h,i)	7
(b,c)	8
(a,h)	8
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

italic: chosen

Kruskal's algorithm - MST

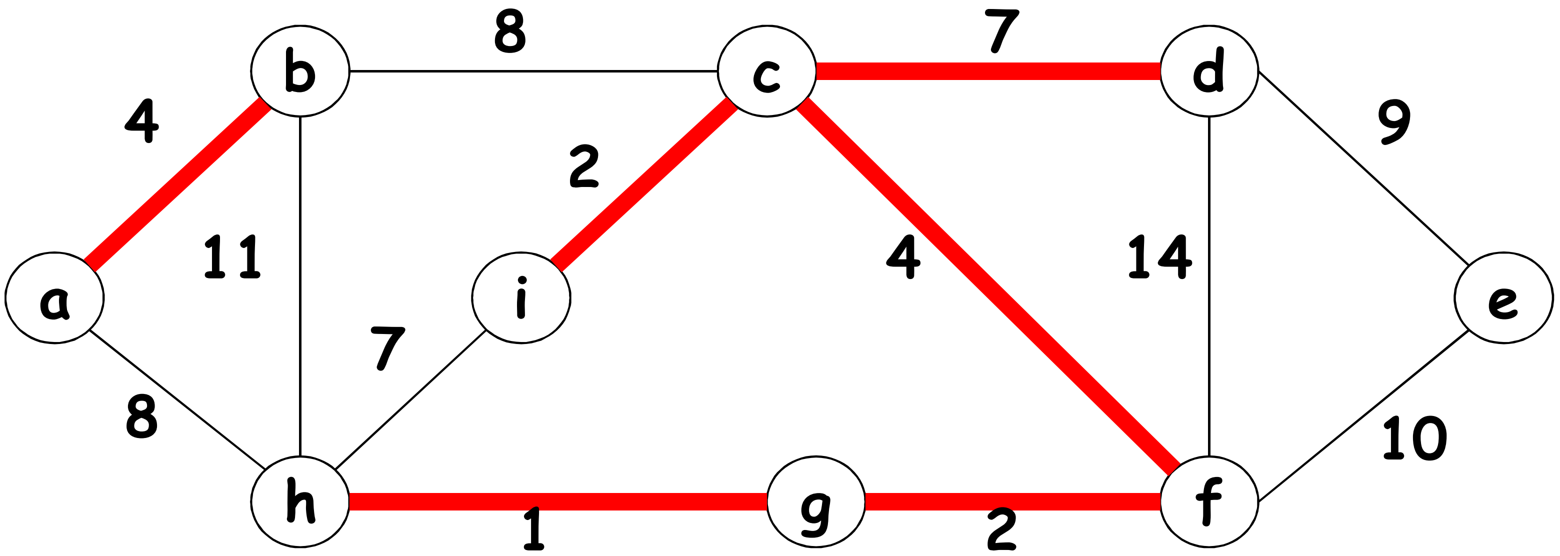


Continue as long as no cycle forms

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
(h,i)	7
(b,c)	8
(a,h)	8
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

italic: chosen

Kruskal's algorithm - MST

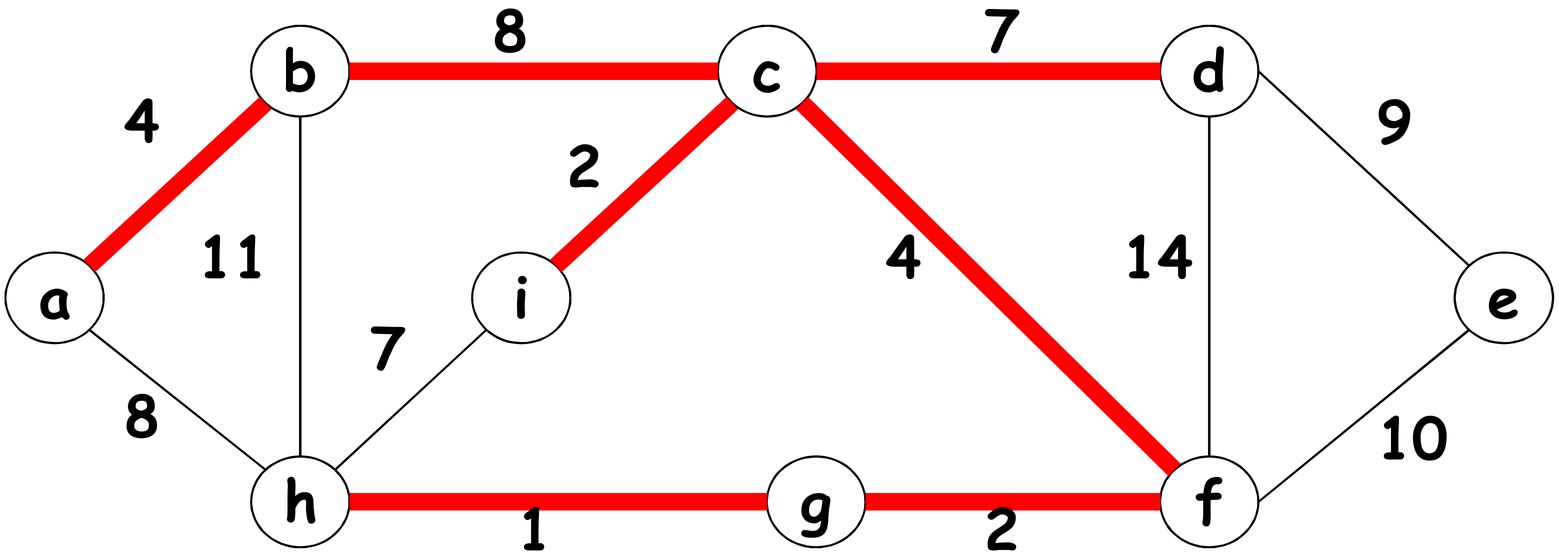


(h,i) cannot be included, otherwise, a cycle is formed

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~(h,i)~~	~~7~~
(b,c)	8
(a,h)	8
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

italic: chosen

Kruskal's algorithm - MST

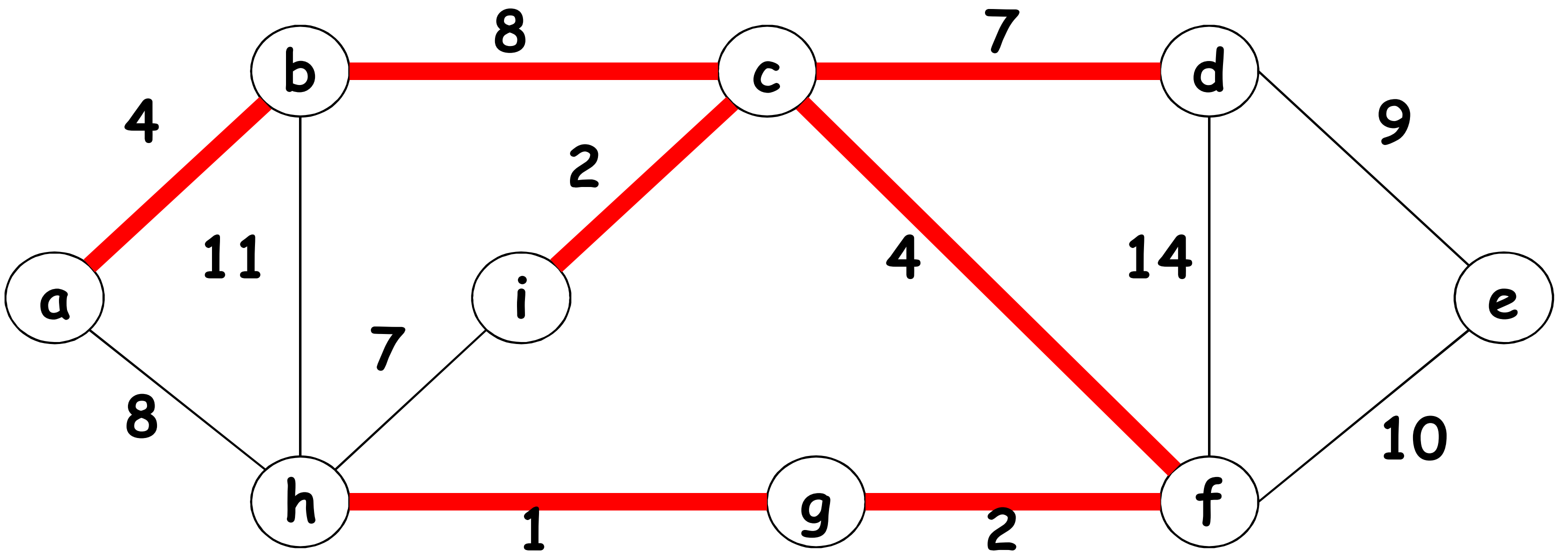


Choose the next minimum weight edge

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~(h,i)~~	~~7~~
(b,c)	*8*
(a,h)	8
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

italic: chosen

Kruskal's algorithm - MST

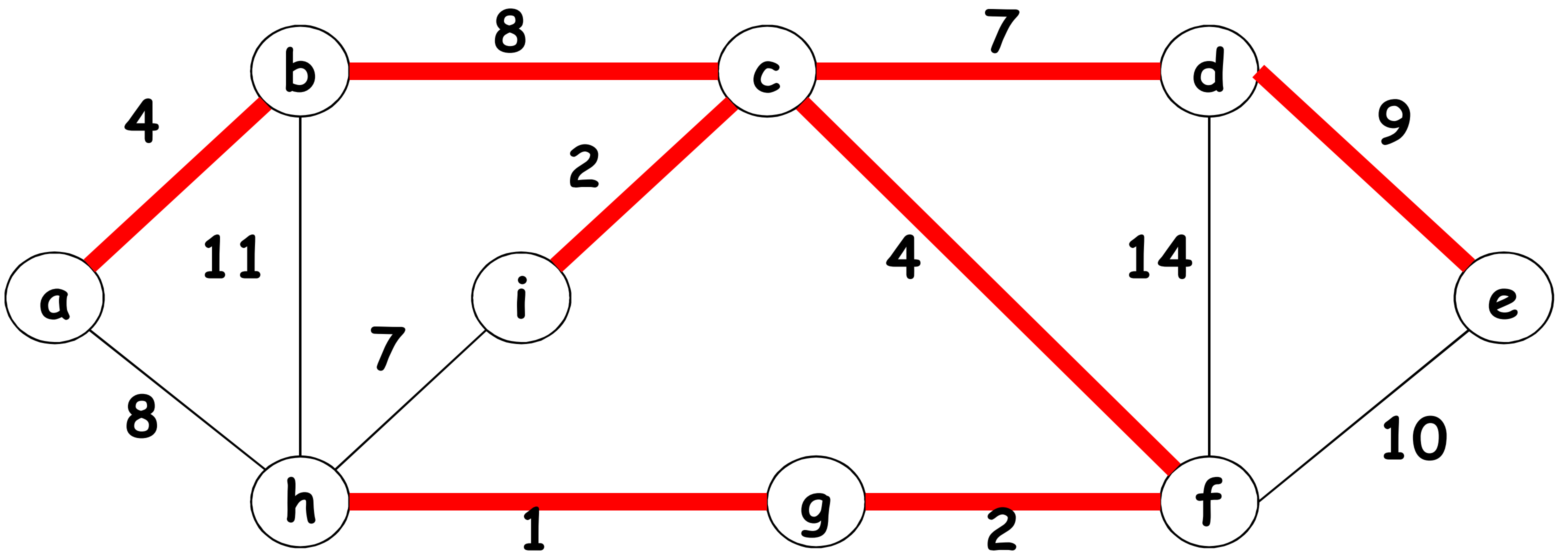


(a,h) cannot be included, otherwise, a cycle is formed

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~(h,i)~~	~~7~~
(b,c)	*8*
~~(a,h)~~	~~8~~
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

italic: chosen

Kruskal's algorithm - MST

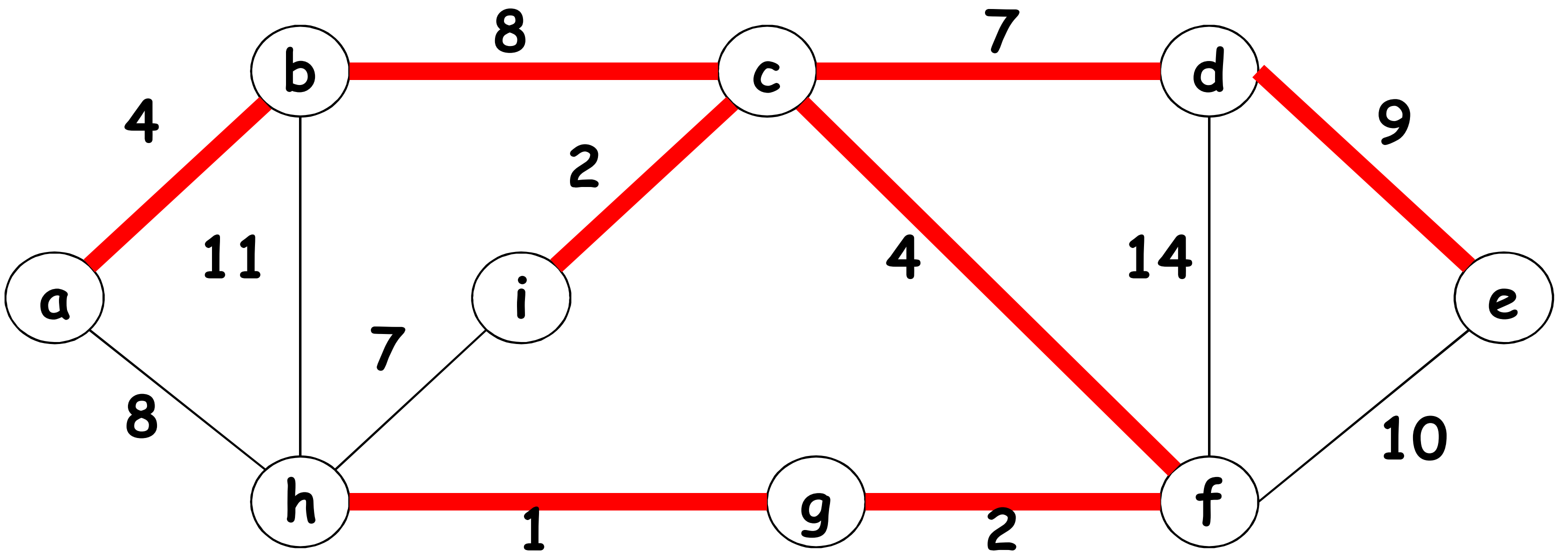


Choose the next minimum weight edge

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~(h,i)~~	~~7~~
(b,c)	*8*
~~(a,h)~~	~~8~~
(d,e)	*9*
(f,e)	10
(b,h)	11
(d,f)	14

italic: chosen

Kruskal's algorithm - MST

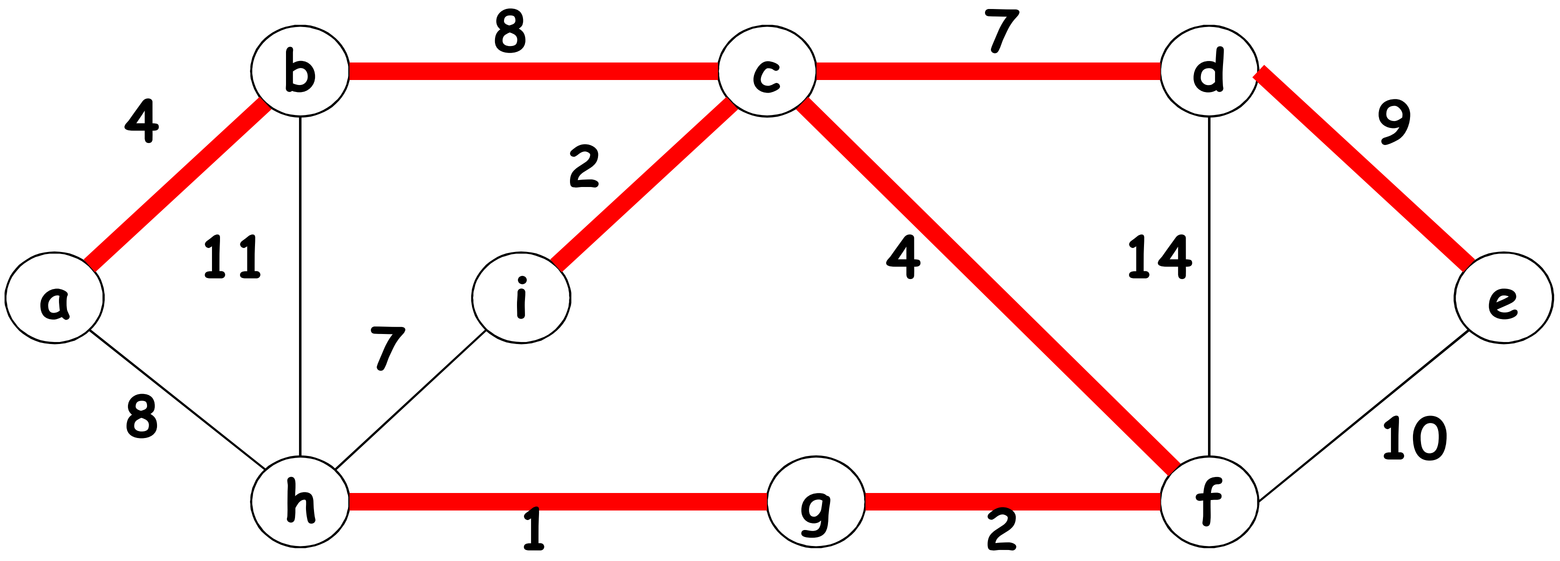


(f,e) cannot be included, otherwise, a cycle is formed

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~(h,i)~~	~~7~~
(b,c)	*8*
~~(a,h)~~	~~8~~
(d,e)	*9*
~~(f,e)~~	~~10~~
(b,h)	11
(d,f)	14

italic: chosen

Kruskal's algorithm - MST

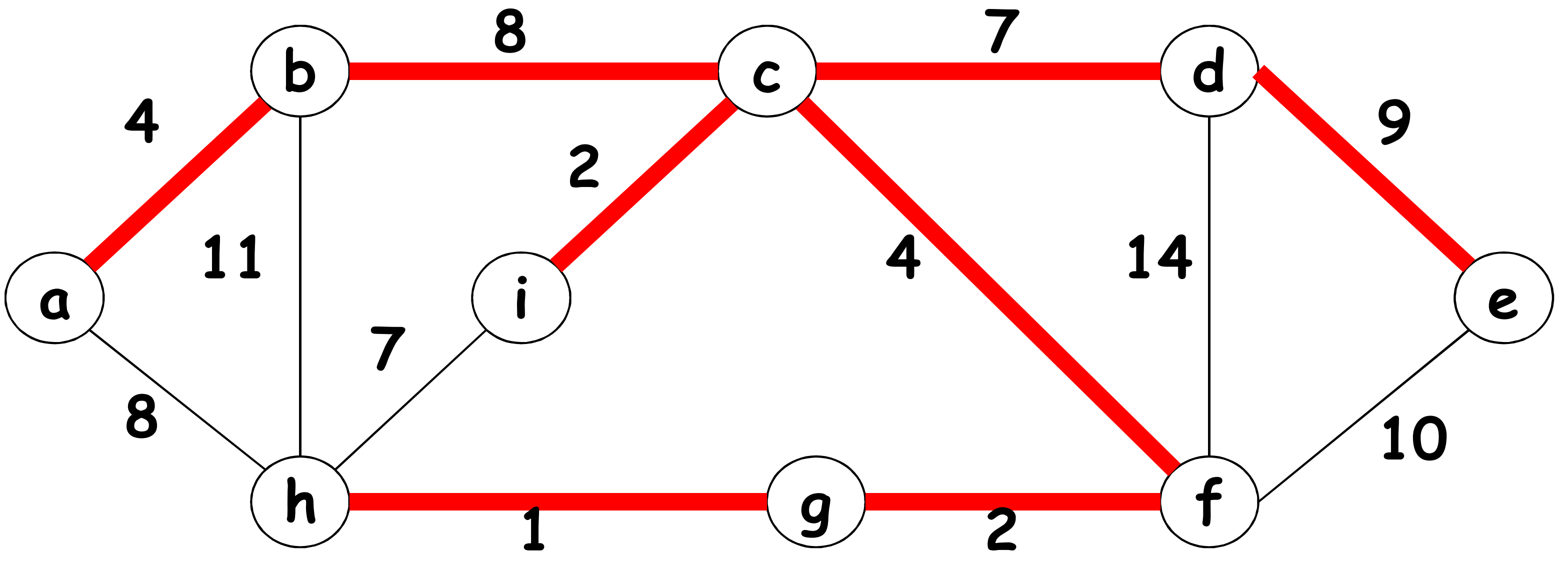


(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~(h,i)~~	~~7~~
(b,c)	*8*
~~(a,h)~~	~~8~~
(d,e)	*9*
~~(f,e)~~	~~10~~
~~(b,h)~~	~~11~~
(d,f)	14

italic: chosen

(b,h) cannot be included, otherwise, a cycle is formed

Kruskal's algorithm - MST

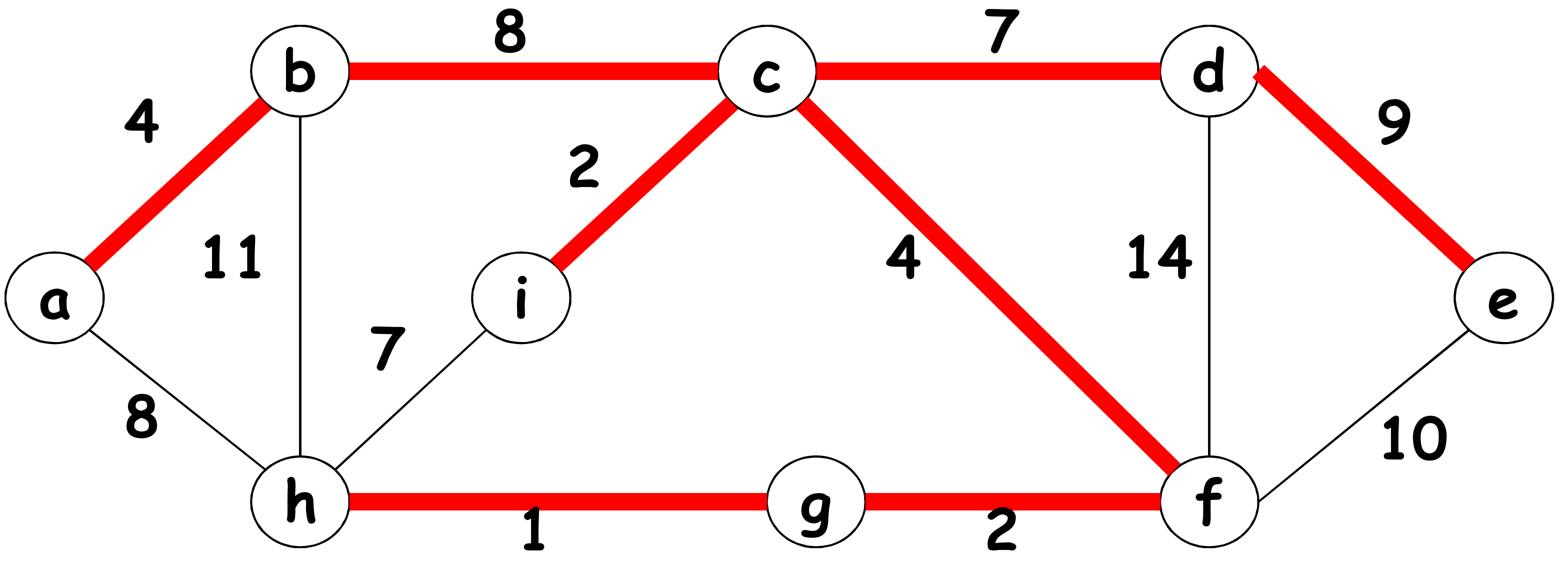


(d,f) cannot be included, otherwise, a cycle is formed

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~(h,i)~~	~~7~~
(b,c)	*8*
~~(a,h)~~	~~8~~
(d,e)	*9*
~~(f,e)~~	~~10~~
~~(b,h)~~	~~11~~
~~(d,f)~~	~~14~~

italic: chosen

Kruskal's algorithm - MST



MST is found when all edges are examined

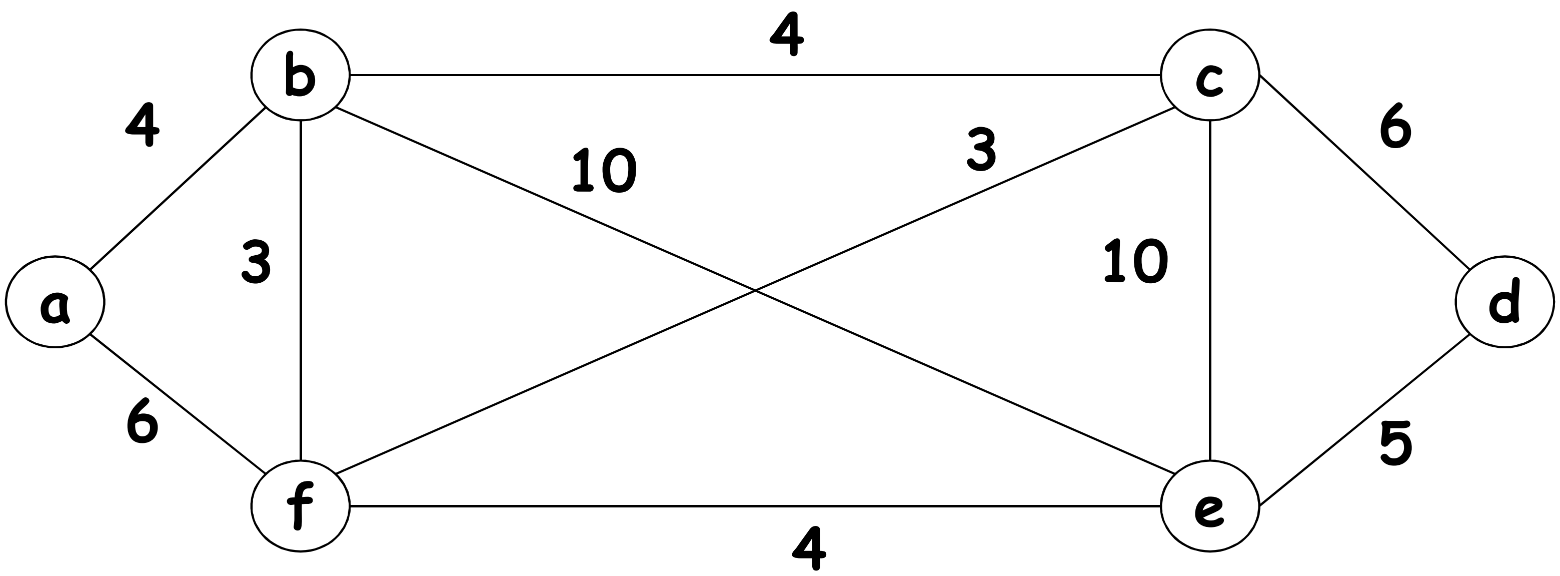
(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~(h,i)~~	~~7~~
(b,c)	*8*
~~(a,h)~~	~~8~~
(d,e)	*9*
~~(f,e)~~	~~10~~
~~(b,h)~~	~~11~~
~~(d,f)~~	~~14~~

italic: chosen

Kruskal's algorithm - MST

Kruskal's algorithm is **greedy** in the sense that it always attempt to select the **smallest** weight edge to be included in the MST

Exercise – Find MST for this graph



order of (edges) selection:

Pseudo code

```
// Given an undirected connected graph G=(V,E)
T = ∅ and E' = E
while E' ≠ ∅ do
begin
    pick an edge e in E' with minimum weight
    if adding e to T does not form cycle then
        add e to T, i.e., T = T ∪ { e }
    remove e from E', i.e., E' = E' \ { e }
end
```

Time complexity?

Can be tested by marking vertices

Dijkstra's algorithm …

Single-source shortest-paths

Consider a (un)directed connected graph G

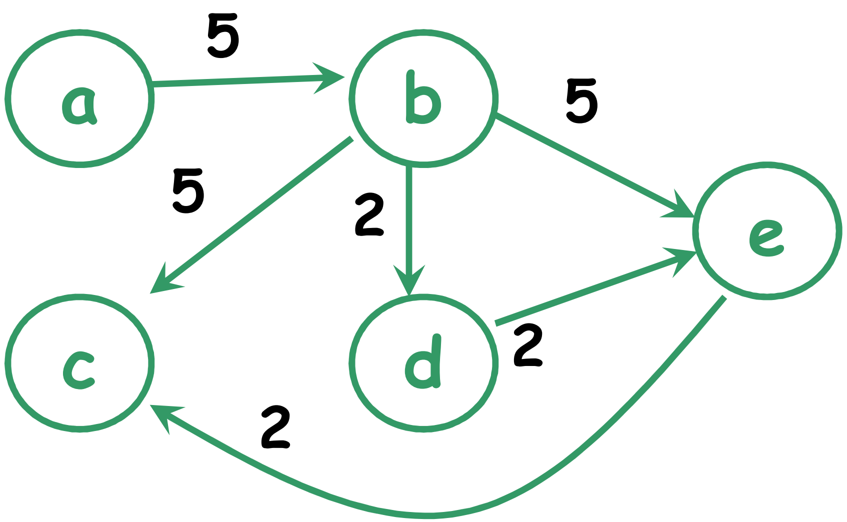
- The edges are labelled by weight

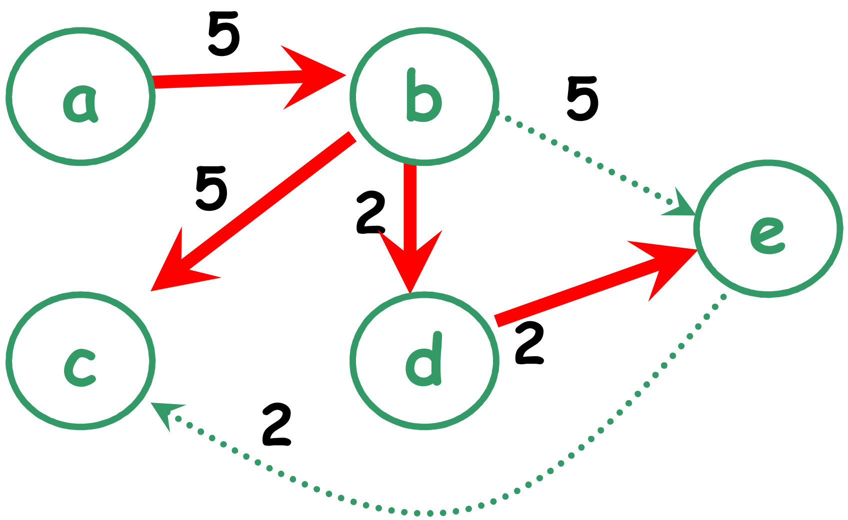
Given a particular vertex called the ***source***

- Find **shortest paths** from the source to all other vertices (shortest path means the total weight of the path is the smallest)

Example

Directed Graph G
(edge label is weight)
a is source vertex

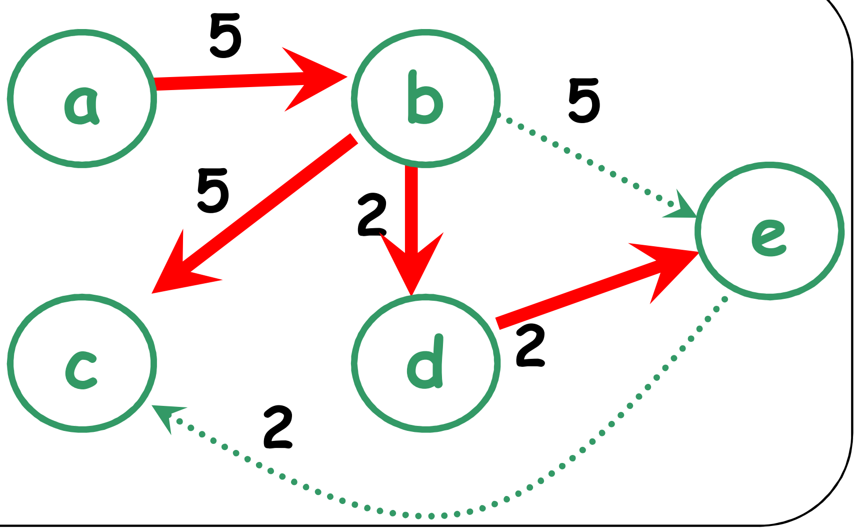




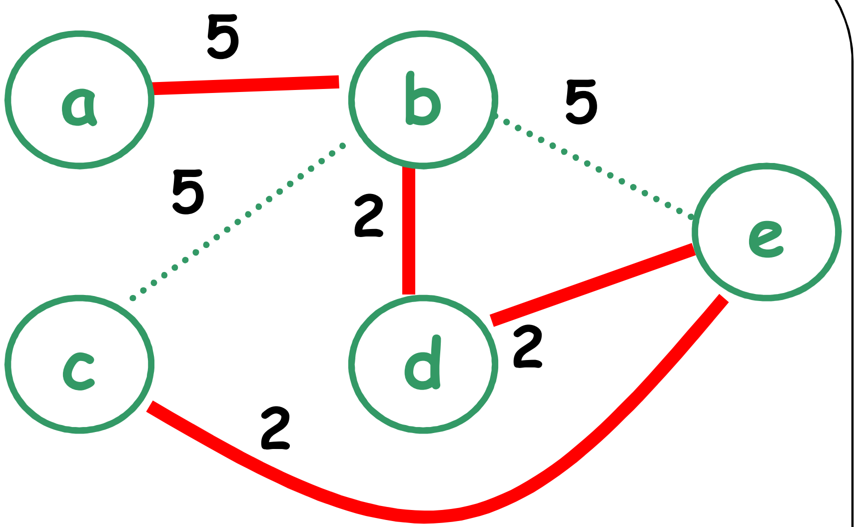
thick lines: shortest path
dotted lines: not in shortest path

Single-source shortest paths vs MST

Shortest paths from a



What is the difference between MST and shortest paths from a?



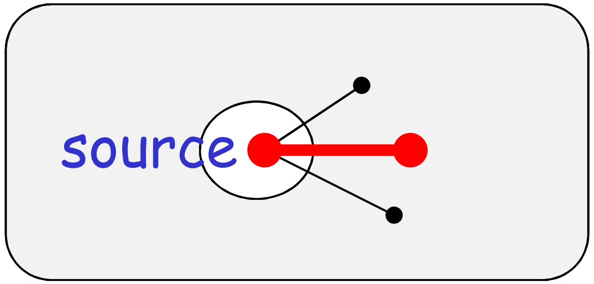
MST

Algorithms for shortest paths

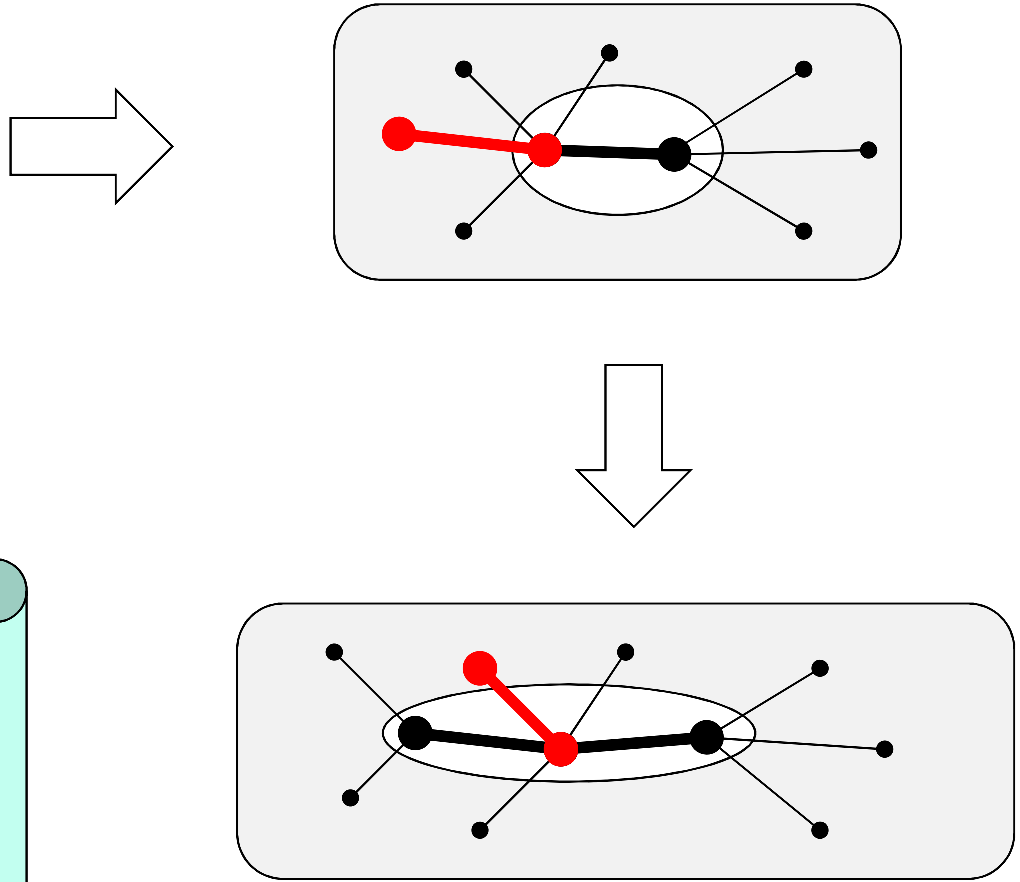
Algorithms

- there are many algorithms to solve this problem, one of them is **Dijkstra's** algorithm, which assumes the weights of edges are **non-negative**

Idea of Dijkstra’s algorithm



choose the edge leading to vertex s.t. cost of path to source is min



Mind that the edge added is ***NOT*** necessarily the minimum-cost one

Dijkstra's algorithm

Input: A directed connected weighted graph G and a source vertex *s*

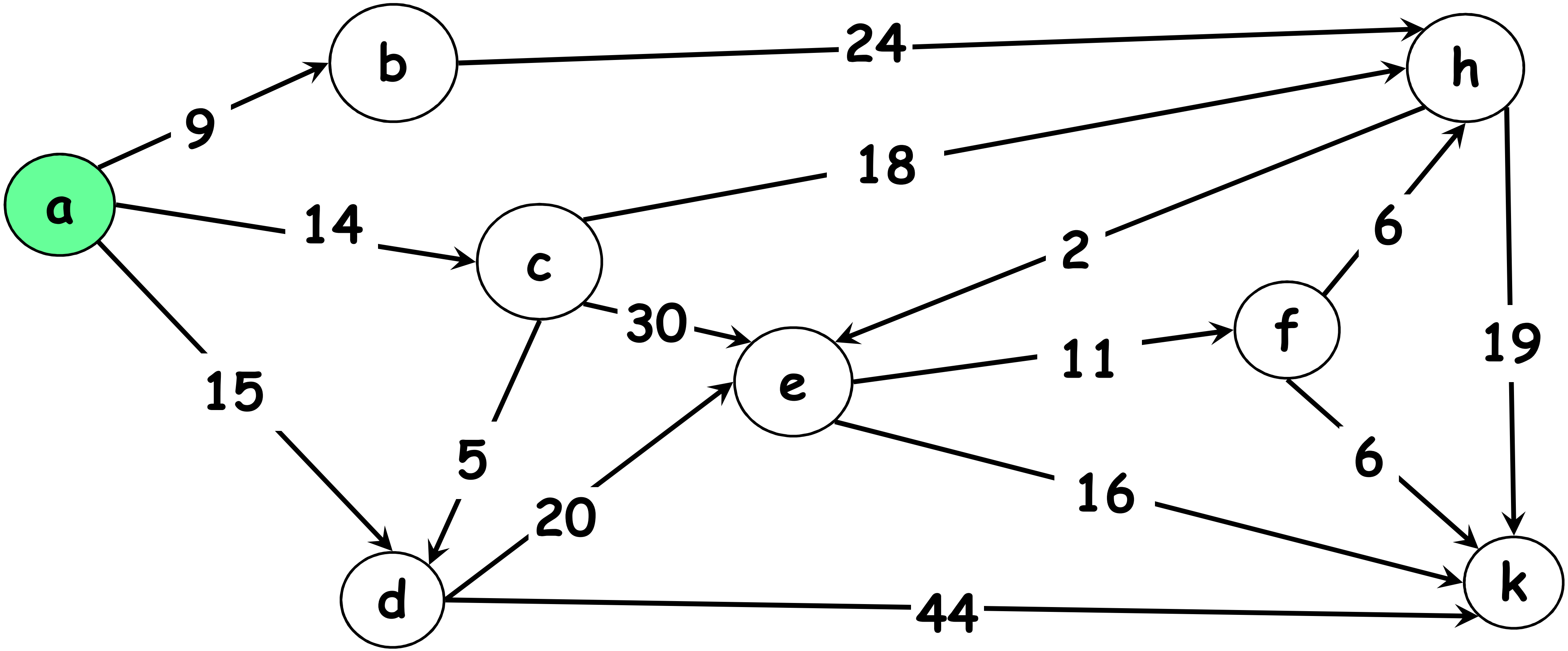
Output: For every vertex *v* in G, find the shortest path from *s* to *v*

Dijkstra's algorithm runs in iterations:

- in the i-th iteration, the vertex which is the i-th closest to *s* is found,
- for every remaining vertices, the current shortest path to *s* found so far (this shortest path will be updated as the algorithm runs)

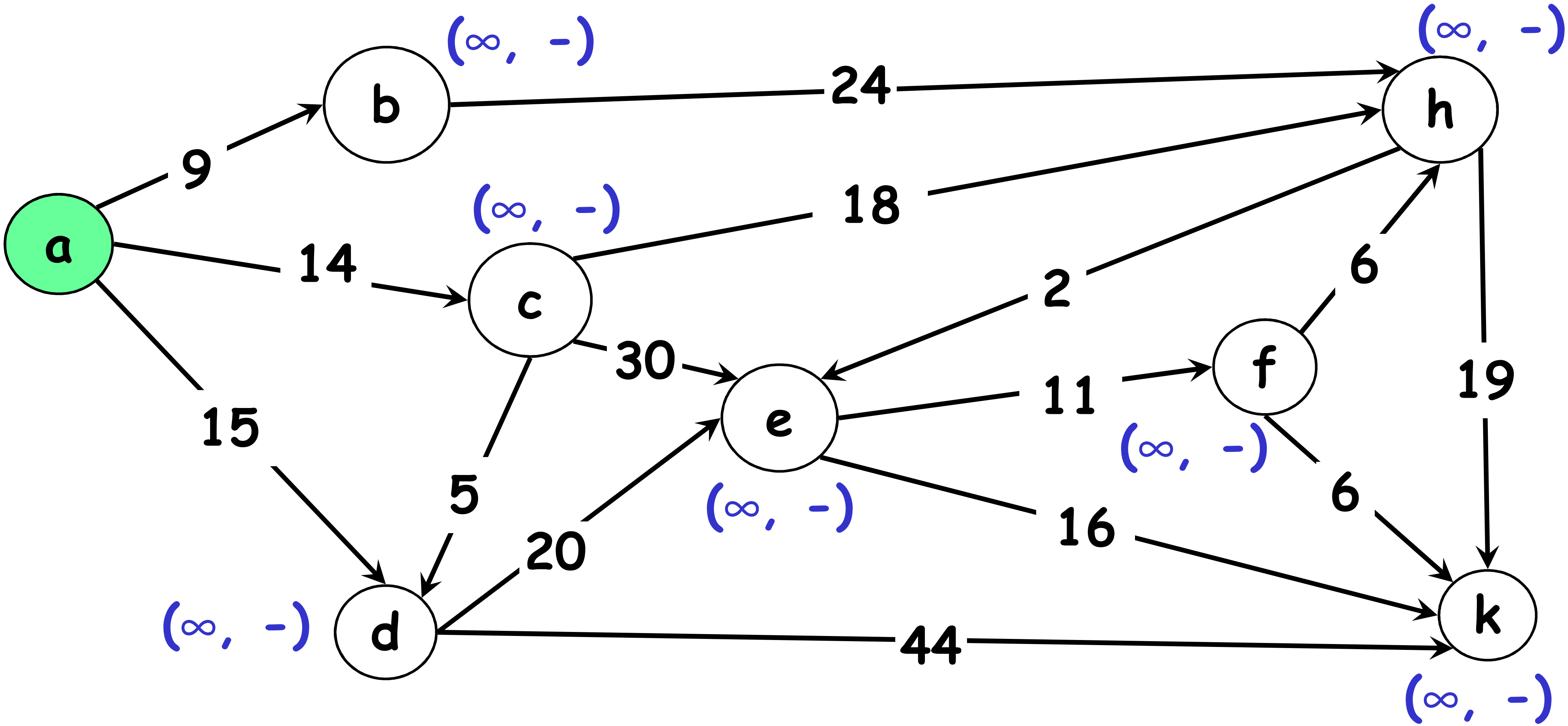
Dijkstra's algorithm

Suppose vertex *a* is the source, we now show how Dijkstra's algorithm works



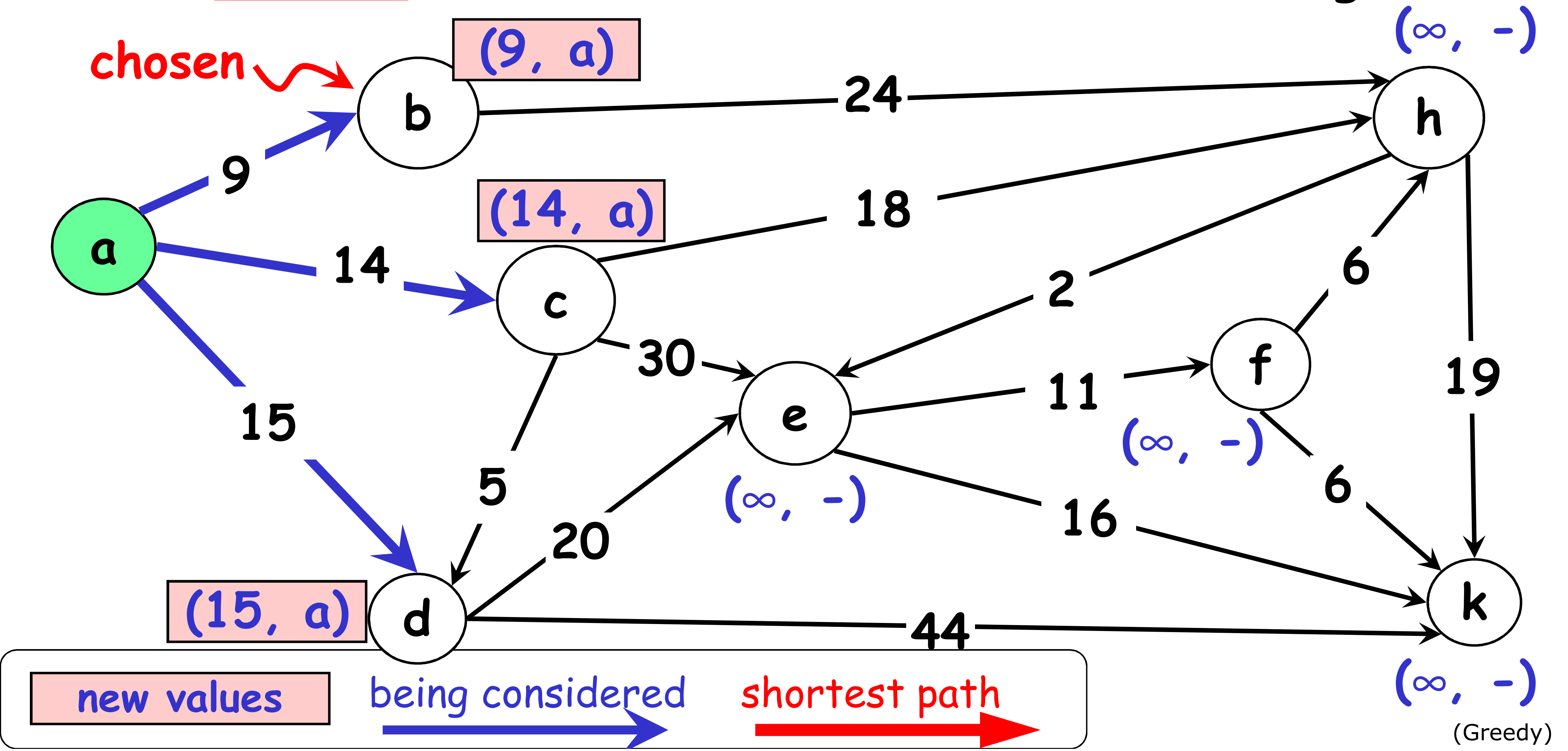
Dijkstra's algorithm

Every vertex v keeps 2 labels: (1) the weight of the current shortest path from a; (2) the vertex leading to v on that path, initially as (∞, -)



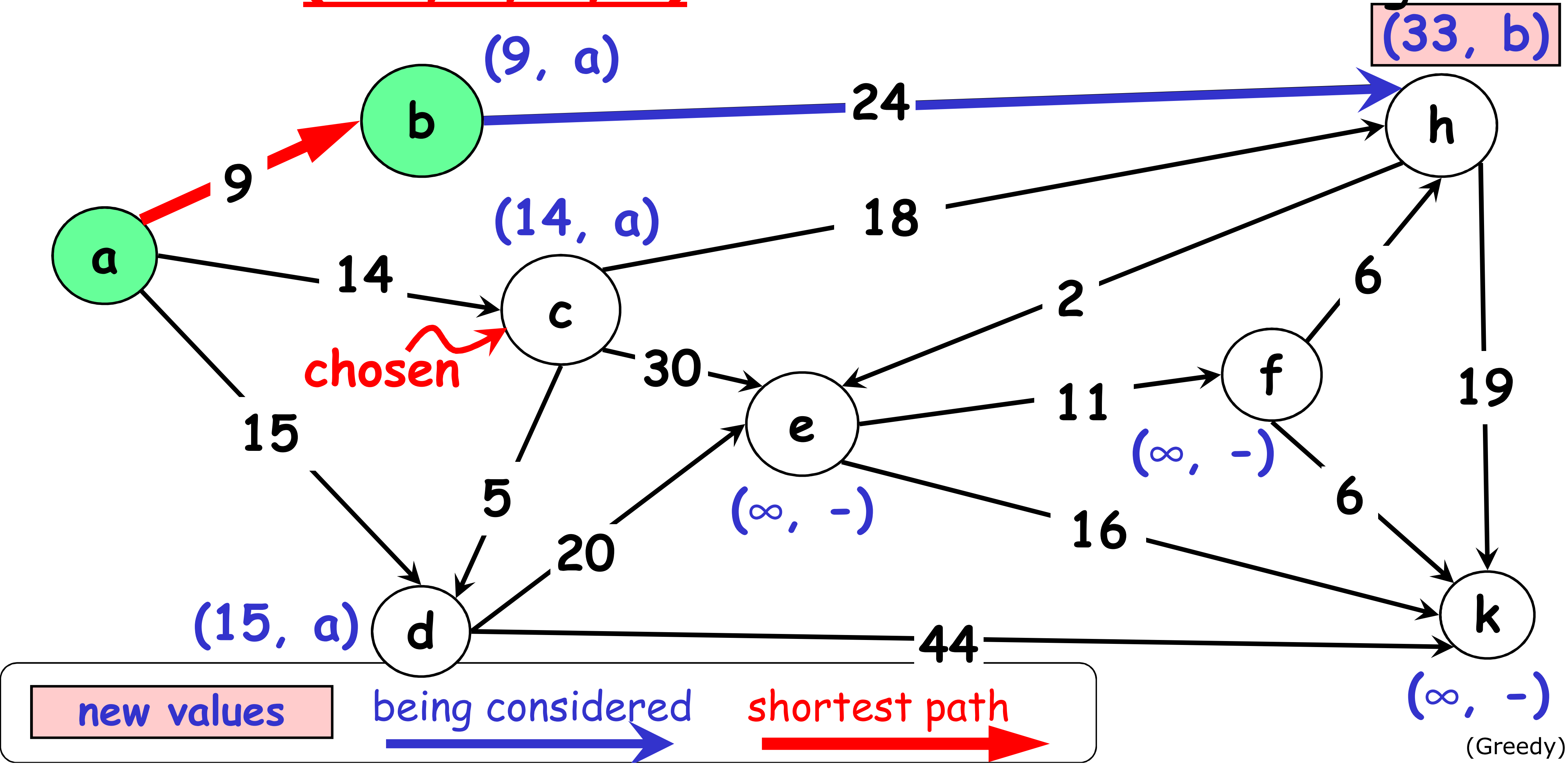
Dijkstra's algorithm

For every neighbor *u* of *a*, update the weight to the weight of *(a,u)* and the leading vertex to *a*. Choose from <u>*b, c, d*</u> the one with the smallest such weight.



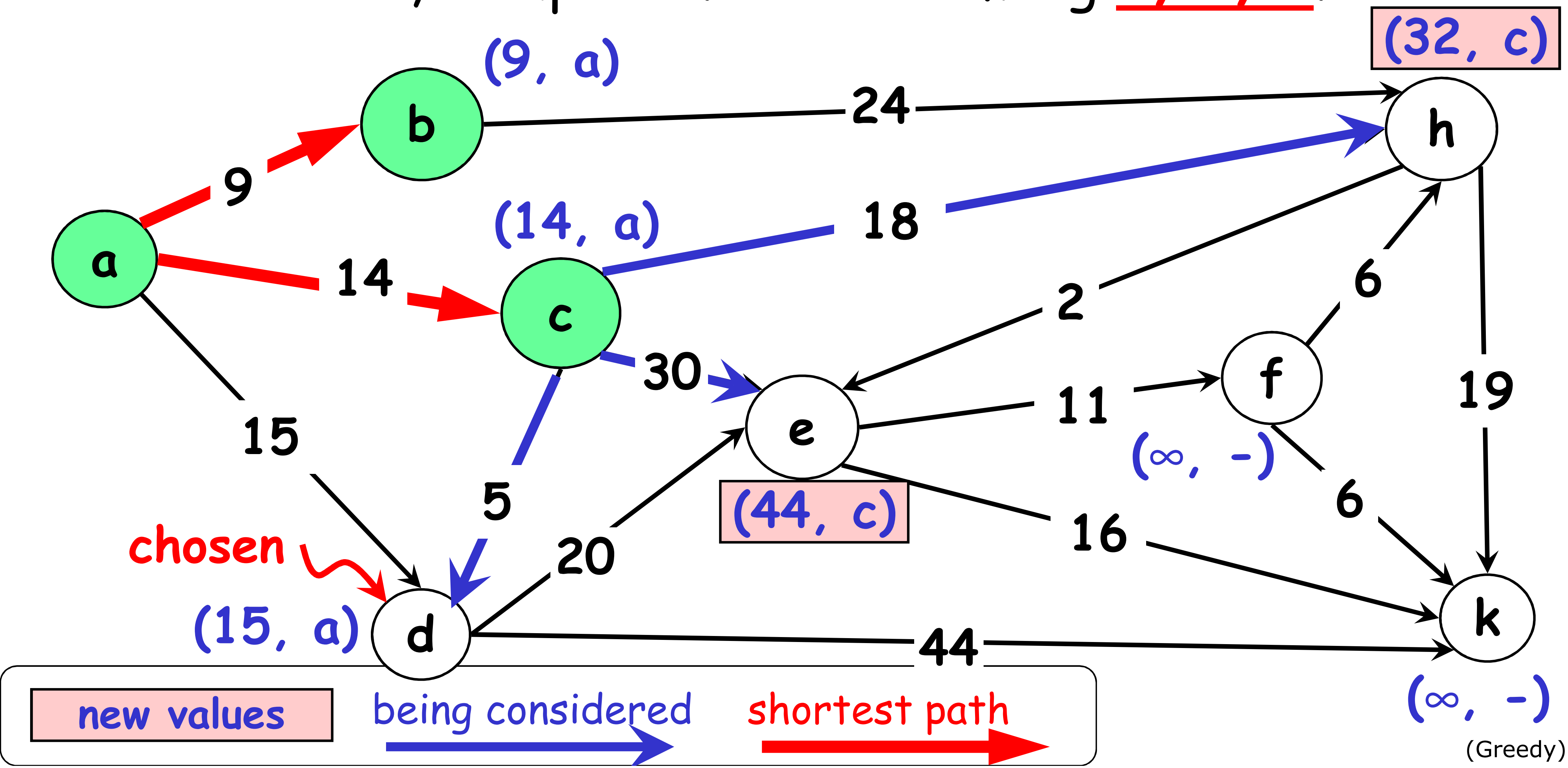
Dijkstra's algorithm

For every un-chosen neighbor of vertex *b*, update the weight and leading vertex. Choose from **ALL** un-chosen vertices (i.e., *c, d, h*) the one with smallest weight.



Dijkstra's algorithm

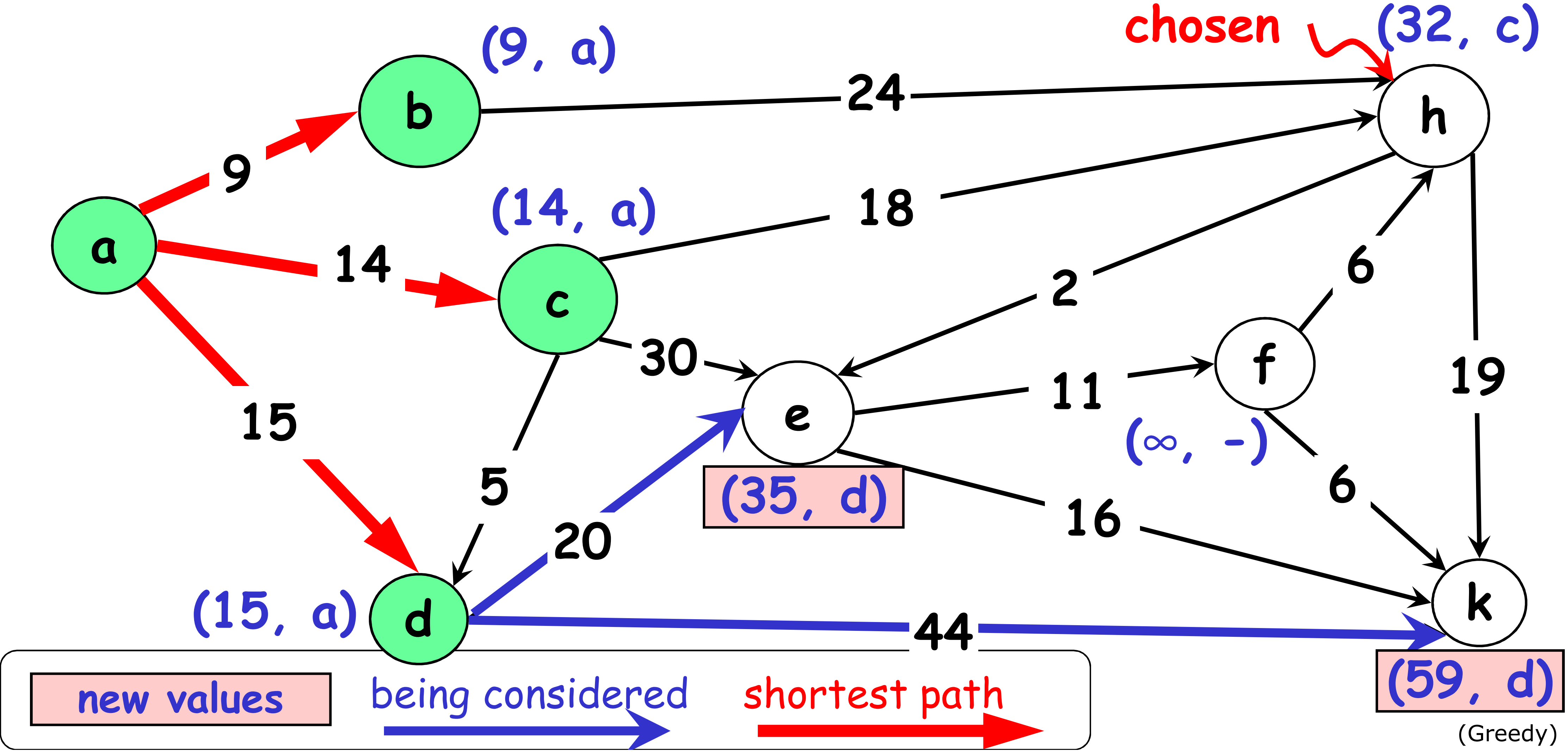
If a new path with smallest weight is discovered, e.g., for vertices *e, h*, the weight is updated. Otherwise, like vertex *d*, no update. Choose among *d, e, h*.



(Greedy)

Dijkstra's algorithm

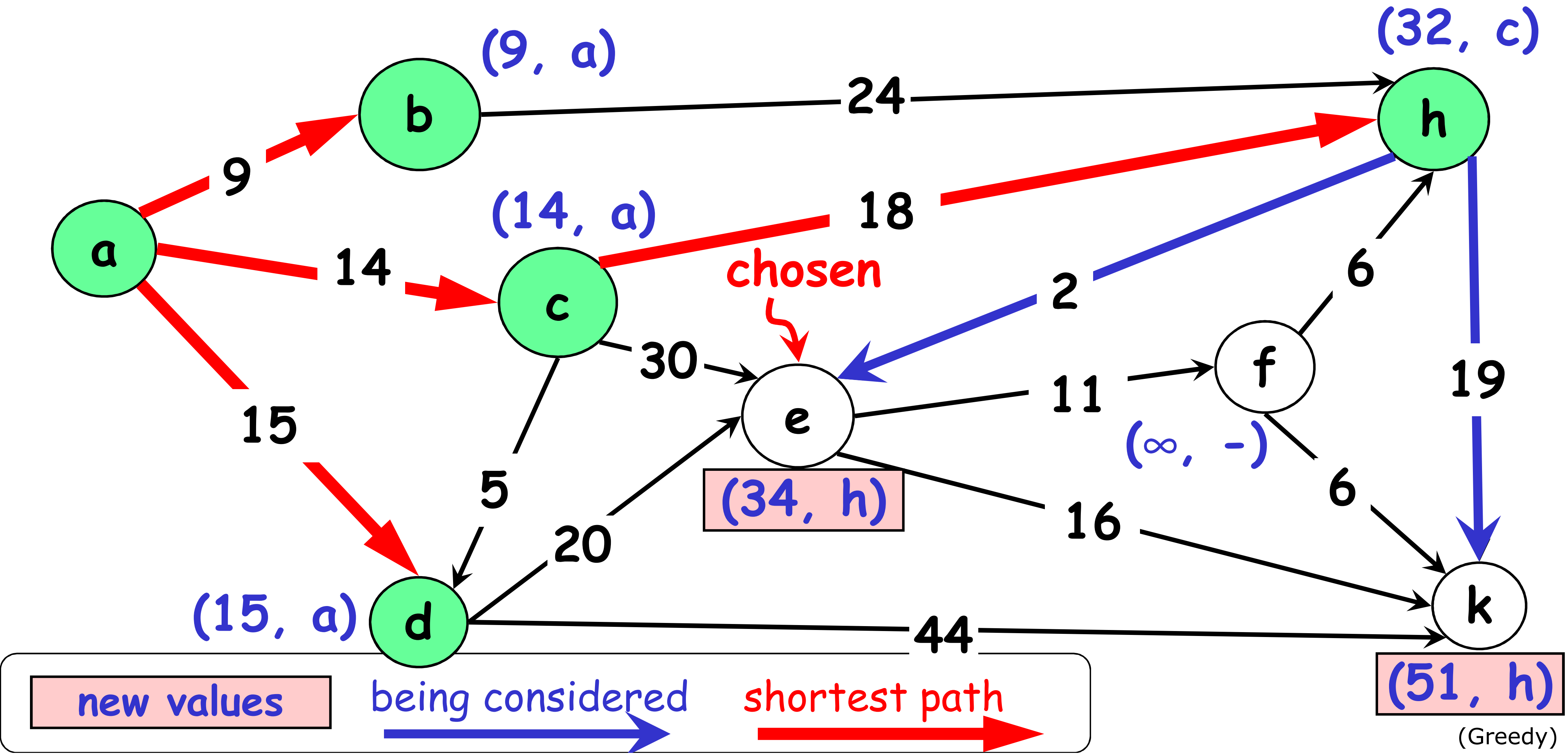
Repeat the procedure. After *d* is chosen, the weight of *e* and *k* is updated. Choose among *e, h, k*. Next vertex chosen is *h*.



(Greedy)

Dijkstra's algorithm

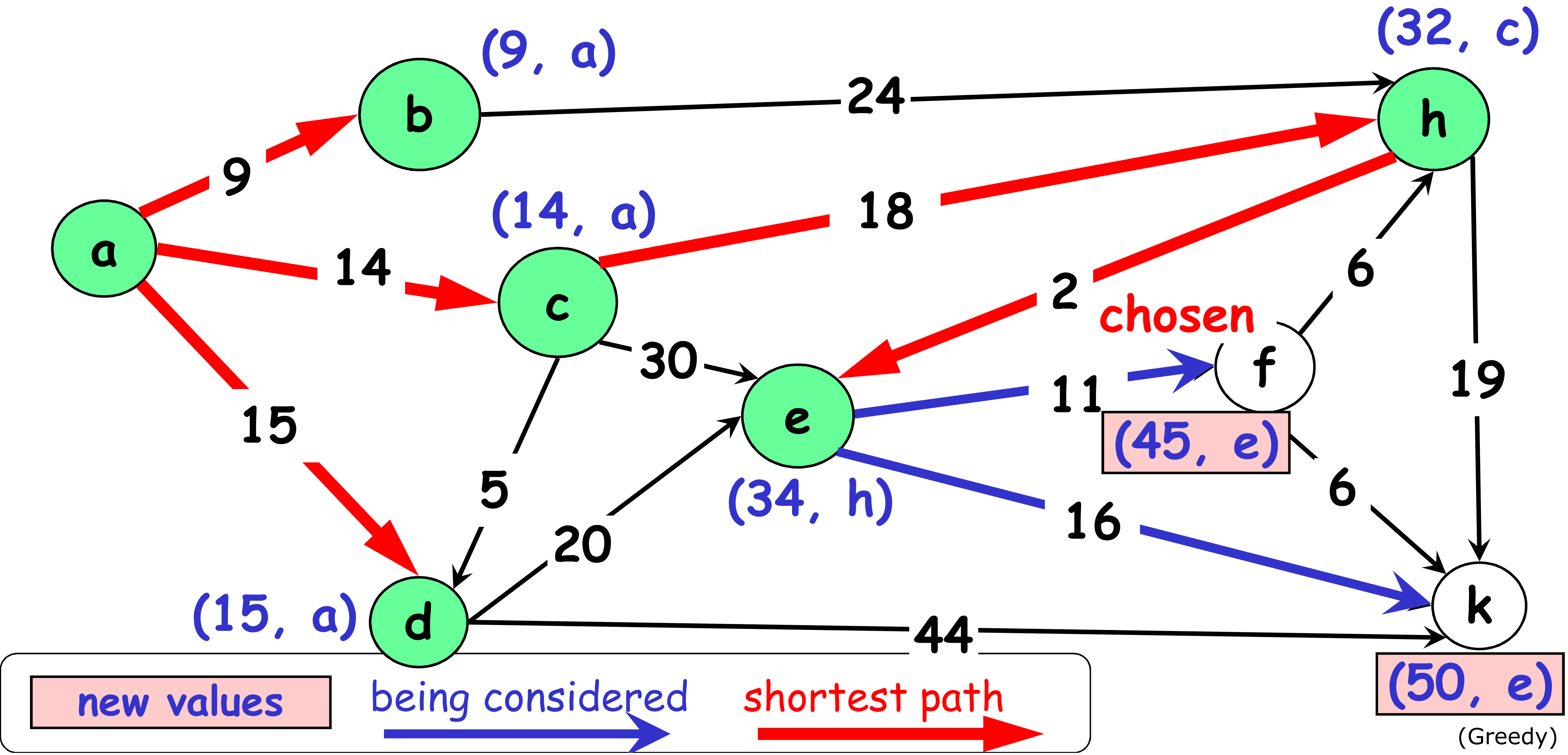
After *h* is chosen, the weight of *e* and *k* is updated again. Choose among e, k. Next vertex chosen is *e*.



(Greedy)

Dijkstra's algorithm

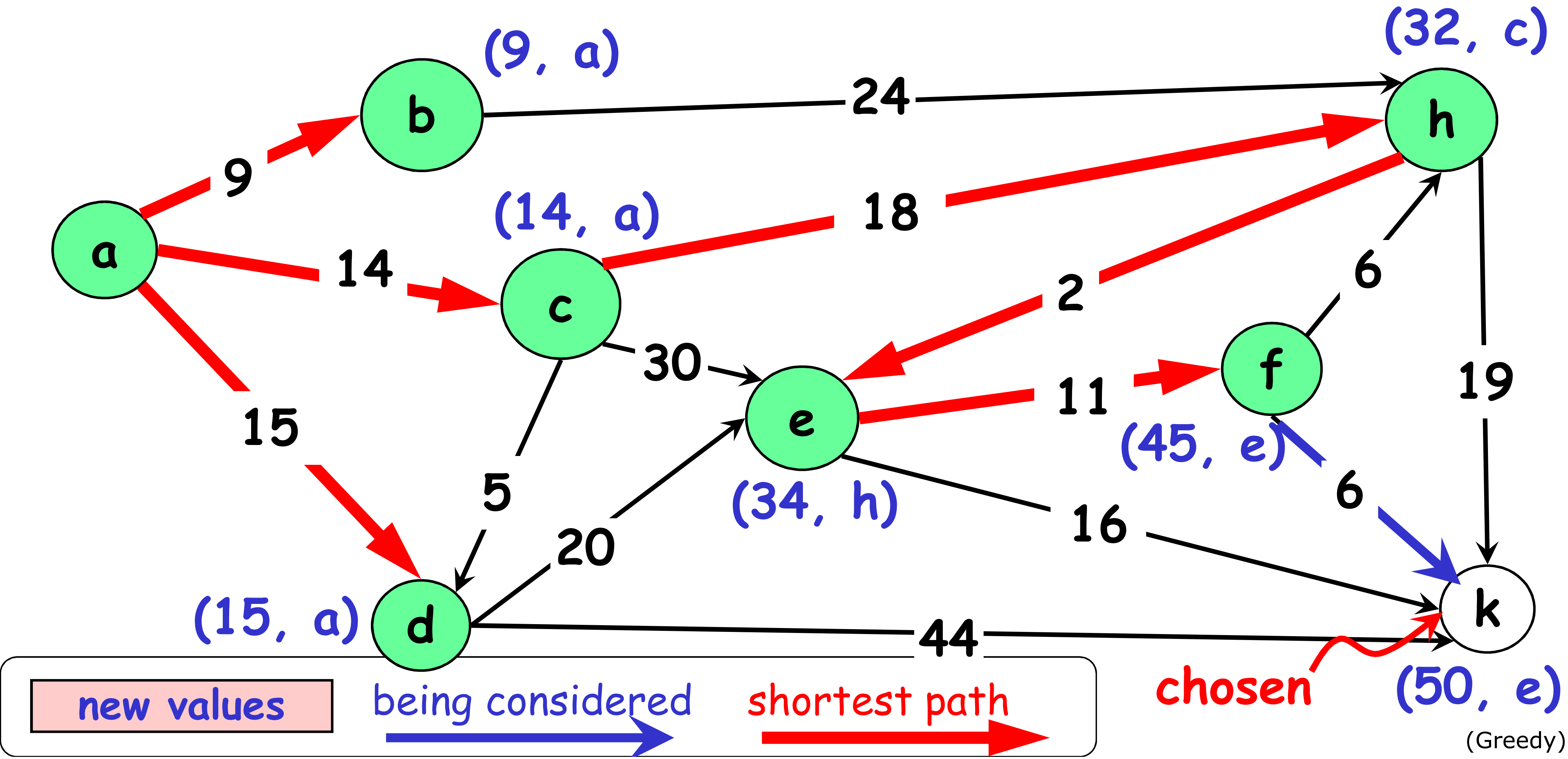
After *e* is chosen, the weight of *f* and *k* is updated again. Choose among <u>*f, k*</u>. Next vertex chosen is *f*.



(Greedy)

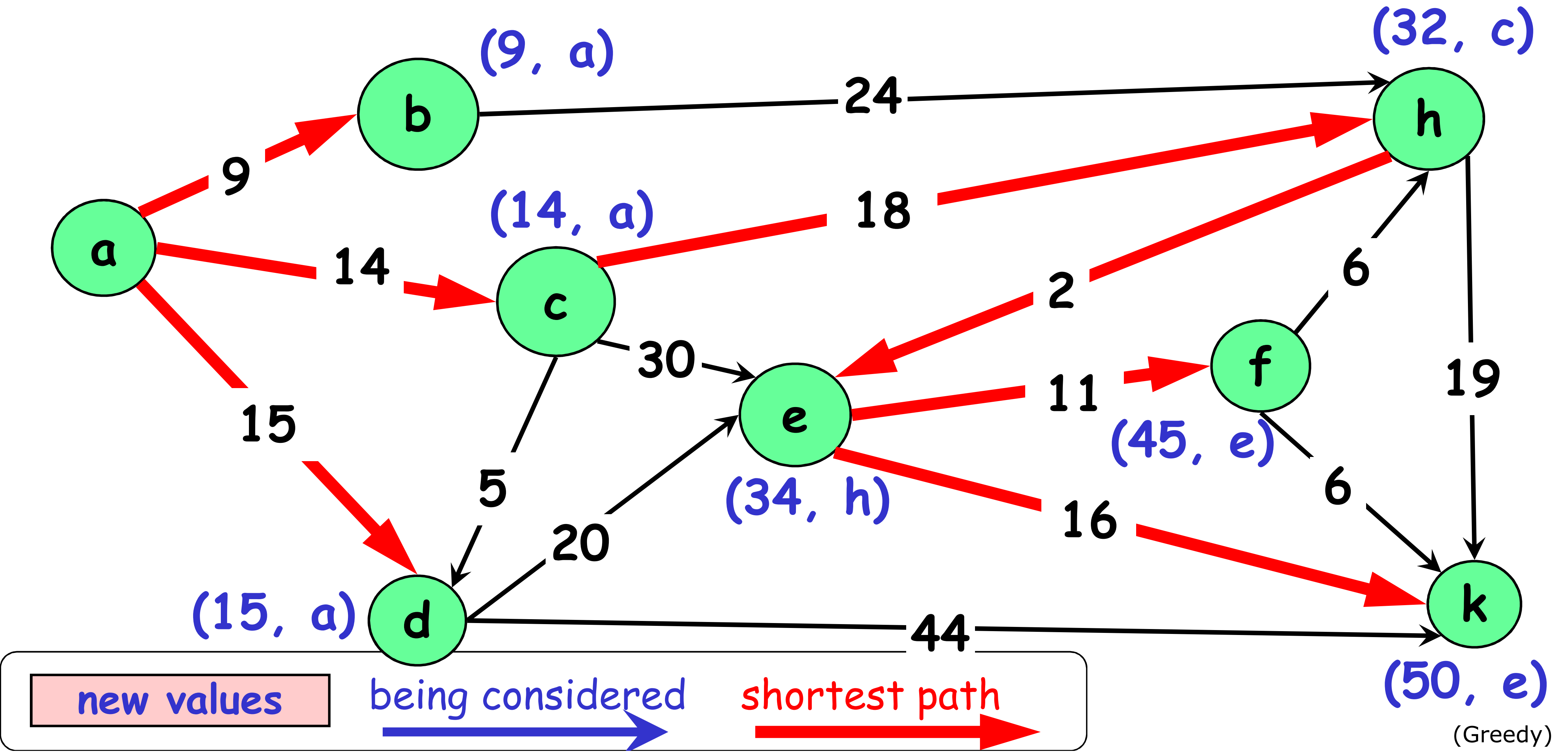
Dijkstra's algorithm

After *f* is chosen, it is NOT necessary to update the weight of *k*. The final vertex chosen is *k*.

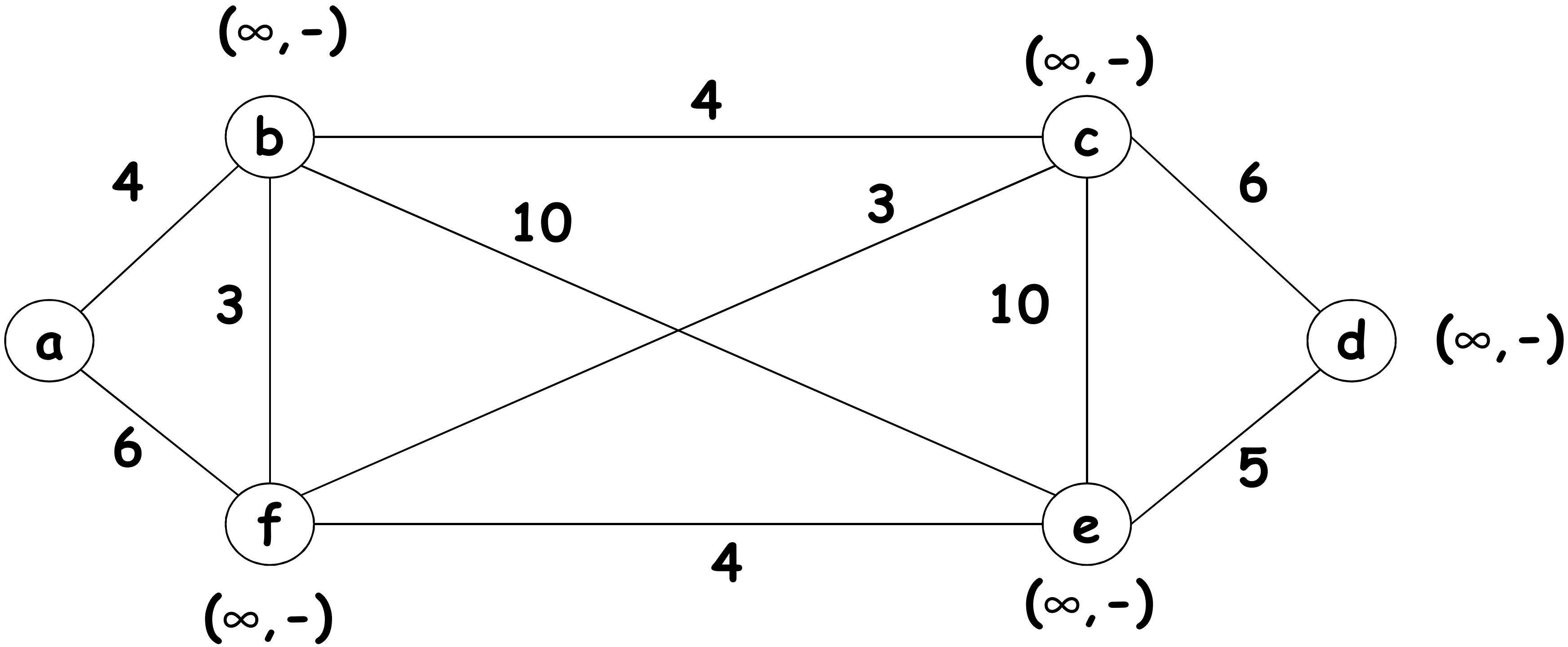


Dijkstra's algorithm

At this point, all vertices are chosen, and the shortest path from *a* to every vertex is discovered.



Exercise – Shortest paths from a



order of (edges) selection:

Compare the solution with slide #26

Dijkstra's algorithm

To describe the algorithm using pseudo code, we give some notations

Each vertex *v* is labelled with two labels:

- a **numeric label** ***d(v)*** indicates the length of the shortest path from the source to *v* found so far
- another label ***p(v)*** indicates next-to-last vertex on such path, i.e., the vertex immediately before *v* on that shortest path

Pseudo code

// Given a graph $G=(V,E)$ and a source vertex s

Time complexity?

for every vertex v in the graph **do**

    set $d(v) = \infty$ and $p(v) = \text{null}$

set $d(s) = 0$ and $V_T = \emptyset$

while $V \setminus V_T \neq \emptyset$ **do** // there is still some vertex left

begin

    choose the vertex u in $V \setminus V_T$ with minimum $d(u)$

    set $V_T = V_T \cup \{ u \}$

    for every vertex v in $V \setminus V_T$ that is a neighbor of u **do**

        if $d(u) + w(u,v) < d(v)$ **then** *// a shorter path is found*

            set $d(v) = d(u) + w(u,v)$ and $p(v) = u$

end

Does Greedy algorithm always return the best solution?

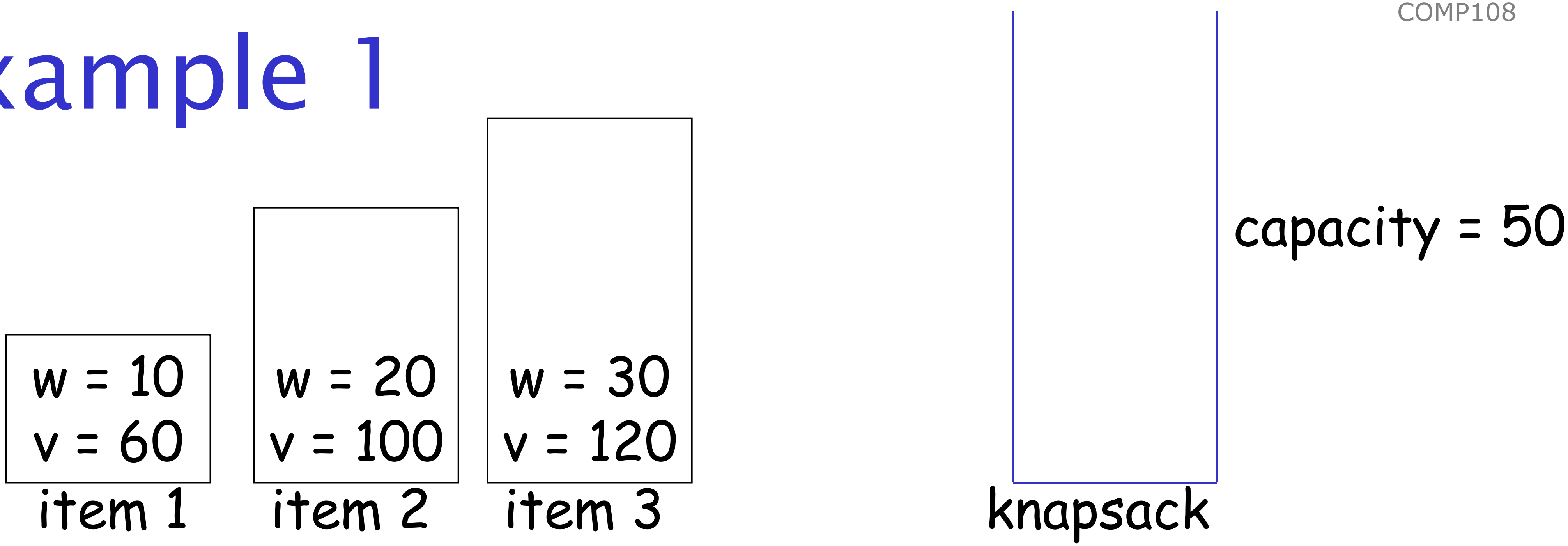
Knapsack Problem

Input: Given n items with weights $w_1, w_2, \ldots, w_n$ and values $v_1, v_2, \ldots, v_n$, and a knapsack with capacity W.

Output: Find the most valuable subset of items that can fit into the knapsack

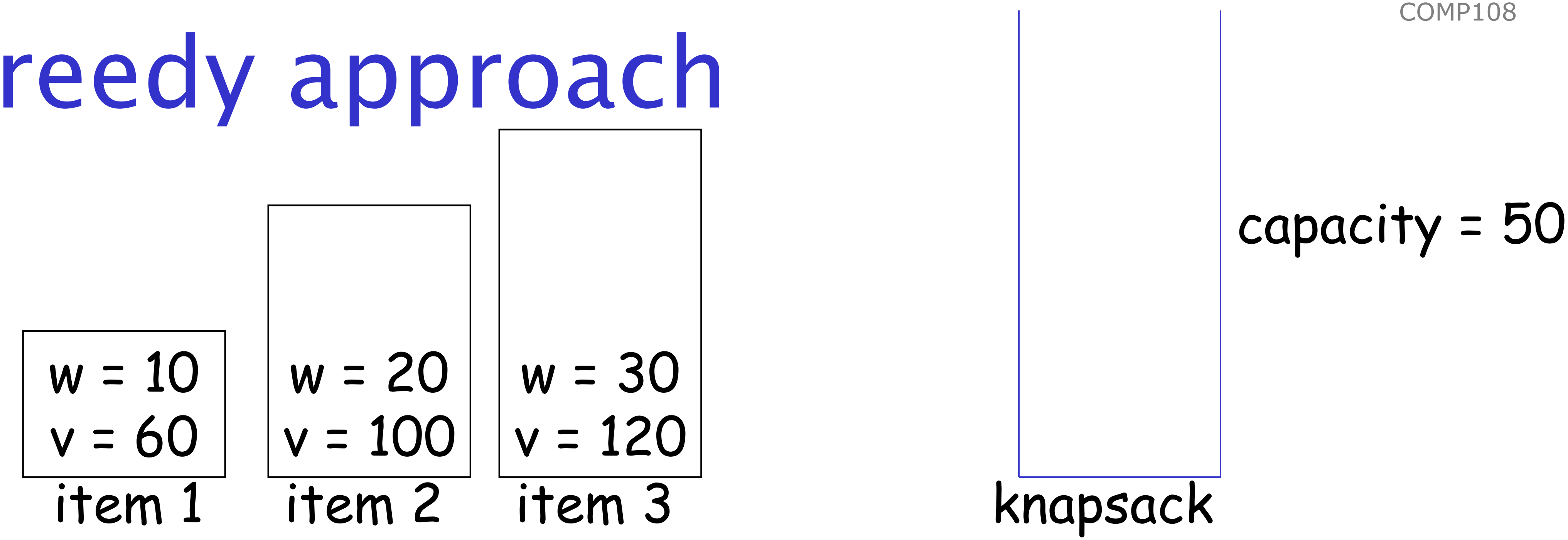
Application: A transport plane is to deliver the most valuable set of items to a remote location without exceeding its capacity

Example 1



subset	total weight	total value
ϕ	0	0
{1}	10	60
{2}	20	100
{3}	30	120
{1,2}	30	160
{1,3}	40	180
{2,3}	**50**	**220**
{1,2,3}	60	N/A

Greedy approach



Greedy: pick the item with the next largest value if total weight ≤ capacity.

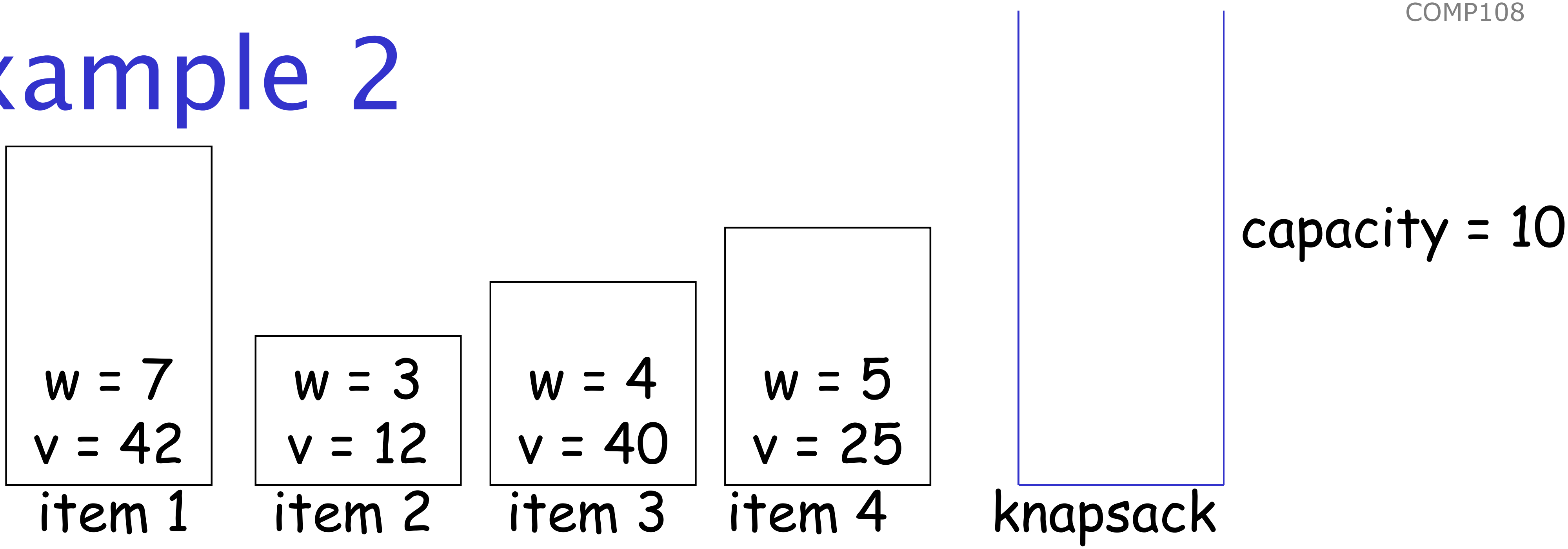
Time complexity?

Result:

- item 3 is taken, total value = 120, total weight = 30
- item 2 is taken, total value = 220, total weight = 50
- item 1 cannot be taken

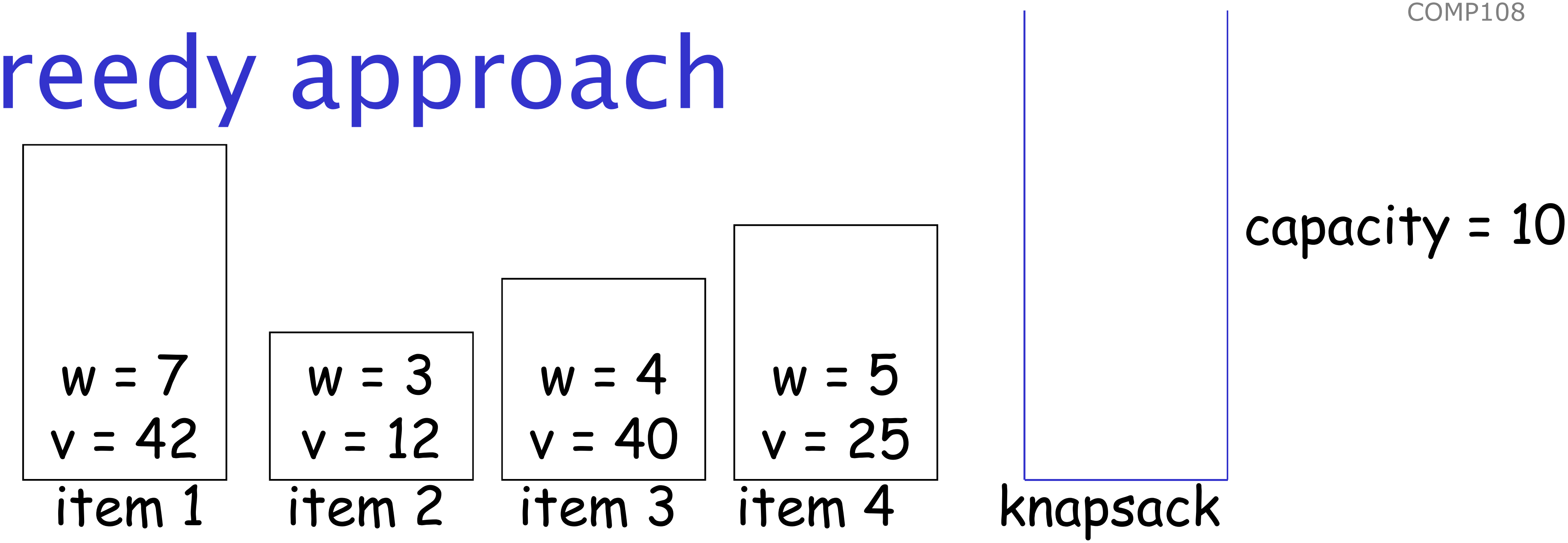
Does this always work?

Example 2



subset	total weight	total value	subset	total weight	total value
ϕ	0	0	{2,3}	7	52
{1}	7	42	{2,4}	8	37
{2}	3	12	**{3,4}**	**9**	**65**
{3}	4	40	{1,2,3}	14	N/A
{4}	5	25	{1,2,4}	15	N/A
{1,2}	10	54	{1,3,4}	16	N/A
{1,3}	11	N/A	{2,3,4}	12	N/A
{1,4}	12	N/A	{1,2,3,4}	19	N/A

Greedy approach

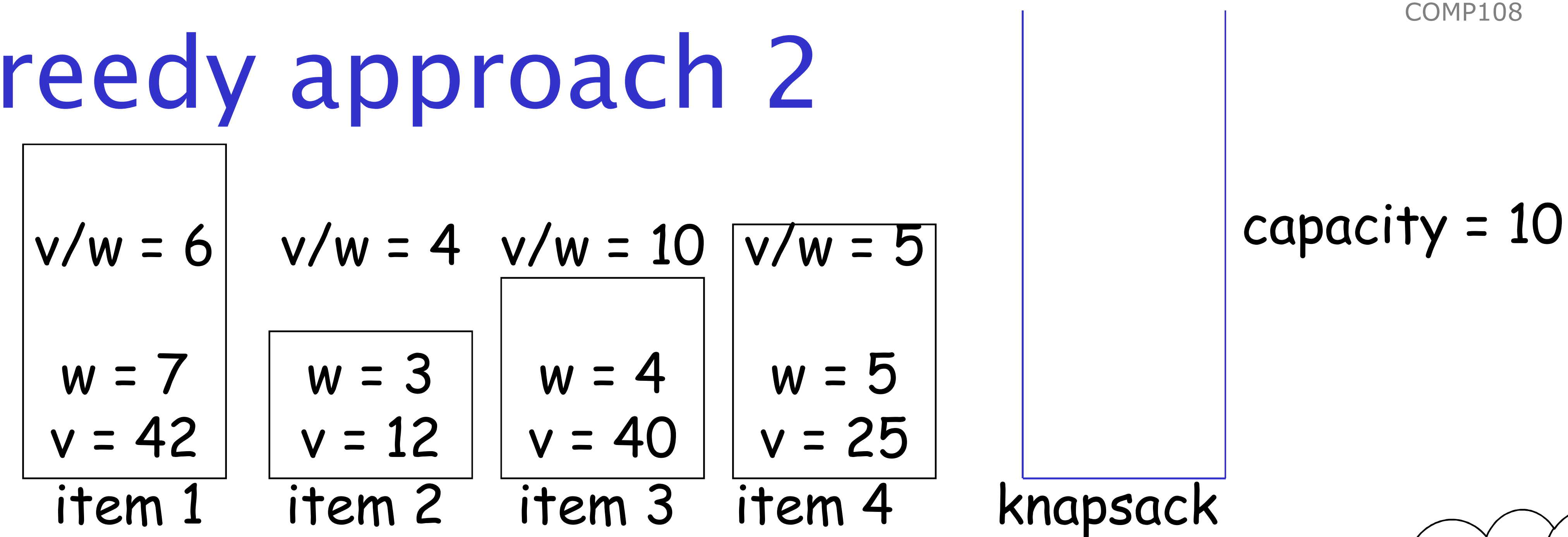


Greedy: pick the item with the next largest value if total weight ≤ capacity.

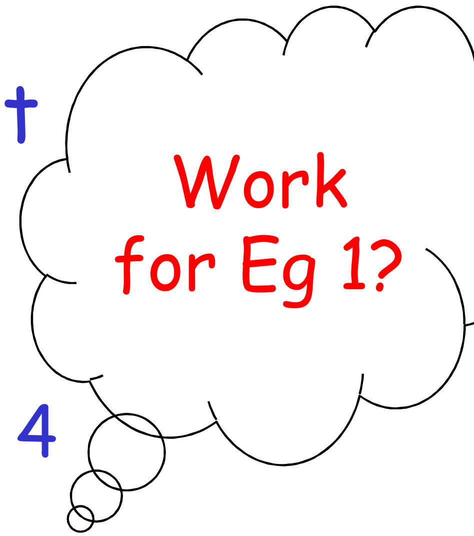
Result:

- item 1 is taken, total value = 42, total weight = 7
- item 3 cannot be taken
- item 4 cannot be taken
- item 2 is taken, total value = 54, total weight = 10

Greedy approach 2



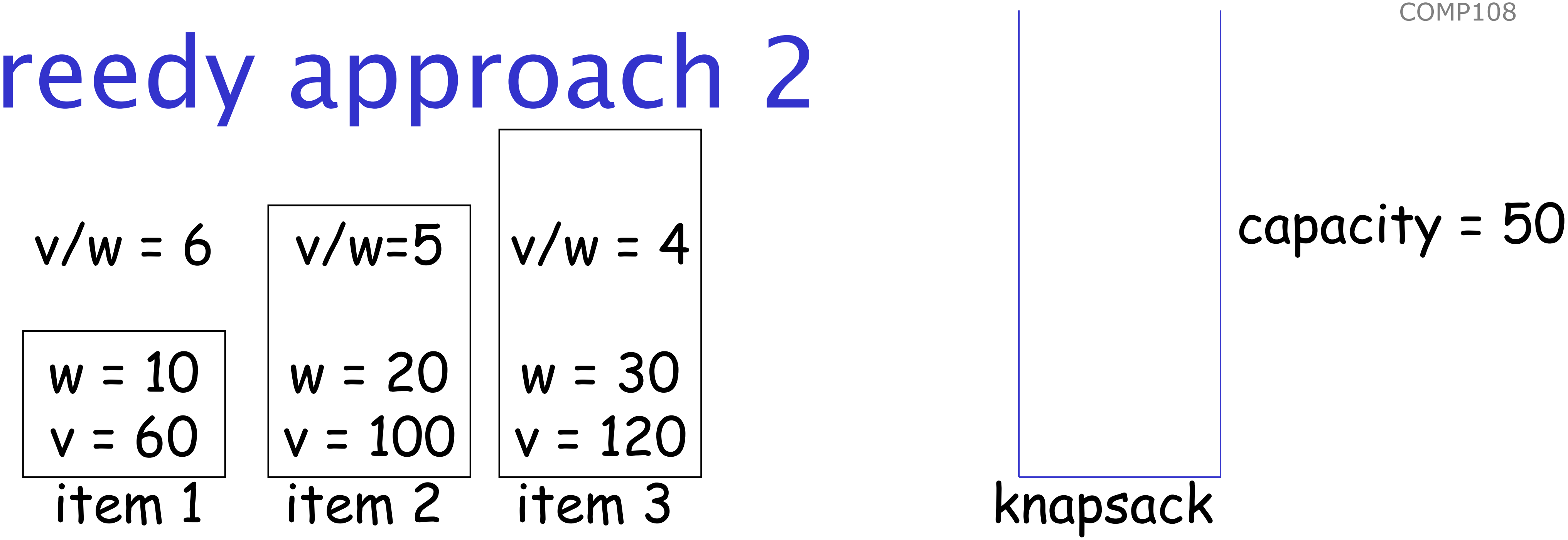
Greedy 2: pick the item with the next largest (value/weight) if total weight ≤ capacity.



Result:

- item 3 is taken, total value = 40, total weight = 4
- item 1 cannot be taken
- item 4 is taken, total value = 65, total weight = 9
- item 2 cannot be taken

Greedy approach 2



Greedy: pick the item with the next largest (value/weight) if total weight ≤ capacity.

Result:

- item 1 is taken, total value = 60, total weight = 10
- item 2 is taken, total value = 160, total weight = 30
- item 3 cannot be taken

Lesson Learned: Greedy algorithm does **NOT** always return the best solution