COMP108
Algorithmic Foundations

Greedy methods

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http://www.csc.liv.ac.uk/~pwong/teaching/comp108/201617

Coin Change Problem

Suppose we have 3 types of coins

10p 20p 50p

Minimum number of coins to make £0.8, £1.0, £1.4 ?

Greedy method

Learning outcomes

- Understand what greedy method is
- Able to apply Kruskal's algorithm to find minimum spanning tree
- Able to apply Dijkstra's algorithm to find single-source shortest-paths
- Able to apply greedy algorithm to find solution for Knapsack problem

Greedy methods

How to be greedy?

- At every step, make the best move you can make
- Keep going until you're done

Advantages

- Don't need to pay much effort at each step
- Usually finds a solution very **quickly**
- The solution found is usually **not bad**

Possible problem

- The solution found may **NOT** be the best one

Greedy methods - examples

Minimum spanning tree

- Kruskal's algorithm

Single-source shortest-paths

- Dijkstra's algorithm

Both algorithms find one of the BEST solutions

Knapsack problem

- greedy algorithm does NOT find the BEST solution

Kruskal's algorithm ...

Minimum Spanning tree (MST)

Given an undirected connected graph G

- The edges are labelled by weight

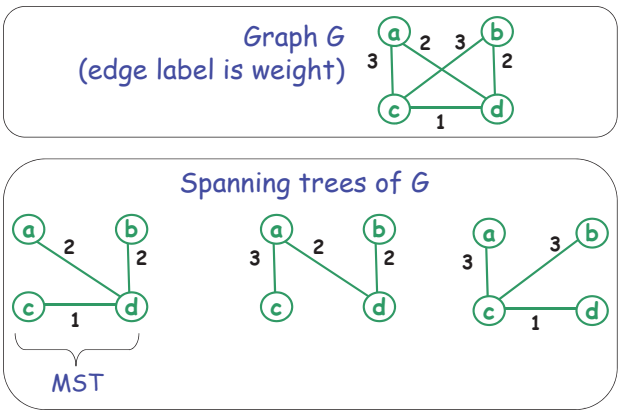
Spanning tree of G

- a tree containing all vertices in G

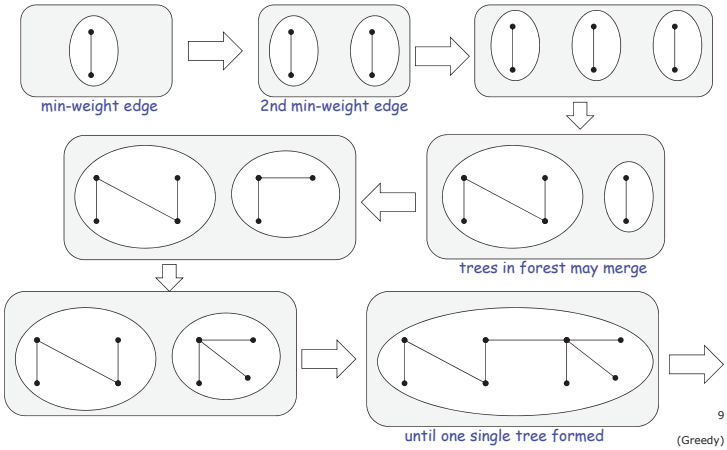
Minimum spanning tree of G

- a spanning tree of G with minimum weight

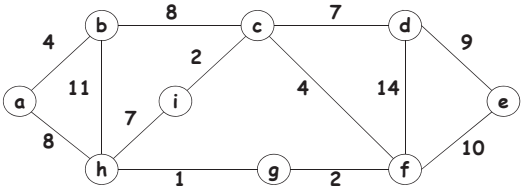
Examples



Idea of Kruskal's algorithm - MST



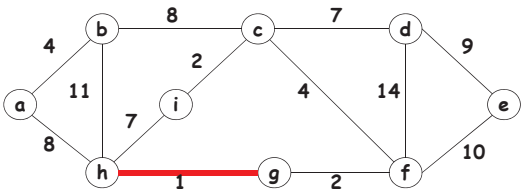
Kruskal's algorithm - MST



Arrange edges from smallest to largest weight

(h,g)	1
(i,c)	2
(g,f)	2
(a,b)	4
(c,f)	4
(c,d)	7
(h,i)	7
(b,c)	8
(a,h)	8
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

Kruskal's algorithm - MST



Choose the minimum weight edge

(h,g)	*1*
(i,c)	2
(g,f)	2
(a,b)	4
(c,f)	4
(c,d)	7
(h,i)	7
(b,c)	8
(a,h)	8
(d,e)	9
(f,e)	10
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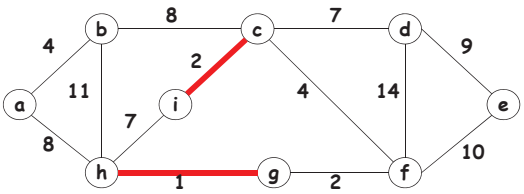
italic: chosen

Kruskal's algorithm - MST



Choose the next minimum weight edge

(h,g)	*1*
(i,c)	*2*
(g,f)	2
(a,b)	4
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(c,d)	7
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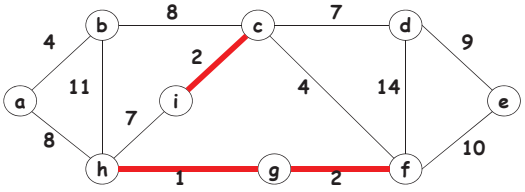
Kruskal's algorithm - MST



Continue as long as no cycle forms

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	4
(c,f)	4
(c,d)	7
(h,i)	7
(b,c)	8
(a,h)	8
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

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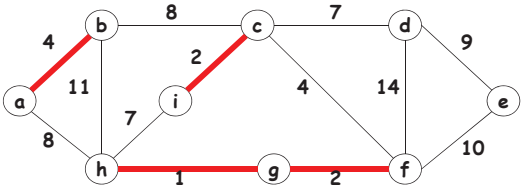
Kruskal's algorithm - MST



Continue as long as no cycle forms

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	4
(c,d)	7
(h,i)	7
(b,c)	8
(a,h)	8
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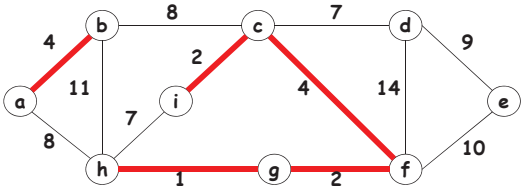
Kruskal's algorithm - MST



Continue as long as no cycle forms

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	7
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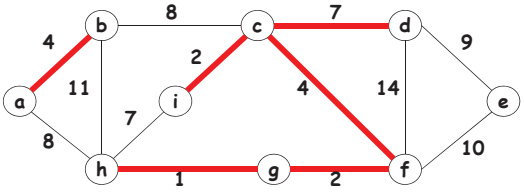
Kruskal's algorithm - MST



Continue as long as no cycle forms

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
(h,i)	7
(b,c)	8
(a,h)	8
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

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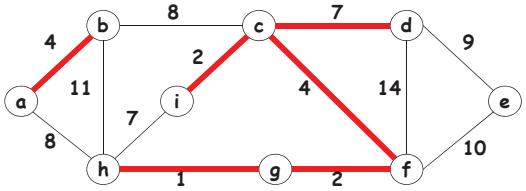
Kruskal's algorithm - MST



(h,i) cannot be included, otherwise, a cycle is formed

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~*(h,i)*~~	~~*7*~~
(b,c)	8
(a,h)	8
(d,e)	9
(f,e)	10
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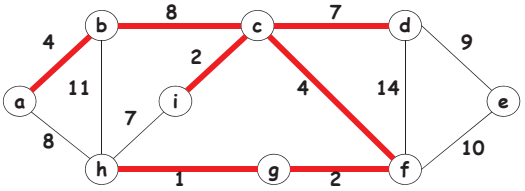
Kruskal's algorithm - MST



Choose the next minimum weight edge

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~*(h,i)*~~	~~*7*~~
(b,c)	*8*
(a,h)	8
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(f,e)	10
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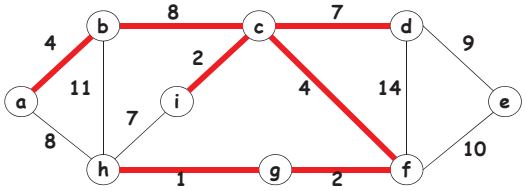
italic: chosen

Kruskal's algorithm - MST

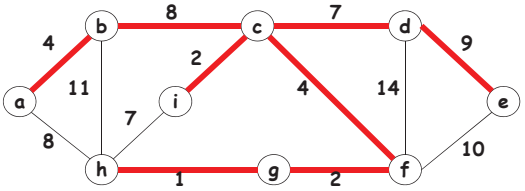


(a,h) cannot be included, otherwise, a cycle is formed

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~*(h,i)*~~	~~*7*~~
(b,c)	*8*
~~*(a,h)*~~	~~*8*~~
(d,e)	9
(f,e)	10
(b,h)	11
(d,f)	14

italic: chosen

Kruskal's algorithm - MST

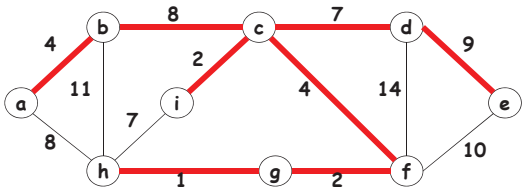


Choose the next minimum weight edge

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~*(h,i)*~~	~~*7*~~
(b,c)	*8*
~~*(a,h)*~~	~~*8*~~
(d,e)	*9*
(f,e)	10
(b,h)	11
(d,f)	14

italic: chosen

Kruskal's algorithm - MST

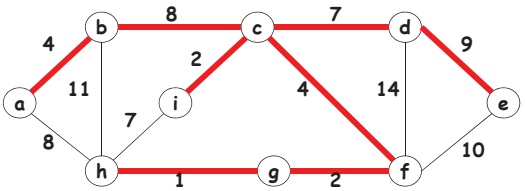


(f,e) cannot be included, otherwise, a cycle is formed

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~*(h,i)*~~	~~*7*~~
(b,c)	*8*
~~*(a,h)*~~	~~*8*~~
(d,e)	*9*
~~*(f,e)*~~	~~*10*~~
(b,h)	11
(d,f)	14

italic: chosen

Kruskal's algorithm - MST

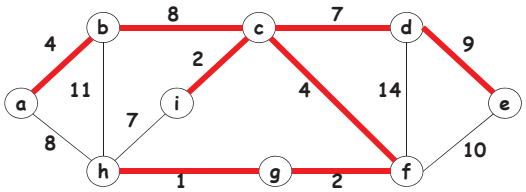


(b,h) cannot be included, otherwise, a cycle is formed

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~*(h,i)*~~	~~*7*~~
(b,c)	*8*
~~*(a,h)*~~	~~*8*~~
(d,e)	*9*
~~*(f,e)*~~	~~*10*~~
~~*(b,h)*~~	~~*11*~~
(d,f)	14

italic: chosen

Kruskal's algorithm - MST

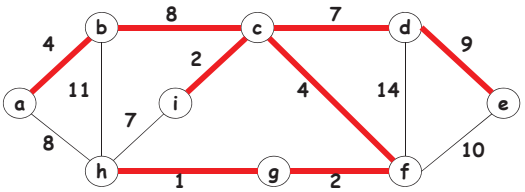


(d,f) cannot be included, otherwise, a cycle is formed

(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~*(h,i)*~~	~~*7*~~
(b,c)	*8*
~~*(a,h)*~~	~~*8*~~
(d,e)	*9*
~~*(f,e)*~~	~~*10*~~
~~*(b,h)*~~	~~*11*~~
~~*(d,f)*~~	~~*14*~~

italic: chosen

Kruskal's algorithm - MST



MST is found when all edges are examined

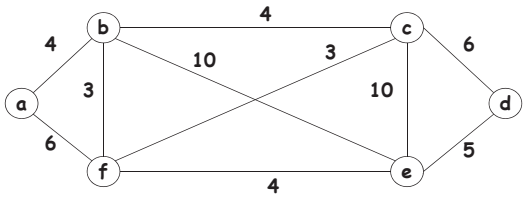
(h,g)	*1*
(i,c)	*2*
(g,f)	*2*
(a,b)	*4*
(c,f)	*4*
(c,d)	*7*
~~*(h,i)*~~	~~*7*~~
(b,c)	*8*
~~*(a,h)*~~	~~*8*~~
(d,e)	*9*
~~*(f,e)*~~	~~*10*~~
~~*(b,h)*~~	~~*11*~~
~~*(d,f)*~~	~~*14*~~

italic: chosen

Kruskal's algorithm - MST

Kruskal's algorithm is **greedy** in the sense that it always attempt to select the **smallest** weight edge to be included in the MST

Exercise – Find MST for this graph



order of (edges) selection:

Pseudo code

// Given an undirected connected graph G=(V,E)
T = Ø and E' = E
while E' ≠ Ø **do**
begin
pick an edge e in E' with minimum weight
if adding e to T does not form cycle **then**
add e to T, i.e., T = T ∪ { e }
remove e from E', i.e., E' = E' \ { e }
end

Time complexity?

Can be tested by marking vertices

Dijkstra's algorithm ...

Single-source shortest-paths

Consider a (un)directed connected graph G

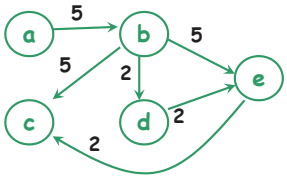
- The edges are labelled by weight

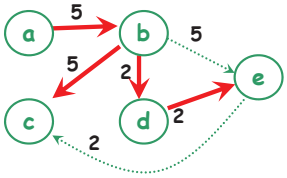
Given a particular vertex called the ***source***

- Find **shortest paths** from the source to all other vertices (shortest path means the total weight of the path is the smallest)

Example

Directed Graph G
(edge label is weight)
a is source vertex

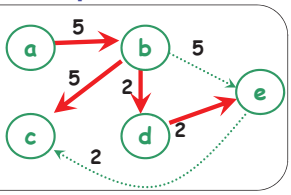




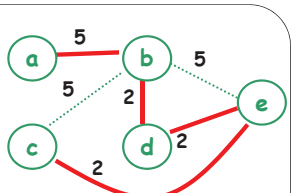
thick lines: shortest path
dotted lines: not in shortest path

Single-source shortest paths vs MST

Shortest paths from a



What is the difference between MST and shortest paths from a?



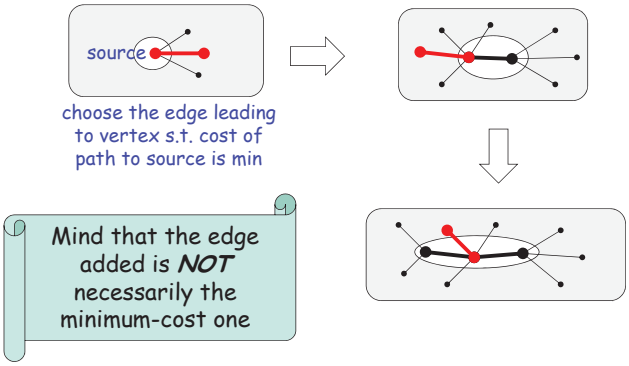
MST

Algorithms for shortest paths

Algorithms

- there are many algorithms to solve this problem, one of them is **Dijkstra's** algorithm, which assumes the weights of edges are **non-negative**

Idea of Dijkstra's algorithm



Dijkstra's algorithm

Input: A directed connected weighted graph G and a source vertex s

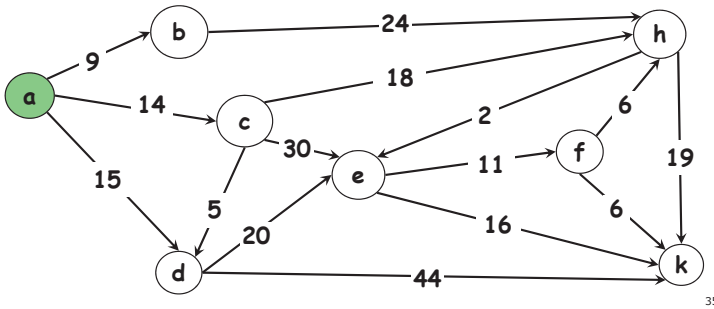
Output: For every vertex v in G, find the shortest path from s to v

Dijkstra's algorithm runs in iterations:

- in the i-th iteration, the vertex which is the i-th closest to s is found,
- for every remaining vertices, the current shortest path to s found so far (this shortest path will be updated as the algorithm runs)

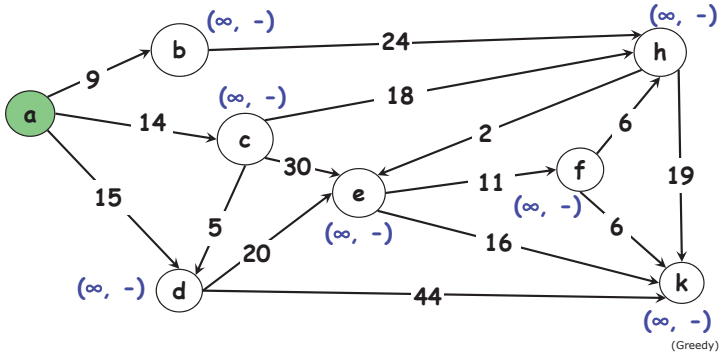
Dijkstra's algorithm

Suppose vertex a is the source, we now show how Dijkstra's algorithm works



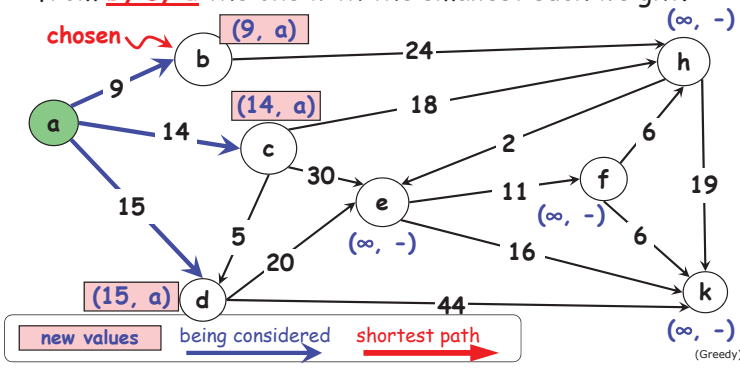
Dijkstra's algorithm

Every vertex v keeps 2 labels: (1) the weight of the current shortest path from a; (2) the vertex leading to v on that path, initially as (∞, -)



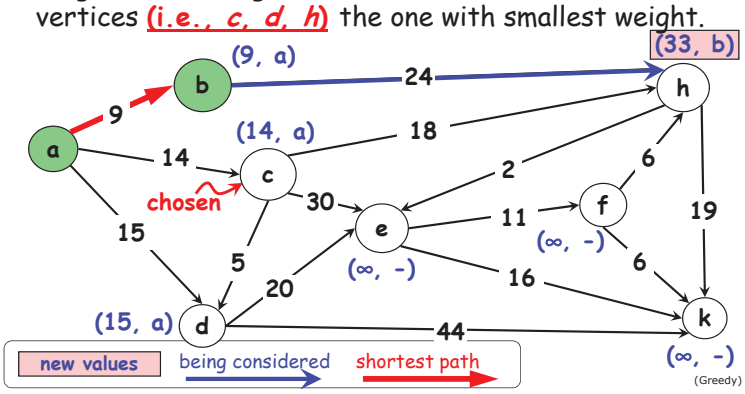
Dijkstra's algorithm

For every neighbor u of a, update the weight to the weight of (a,u) and the leading vertex to a. Choose from b, c, d the one with the smallest such weight.



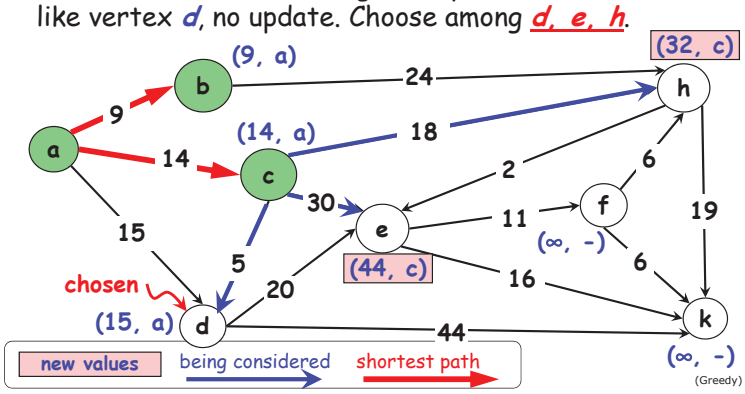
Dijkstra's algorithm

For every un-chosen neighbor of vertex b, update the weight and leading vertex. Choose from **ALL** un-chosen vertices (i.e., c, d, h) the one with smallest weight.



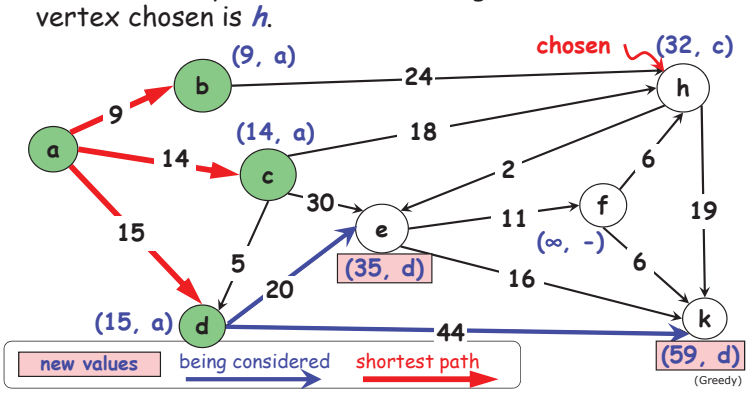
Dijkstra's algorithm

If a new path with smallest weight is discovered, e.g., for vertices e, h, the weight is updated. Otherwise, like vertex d, no update. Choose among d, e, h.



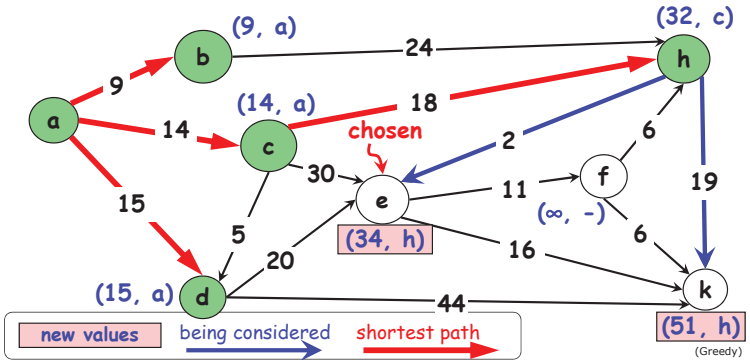
Dijkstra's algorithm

Repeat the procedure. After d is chosen, the weight of e and k is updated. Choose among e, h, k. Next vertex chosen is h.



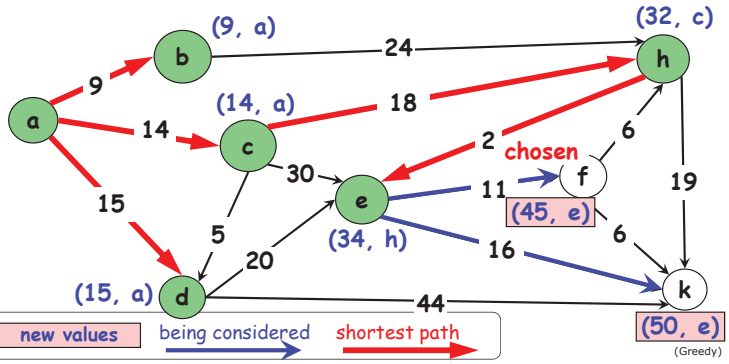
Dijkstra's algorithm

After *h* is chosen, the weight of *e* and *k* is updated again. Choose among *e*, *k*. Next vertex chosen is *e*.



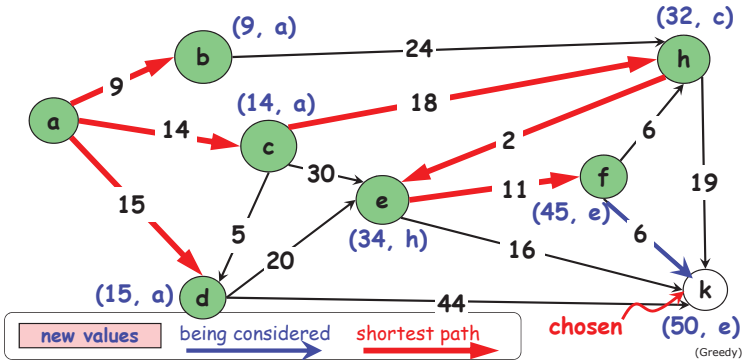
Dijkstra's algorithm

After *e* is chosen, the weight of *f* and *k* is updated again. Choose among *f*, *k*. Next vertex chosen is *f*.



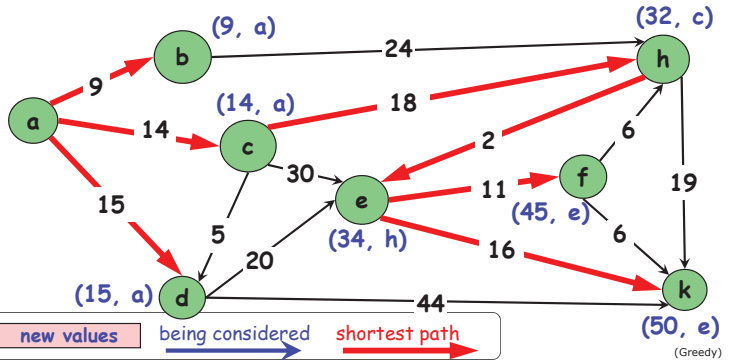
Dijkstra's algorithm

After *f* is chosen, it is NOT necessary to update the weight of *k*. The final vertex chosen is *k*.

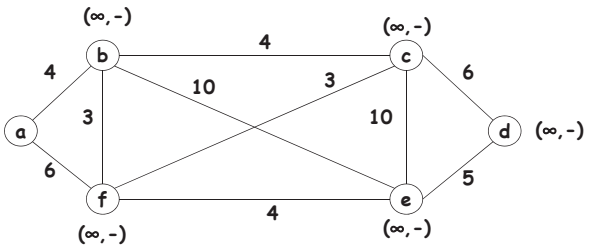


Dijkstra's algorithm

At this point, all vertices are chosen, and the shortest path from *a* to every vertex is discovered.



Exercise – Shortest paths from a



order of (edges) selection:

Compare the solution with slide #26

Dijkstra's algorithm

To describe the algorithm using pseudo code, we give some notations

Each vertex *v* is labelled with two labels:

- a **numeric label** *d(v)* indicates the length of the shortest path from the source to *v* found so far
- another label *p(v)* indicates next-to-last vertex on such path, i.e., the vertex immediately before *v* on that shortest path

Pseudo code

```
// Given a graph G=(V,E) and a source vertex s
for every vertex v in the graph do
    set d(v) = ∞ and p(v) = null
set d(s) = 0 and V_T = ∅
while V \ V_T ≠ ∅ do      // there is still some vertex left
begin
    choose the vertex u in V \ V_T with minimum d(u)
    set V_T = V_T ∪ { u }
    for every vertex v in V \ V_T that is a neighbor of u do
        if d(u) + w(u,v) < d(v) then    // a shorter path is found
            set d(v) = d(u) + w(u,v) and p(v) = u
end
```

Time complexity?

Does Greedy algorithm always return the best solution?

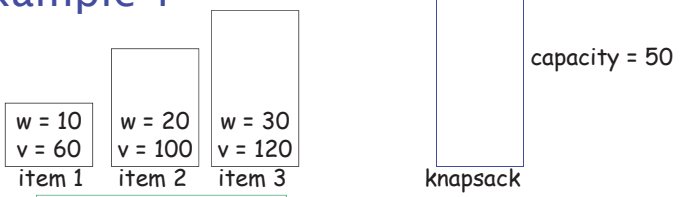
Knapsack Problem

Input: Given n items with weights $w_1, w_2, \ldots, w_n$ and values $v_1, v_2, \ldots, v_n$, and a knapsack with capacity W.

Output: Find the most valuable subset of items that can fit into the knapsack

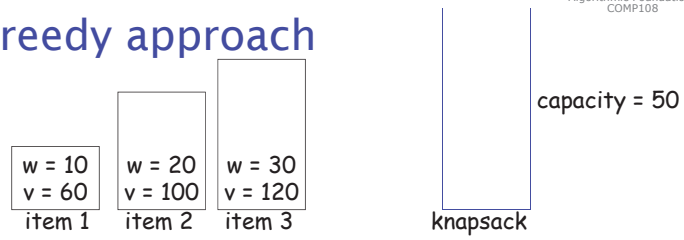
Application: A transport plane is to deliver the most valuable set of items to a remote location without exceeding its capacity

Example 1



subset	total weight	total value
ϕ	0	0
{1}	10	60
{2}	20	100
{3}	30	120
{1,2}	30	160
{1,3}	40	180
{2,3}	**50**	**220**
{1,2,3}	60	N/A

Greedy approach



Greedy: pick the item with the next largest value if total weight ≤ capacity.

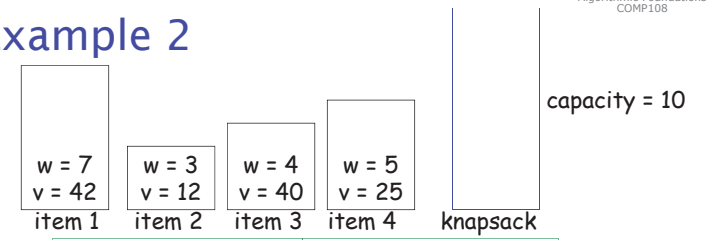
Time complexity?

Result:

- item 3 is taken, total value = 120, total weight = 30
- item 2 is taken, total value = 220, total weight = 50
- item 1 cannot be taken

Does this always work?

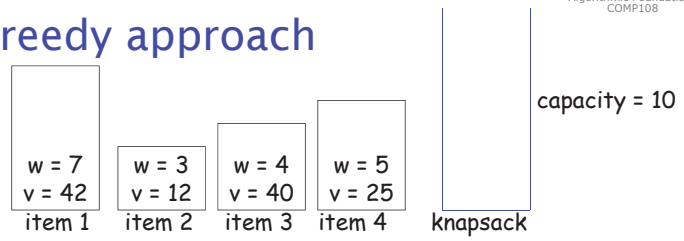
Example 2



subset	total weight	total value
ϕ	0	0
{1}	7	42
{2}	3	12
{3}	4	40
{4}	5	25
{1,2}	10	54
{1,3}	11	N/A
{1,4}	12	N/A

subset	total weight	total value
{2,3}	7	52
{2,4}	8	37
{3,4}	**9**	**65**
{1,2,3}	14	N/A
{1,2,4}	15	N/A
{1,3,4}	16	N/A
{2,3,4}	12	N/A
{1,2,3,4}	19	N/A

Greedy approach



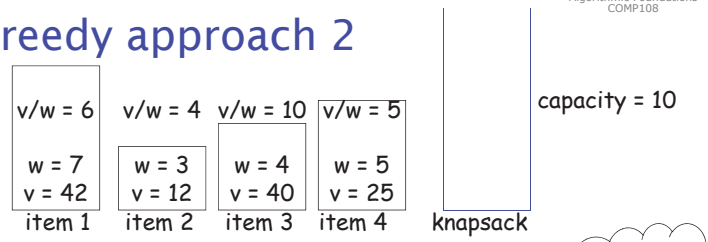
Greedy: pick the item with the next largest value if total weight ≤ capacity.

Result:

- item 1 is taken, total value = 42, total weight = 7
- item 3 cannot be taken
- item 4 cannot be taken
- item 2 is taken, total value = 54, total weight = 10

not the best!!

Greedy approach 2



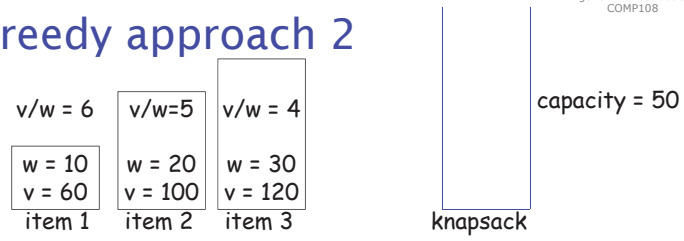
Greedy 2: pick the item with the next largest (value/weight) if total weight ≤ capacity.

Work for Eg 1?

Result:

- item 3 is taken, total value = 40, total weight = 4
- item 1 cannot be taken
- item 4 is taken, total value = 65, total weight = 9
- item 2 cannot be taken

Greedy approach 2



Greedy: pick the item with the next largest (value/weight) if total weight ≤ capacity.

Result:

- item 1 is taken, total value = 60, total weight = 10
- item 2 is taken, total value = 160, total weight = 30
- item 3 cannot be taken

Not the best!!

Lesson Learned: Greedy algorithm does NOT always return the best solution