COMP108 Algorithmic Foundations

Tutorial 2 (Suggested Solution and Feedback) w/c 13th February 2017

1. The problem can be solved by finding the LCM (lowest common multiples) first and then outputting all multiples of the LCM which is at most 100.

```
lcm = x, found = false
while found == false do
begin
    if lcm % y == 0 then
        found = true
    else lcm = lcm + x
end
day = lcm
while day ≤ 100 do
begin
    output day
    day = day + lcm
end
```

Alternatively, we can test each multiple of x and see if each of them is a multiple of y.

```
day = x
while day ≤ 100 do
begin
    if day%y == 0 then
        output day
    day = day + x
end
```

2. **Base case:** When $n = 0$, $L.H.S. = 2^0 = 1$, $R.H.S. = 2^1 - 1 = 1$.
Therefore, the property holds for $n = 0$

Induction hypothesis: Assume that the property holds for some $k \geq 0$,
i.e., assume $2^0 + 2^1 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$.

Induction step: When $n = k + 1$,

$$\begin{aligned} L.H.S. &= 2^0 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} \qquad \text{(by induction hypothesis)} \\ &= 2 \times 2^{k+1} - 1 = 2^{(k+1)+1} - 1 = R.H.S. \end{aligned}$$

Therefore, the property also holds for $n = k + 1$.

By the principle of mathematical induction, the property holds for all integers $n \geq 0$.

N.B.: Remember to put down where the induction hypothesis is used.

Important: It is WRONG if you put equal signs before/while you prove the induction step like the following:

$$\begin{aligned}
2^0 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} &= 2^{(k+1)+1} - 1 && \text{WRONG!} \\
(2^{k+1} - 1) + 2^{k+1} &= 2 \times 2^{k+1} && \text{WRONG!} \\
2^{k+1} + 2^{k+1} &= 2 \times 2^{k+1} && \text{WRONG!} \\
2 \times 2^{k+1} &= 2 \times 2^{k+1} && \text{WRONG!}
\end{aligned}$$

You should work on L.H.S. and R.H.S. independently.

3. ***Base case:*** When $n = 1$, L.H.S. $= 1$ R.H.S. $= 1^2 = 1$

   Therefore, L.H.S. = R.H.S.

   Induction hypothesis: Assume that the property holds for some positive integer k, i.e., assume $1 + 3 + 5 + \cdots + (2k - 1) = k^2$, for some positive integer $k \geq 1$.

   Induction step: When $n = k+1$, we want to prove $1+3+5+\cdots+(2k-1)+(2(k+1)-1) = (k+1)^2$.

$$\begin{aligned}
\text{L.H.S.} &= 1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) \\
&= k^2 + (2(k + 1) - 1) \qquad \text{by hypothesis} \\
&= k^2 + 2k + 2 - 1 \\
&= k^2 + 2k + 1 \\
&= (k + 1)^2 \\
&= \text{R.H.S.}
\end{aligned}$$

   Therefore, L.H.S. = R.H.S. and the property holds for $n = k + 1$.

   By the principle of mathematical induction, the property holds for all positive integers $n \geq 1$.

4. Group the 9 balls evenly into 3 groups, A, B and C. We first weigh twice to find out which group is abnormal and whether it is heavier or lighter. **Weigh A and B: (1)** If they weigh the same, the abnormal ball is in C. We can then find out whether it is heavier or lighter by **weighing A and C. (2)** If A and B are different, all balls in C are normal. Then **weigh A and C**: (a) if same, the abnormal ball is in B and we can tell whether it is heavier or lighter from the outcome of the first weigh; (b) if different, the abnormal ball is in A and we can tell whether it is heavier or lighter from the second weigh (the first weigh as well).

   Then we are left with a group of 3 balls and we know whether it is heavier or lighter. Suppose it is heavier (the case for lighter is similar). Weigh two balls against each other. If different, the heavier one is the abnormal one. If same, the third ball is the abnormal (heavier) one.

5. **Base case:** When $n = 0$, $L.H.S. = (4\times0+1) = 1$, $R.H.S. = (0+1)(2\times0+1) = 1 = L.H.S.$
Therefore, property holds for $n = 0$

Induction hypothesis: Assume the property holds for some $k \geq 0$,
i.e., assume $1 + 5 + 9 + \ldots + (4k + 1) = (k + 1)(2k + 1)$,

Induction step: When $n = k + 1$,

$$\begin{aligned}
L.H.S. &= 1 + 5 + 9 + \cdots + (4k + 1) + (4(k + 1) + 1) \\
&= (k + 1)(2k + 1) + (4k + 5) \qquad \text{(by induction hypothesis)} \\
&= 2k^2 + 3k + 1 + 4k + 5 = 2k^2 + 7k + 6 \\
R.H.S. &= ((k + 1) + 1)(2(k + 1) + 1) \\
&= (k + 2)(2k + 3) = 2k^2 + 7k + 6 = L.H.S.
\end{aligned}$$

Therefore, the property also holds for $n = k + 1$.

By the principle of mathematical induction, the property holds for all integers $n \geq 0$.