*COMP323 – Introduction to Computational Game Theory*

# Strategic Network Formation

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# Outline


1. Strategic network formation: concepts
2. One-sided link formation
3. Two-sided link formation
4. Basic network creation games

1. Strategic network formation: concepts
   - Introduction
   - One-sided links
   - Two-sided links
2. One-sided link formation
3. Two-sided link formation
4. Basic network creation games

# An economic theory of network formation

- Network structure has profound effects on individual behavior and payoffs as well as on aggregate social outcomes.
- Individual entities will have an incentive to form connections with others in advantageous ways.
- This suggests an economic approach in which individuals will trade off the costs and benefits of forming connections.

$\Rightarrow$ We will overview the basic elements and some applications of this economic theory of network formation.

# Network formation games

A network formation game specifies

1. a set of players;
2. the link formation actions available to each player; and
3. the payoffs to each player from the networks that arise out of individual linking decisions.

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3. the payoffs to each player from the networks that arise out of individual linking decisions.

How can a model of network formation be developed?

Basic issues:

- specification of the individual and social values arising from networks
- decision power of individuals in determining links.

# Network formation games

## Models

Individual payoffs:

How should the returns to a player $i$ from a link with $j$ be defined?

- every player pays her share of the costs of linking, or
- the division of costs is derived from a bargaining process.

# Network formation games

## Models

Decision power:

Which players should have the power to decide on a link between $i$ and $j$?

- any player can decide to form the link, or
- only the two players directly involved have a say, or
- the existing current partners of $i$ and $j$ should also have some say (since a link between $i$ and $j$ alters their payoffs).

# Network formation games

## Equilibrium

**Definition (Strategically stable network)**

A network is said to be strategically stable or in equilibrium if there are no incentives for individual players (either acting alone or in groups) to form or delete links and thereby alter the network.

Research has focused on single- and two-player incentives.

- Invoking the incentives and coordination among large groups is not appropriate for networks amenable to subtle and quick transformations via local linking and de-linking activity.
- Networks are complicated objects and even with single- or two-person moves the analysis of network formation is intricate.

# One-sided links

One-sided links (the simplest scenario):

- Each player can form links with any subset of players.
- Link formation is unilateral: an individual $i$ can decide to form a link with $j$ by paying for the link.
- The payoffs to a player depend on her own links as well as the links that other players form.

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Motivation:

- it leads to a simple link formation model which captures essential economic trade-offs and can be analyzed with concepts such as Nash equilibria;
- it is descriptively appropriate in some important instances: forming links on the WWW, citations, making gifts...

# Two-sided links

Two-sided links:

- A link between $i$ and $j$ requires the assent of both the individuals.
- The analysis of bilateral link formation games requires concepts that go beyond Nash equilibrium.
- Pairwise stability looks at incentives to add or delete a single link.
- Other concepts of stability include networks stable w.r.t. addition and deletion of several links by many players.

1 Strategic network formation: concepts

2 One-sided link formation
- A simple model of link formation
- Related themes

3 Two-sided link formation

4 Basic network creation games

# One-sided link formation

We begin with the following one-sided link formation model:

- There are $n$ individuals, each of whom can form a link with any subset of the remaining players.
- Link formation is unilateral: an individual $i$ can decide to form a link with any player $j$ by paying for the link.

# One-sided link formation

Attractive features of the model:

1. It is simple and permits an exploration of the process of network formation.
2. It allows for a fairly general description of payoffs: they are assumed to be
   - increasing in the number of other people accessed via the network and
   - decreasing in the numbers of links formed by an individual.
3. It yields sharp predictions on the architecture of networks: equilibrium networks can be completely characterized.

# A simple model of link formation

Formally:

- The set of players is $N = \{1, \ldots, n\}$ with $n \geq 3$.
- A (pure) strategy of player $i \in N$ is a (row) vector

$$s_i = (s_{i,1}, \ldots, s_{i,i-1}, s_{i,i+1}, \ldots, s_{i,n})$$

  where $s_{i,j} \in \{0, 1\}$ for each $j \in N \setminus \{i\}$.
- Player $i$ has a link with $j$ iff $s_{i,j} = 1$.
- The set of (pure) strategies of player $i$ is $S_i = \{0, 1\}^{n-1}$.
- A strategy profile is denoted by $s = (s_1, \ldots, s_n) \in S = \times_{i \in N} S_i$.

# A simple model of link formation

- There is an equivalence between a strategy profile and a directed network on $n$ nodes.
- Links are one-sided:
  - they can be formed on individual initiative;
  - the individual forming the link incurs the cost of forming links.
  - Links can be seen as information channels.
- Benefits flow between two players so long as one of the two has formed a link with the other, as in, e.g.,
  - a telephone call between two players;
  - a social relation involving gifts and reciprocal favors.

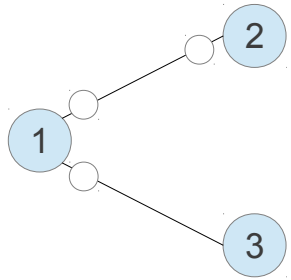
# A simple model of link formation

To capture this two-way flow model:

- For a strategy profile $g$, define $\hat{g}_{ij} = \max\{g_{ij}, g_{ji}\}$.
- Given $g$, it is straightforward to define a corresponding network $\hat{g}$, using the above operation.
- Link $\hat{g}_{ij} = 1$ is represented by an edge between $i$ and $j$.
- A circle on the edge near player $i$ indicates that it is the player who has initiated the link.

# Example

Example: Network representation of $\hat{g}$




- player 1 has formed links with players 2 and 3;
- player 2 has formed a link with player 1;
- player 3 does not link up with any other player.

Note:

- Since players choose strategies independently, two players may simultaneously initiate a two-way link.
- Every strategy profile $g$ has a unique representation in the manner shown in the figure.

# Payoffs

- Denote the set of nonnegative integers by $\mathbb{Z}_+$.
- Let $\phi : \mathbb{Z}_+^2 \to \mathbb{R}$ be a real-valued function.
- Define each player's payoff function $\Pi_i : \mathcal{G} \to \mathbb{R}$ as

$$\Pi_i(g) = \phi(\hat{n}_i(g), \eta_i(g)) \ ,$$

where

- $\mathcal{G}$ is the set of networks on $n$ nodes;
- $\hat{n}_i(g)$ is the number of players accessed by $i$ in the undirected version of $g$ (i.e., the players to whom $i$ has a path, including herself);
- $\eta_i(g)$ is the number of links formed by player $i$ in network $g$.

# Payoffs

**Assumption**

The payoff function $\Pi_i(g) = \phi(\hat{n}_i(g), \eta_i(g))$ is strictly increasing in $\hat{n}_i(g)$ and strictly decreasing in $\eta_i(g)$.

- $\hat{n}_i(g)$ is naturally interpreted as the number of people that player $i$ accesses in the network.
- $\eta_i(g)$ measures the cost associated with maintaining her links.
- This assumption about $\phi(\cdot, \cdot)$ is quite mild.

# Payoffs

An example of linear payoffs satisfying the assumption:

$$\Pi_i(g) = \hat{n}_i(g) - \eta_i(g) \cdot c \ .$$

The ranges of parameter $c$ have an important role:

- If $c \in (0, 1)$, then player $i$ will be willing to form a link with player $j$ for the sake of $j$'s value alone.
- If $c \in (1, n-1)$, player $i$ will require $j$ to access some additional players to induce her to form a link with $j$.
- If $c > n-1$, then the cost of link formation exceeds the total access benefit available from the rest of society: it is optimal for $i$ not to form a link with any player, no matter what the other players do.

# Nash networks

**Theorem (Bala and Goyal, 2000)**

*Suppose that the payoffs are given by*

$$\Pi_i(g) = \phi(\hat{n}_i(g), \eta_i(g))$$

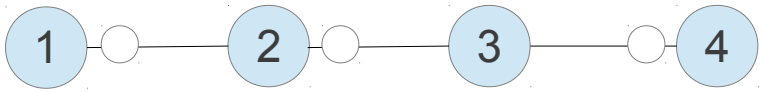
*and $\phi$ is strictly increasing in $\hat{n}_i(g)$ and strictly decreasing in $\eta_i(g)$. Then a Nash network is either empty or minimally connected.*

Note:

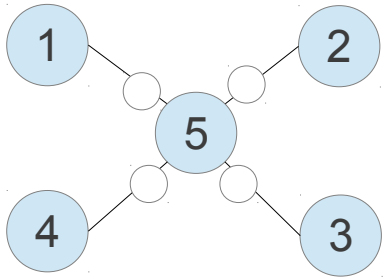
- An empty network has $n$ nodes and no links.
- A minimally connected network has one and only one "undirected" path between every pair of players.

# Nash networks

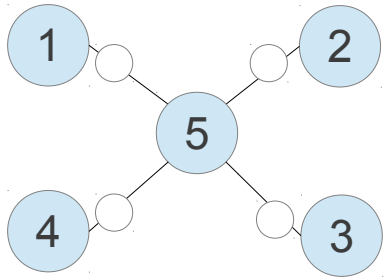
## Examples of minimally connected networks:



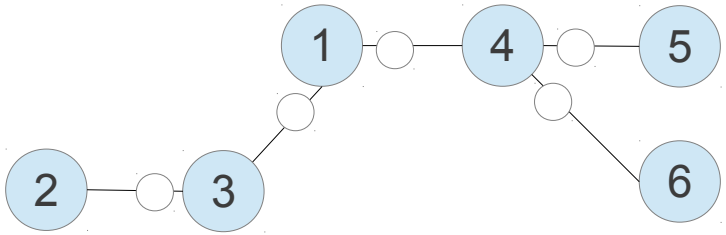

A line network



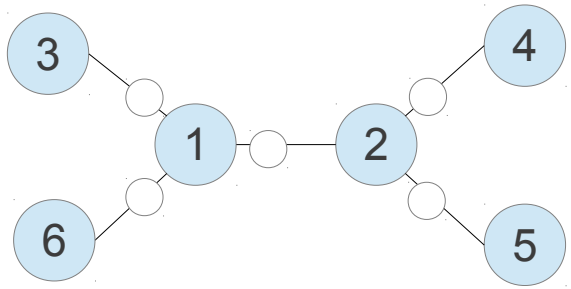

Center-sponsored star




Periphery-sponsored star




Line + star network



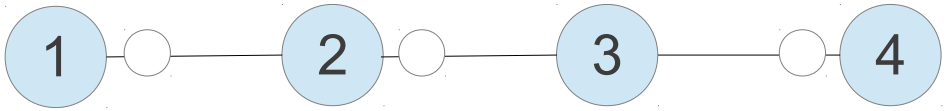

Interlinked star

# Nash networks

- Minimality is a direct consequence of the assumption that the advantages that $i$ enjoys form accessing $j$ do not depend on the length of the paths in the network.
- A non-empty network with two or more components can not be in equilibrium: Consider a network with two components.
  - There is at least one player in each component who forms a link. Number these players 1 and 2. W.l.o.g., suppose player 1 earns weakly higher payoff.
  - If player 2 imitates the linking strategy of 1, she will access all players that 1 is accessing, while she will form the same number of links as 1.
  - Given that the payoff functions are the same and strictly increasing in the number of players accessed, player 2 can strictly increase her payoff by deleting all existing links and instead imitating player 1. □

# Strict Nash networks

Minimally connected networks have very different structures.
Are they equally robust? Consider the line network:




- Player 1 would be indifferent between linking with 2 or 3.
- In a dynamic context, this suggests that society may move out of the line network, purely by chance.
- This possibility raises a number of questions:
  - Are there networks not vulnerable to such a switching argument?
  - If individuals revise links over time, does the process converge?
  - If yes, then what are the stable networks?

# Strict Nash networks

The notion of strict equilibrium addresses the first question:

**Definition**

A Nash equilibrium is strict if every player is playing a strategy which yields payoffs that are strictly higher than the payoffs from an alternative strategy.

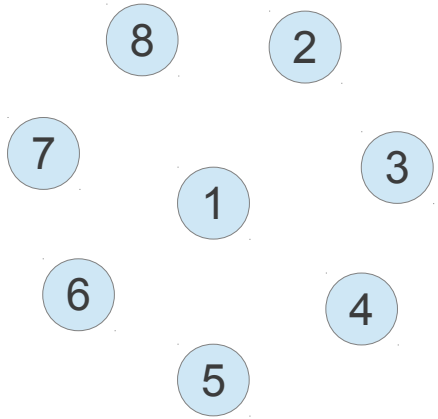
The following provides a complete characterization of the architecture of strict Nash networks:

**Theorem (Bala and Goyal, 2000)**

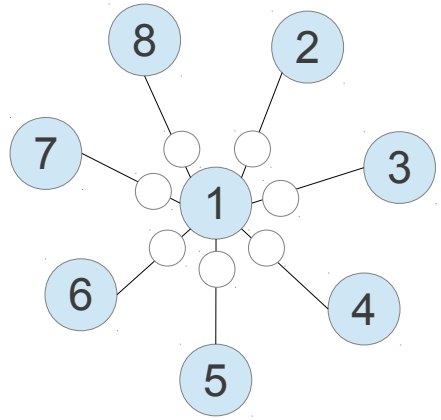
*Suppose that the payoffs are given by* $\Pi_i(g) = \phi(\hat{n}_i(g), \eta_i(g))$ *and* $\phi$ *is strictly increasing in* $\hat{n}_i(g)$ *and strictly decreasing in* $\eta_i(g)$*. Then a Nash network is either a center-sponsored star or the empty network.*

# Strict Nash networks

Example: Strict equilibrium networks for $n = 8$.




The empty network




Center-sponsored star

In the linear payoffs case where $\Pi_i(g) = \hat{n}_i(g) - \eta_i(g) \cdot c$:

- If $0 < c < 1$ then the center-sponsored star is the unique strict equilibrium.
- If $c > 1$ then the empty network is the unique strict equilibrium.

# Strict Nash networks

**Theorem (Bala and Goyal, 2000)**

*A Nash network is either a center-sponsored star or the empty network.*

This result tells us that strategic considerations in link formation lead to networks with

- small average degree;
- an unequal degree distribution; and
- short average distances between players.

Proof. The proof rests on a simple **switching argument**.

# Strict Nash networks

Proof (continued).

- If player $i$ has a link with $j$, no other player can have a link with $j$.
  - If $j$ did have a link with some player $k$, then $i$ would be indifferent between forming a link with $k$ or forming a link with $j$.
  - This implies that a strategy involving a link with $j$ is not strictly better than all other strategies.
  - Thus such pattern of links cannot be supported in a strict Nash equilibrium.
- Similarly, if $i$ has a link with $j$, no other player can have a link with $i$.
- Since the network is a nonempty equilibrium network, it must be connected.
- So player $i$ must form links with all players, so the network is a center-sponsored star.

□

# Strict Nash networks

**Theorem (Bala and Goyal, 2000)**

*A Nash network is either a center-sponsored star or the empty network.*

- The above theorem shows that individual incentives restrict the range of possible network architectures.
- This characterization of equilibrium networks leads to the questions:
  - How do individuals choose links if they start with some other network?
  - Is there some pressure moving the network to the equilibrium networks?
- In other words, will individuals learn to coordinate their link formation activity and arrive at a star network?

# Myopic best-response

Network formation dynamics:

- Individuals periodically get an opportunity to revise their links (form new, delete or maintain existing).
- Decisions on linking at any point in time will be guided by a number of considerations:
  - The structure of the existing network determines potential rewards from different links.
  - The possibility that current linking activity will alter incentives for others is also important.
- Simplest and best understood case: **myopic best-response**.

# Myopic best-response

- Time is a discrete variable indexed by $t = 1, 2, 3 \ldots$.
- $g^t$ is the network (strategy profile) at the start of time $t$.
- In each period, with probability $p \in (0, 1)$, a player gets the opportunity to revise her strategy.
- She is assumed to know the network, and she chooses a set of links maximizing her payoff, assuming that the links of other players do not change.
- If more than one strategy is optimal, she randomizes across the optimal strategies.
- A strategy of player $i$ in period $t$ is denoted by $g_i^t$.
- If $i$ is not active in period $t$, it follows that $g_i^t = g_i^{t-1}$.

# Myopic best-response

This simple best-response strategy revision rule generates a transition probability function

$$P_{g,g'} : \mathcal{G} \to \mathcal{G} \to [0,1] \quad \text{with} \quad \sum_{g'} Pg, g' = 1 \ .$$

- This is a Markovian dynamic process in which the probability of moving from a state (network) $g$ at time $t$ to state $g'$ at time $t+1$ depends only on the state at time $t$.
- A state $g$ is absorbing if the dynamic process cannot escape from the state once it reaches it, i.e.,

$$P_{g,g} = 1 \ .$$

# Myopic best-response

Note:

- In general, the myopic best-response process may not converge.
  - In such cases, we talk of absorbing set(s) of networks: a collection of networks such that once the process reaches one of the networks in it, then it remains within this set forever.
  - The set of all networks $\mathcal{G}$ is clearly an absorbing set of the process.
- In some applications, there exist multiple absorbing states.
  - This leads to investigate the relative stability of different networks.
- Practically, all the research on dynamic models of network formation assumes myopic players.
  - However, individual linking alters the incentives for linking by others.
  - This suggests that players should wait or preempt others in linking decisions.
  - The characterization of networks arising when players are far-sighted is an open problem.

# Myopic best-response

Myopic best-response dynamics are well-behaved for our one-sided link formation, two-way flow model:

**Theorem (Bala and Goyal, 2000)**

*Suppose that the payoffs are given by* $\Pi_i(g) = \phi(\hat{n}_i(g), \eta_i(g))$ *and* $\phi$ *is strictly increasing in* $\hat{n}_i(g)$ *and strictly decreasing in* $\eta_i(g)$.

- *If* $\phi(x+1, y+1) > \phi(x, y)$ *for all* $y \in \{0, 1, \ldots, n-2\}$ *and* $x \in \{y+1, y+2, \ldots, n-1\}$, *then the dynamic process converges to the center-sponsored star in finite time.*
- *If* $\phi(x+1, y+1) < \phi(x, y)$ *for all* $y \in \{0, 1, \ldots, n-2\}$ *and* $x \in \{y+1, y+2, \ldots, n-1\}$, *then the dynamic process converges to the empty network in finite time.*

For the linear model: if $0 < c < 1$ the process converges to the center-sponsored star, and if $c > 1$ it converges to the empty network.

# Myopic best-response

Proof. The proof uses three general arguments:

1. Starting with any network, the dynamics either converge to the empty network or a minimally connected network.
2. Starting from any minimally connected connected network, peripheral players de-link from "remote" players and move towards a central player. This results in a network with short average distance.
3. Miscoordination in linking arises when two players $i$ and $j$ linked to a common player $k$ simultaneously choose to delete their link with $k$ and form a link with each other instead.
   - This leads to $i$ and $j$ forming a redundant link among themselves and isolating themselves from $k$.
   - This facilitates the sponsoring of additional links by player $k$, which, in due course, leads to the emergence of a center-sponsored star.

□

# Efficient networks

We will examine the nature of networks that maximize aggregate welfare, i.e., the sum of individual payoffs.

Observations:

- Efficient networks are minimal (follows from the assumptions that access benefits do not depend on the length of the path between players).
- Disconnected networks can be efficient for some payoff functions:

**Example**

Suppose $n = 3$ and $\phi(1,0) = 6.4, \phi(2,0) = 7, \phi(3,0) = 7.1, \phi(2,1) = 6, \phi(3,1) = 6.1, \phi(3,2) = 0$ (satisfying the assumptions). Then a network $g$ in which $g_{1,2} = 1$ and $g_{i,j} = 0$ for all other pairs of players is efficient.

- This complicates a characterization of efficient networks.

# Efficient networks

However: If the payoff function is such that the marginal gains from accessing a player are greater than the marginal costs of forming the link, then an efficient network is connected:

**Proposition**

Any efficient network is minimal. If $\phi(x+1, y+1) \geq \phi(x+y)$ for all $y \in \{0, 1, \ldots, n-2\}$ and $x \in \{y+1, y+2, \ldots, n-1\}$, then an efficient network is connected.

Proof.

- Minimality follows from the lack of decay (only the existence and not the lengths of paths affect payoffs).
- The rest of the result follows directly from the conditions on payoffs.

□

# Distributional properties

- The distribution of costs is such that the center pays for all the links and therefore gets a much lower payoff than the peripheral players.
- In the linear model (payoffs are given by $\Pi_i(g) = \hat{n}_i(g) - \eta_i(g) \cdot c$)
  - the center earns $(n-1)(1-c)+1$ and
  - the peripheral players each earn $n$, so for large values of $c$ and $n$ the payoff difference can be very large.
- To summarize, the analysis of the basic model of link formation yields that (strict) equilibrium networks
  - are stars or are empty;
  - exhibit significant payoff inequality.

# Related themes

- Strategic network formation has been a very active area of research and a number of themes have been explored.
- The two-way flow model we have presented offers a simple and general framework to discuss three of these themes:
  1. local network knowledge;
  2. imperfect reliability of links;
  3. alternative models of benefit flow.

# Local knowledge of the network

- In the basic model, individuals choose links simultaneously and once links are in place everyone can observe the entire network $\Longrightarrow$ **focus on Nash equilibrium**.
- However, in many applications (the WWW, coauthor networks, networks of friends...) the number of players is very large and individuals have only local knowledge about the network.
- Individuals typically know their own neighbors and have limited knowledge of the rest of the network.
- Does limited knowledge lead individuals to form very different networks than they would form under global knowledge?

# Local knowledge of the network

The answer is yes.

- Suppose payoffs are given by $\Pi_i(g) = \hat{n}_i(g) - \eta_i(g) \cdot c$ (linear case) and assume $c < 1$.
- In the full information case, a Nash network is minimally connected and a strict Nash network is a center-sponsored star.
- (McBride, 2006) shows that, with local knowledge of the network structure, cycles and disconnected networks can be supported in equilibrium.
- Thus local knowledge can lead to equilibrium networks which may be over-connected or under-connected relative to the equilibrium networks under full information.

# Reliability

- In the basic model, it is assumed that links convey benefits with certainty.
- In many contexts, e.g., communication or flow of traffic over the Internet, a link might not function perfectly.
- Reliability of links is an important issue and affects equilibrium outcomes and socially optimal structures.
- (Bala and Goyal, 2000) formulated reliability in terms of a probability $r \in [0, 1]$ that a link works.
- We assume that the probability of successful link operation is independent across links.

# Reliability

- Given a network $g$, the probability of a sub-network $g' \subset g$ being operational or being realized is

$$\lambda(g' \mid g) = r^{L(g')}(1-r)^{L(g)-L(g')}$$

where $L(\cdot)$ is the number of links in a network.

- The payoff to $i$ from $g$ can be written as

$$\Pi_i(g) = \sum_{g' \subset g} \lambda(g' \mid g) \cdot \hat{n}_i(g') - \eta_i(g) \cdot c \ .$$

# Reliability

- For fixed $r$, if the number of players is large enough, equilibrium networks and efficient networks will be super-connected: remain connected upon the deletion of any single link.
- Intuitively, consider the star network:
  - A peripheral player depends on a single link for accessing the entire network.
  - Given $r$ and a cost of forming links $c$, the marginal value from linking with another peripheral player is larger than the cost of the link, for large enough $n$.
  - A similar argument also implies that efficient networks are also super-connected.

# Flow of benefits

- We have focused so far on the case where flow of benefits is two-way.
- In some settings, e.g., the WWW, benefits accrue to players who incur the cost for setting up the link.
- This leads to a model of one-way flow of benefits.
- The payoff to player $i$ is given by

$$\Pi_i(g) = \phi(n_i(g), \eta_i(g))$$

where $n_i(g)$ is the number of nodes accessed by $i$ and $\eta_i(g)$ is the number of links set up by $i$.

- $\phi(\cdot, \cdot)$ is assumed to be strictly increasing in $n_i(g)$ and strictly decreasing in $\eta_i(g)$.
- Under these assumptions, we will show that

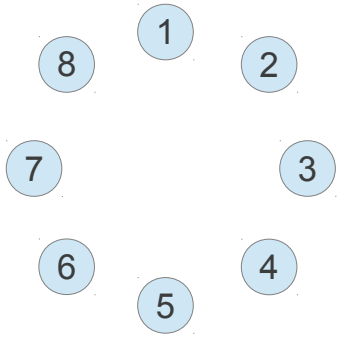
**Theorem**

*A strict Nash network is either the cycle or the empty network.*

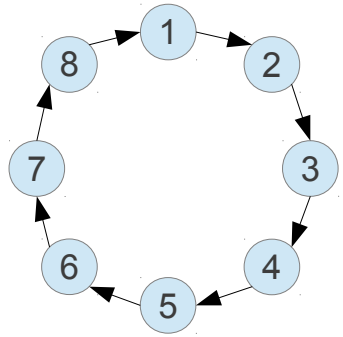
# One-way flow of benefits

## Examples

- Equilibrium networks for $n = 8$ (a link formed by $i$ to $j$ is represented by an arrow towards $j$):




The empty network




The cycle network

# One-way flow of benefits

## Examples

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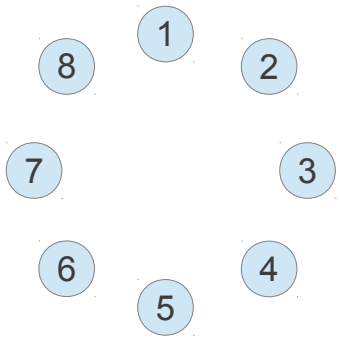
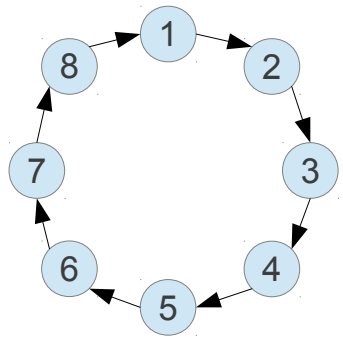



The empty network




The cycle network

- In the linear model, where $\Pi_i(g) = n_i(g) - \eta_i(g) \cdot c$, player $i$'s payoff is the number of players she observes minus the total cost of link formation, and:
  - If $c < 1$ the cycle is the unique equilibrium network.
  - If $1 < c < n - 1$ the cycle and the empty network are the only equilibria.
  - If $c > n - 1$ the empty network is the unique equilibrium network.

# One-way flow of benefits

**Theorem**

*A strict Nash network is either the cycle or the empty network.*

Proof. The proof is based on two main arguments:

1. A Nash network is either empty or connected. But there are many connected networks with a wide range of architectures.
2. By exploiting the refinement of strictness all connected networks other than the cycle are ruled out.
   - If $i$ and $j$ have a link with the same player $k$, then $i$ will be indifferent between forming a link with $k$ or instead forming a link with $j$.
   - Thus every player has **one and only one** player who initiates a link with her.
   - So a (nonempty) strict Nash network has exactly $n$ links.
   - The cycle is the only connected directed network with exactly $n$ links.

□

1 Strategic network formation: concepts

2 One-sided link formation

3 Two-sided link formation
- The model
- Pairwise stability
- Extensions of pairwise stability

4 Basic network creation games

# Two-sided link formation

In many social and economic contexts the creation of a link between two individuals requires that both agree to the link.

Examples:

- friendships,
- coauthorships in research papers,
- collaboration between firms,
- free trade agreements between nations.

Questions:

1. What is the architecture of the resulting networks?
2. Are these networks socially efficient?
3. What are the distributional properties of the networks that arise?

# Two-sided link formation

The simplest way to think about two-sided link formation is to imagine an announcement game:

- Each player announces a set of intended links.
- An intended link is a binary variable $s_{ij} \in \{0, 1\}$ where $s_{ij} = 1$ means that $i$ intends to form a link with $j$.
- A (pure) strategy for $i$ is $s_i = (s_{ij})_{j \neq i}$.
- A strategy profile for all players is denoted $s = (s_i)_{i \in N}$.
- Define $g_{ij} = \min\{s_{ij}, s_{ji}\}$. Clearly, $g_{ij} = 1$ iff $s_{ij} = s_{ji} = 1$.
- Therefore every strategy profile $s$ induces a corresponding undirected network $g(s)$.

# The connections model

The connections model provides a simple example of such a game:

- Let $\delta \in [0,1]$ be the rate of decay in value as it moves across links.
- Given a strategy profile $s$, the payoff to player $i$ in network $g = g(s)$ is

$$\Pi_i(g) = 1 + \sum_{j \in N_i(g), j \neq i} \delta^{d(i,j)} - \eta_i(g)c \ ,$$

where

- $c > 0$ is the cost of forming a link,
- $d(i,j)$ is the distance between $i$ and $j$ in $g$,
- $N_i(g)$ is the set of players to whom $i$ has a path, and
- $\eta_i(g)$ is the number of connections (links) of $i$ in $g$.

# The connections model

$$\Pi_i(g) = 1 + \sum_{j \in N_i(g), j \neq i} \delta^{d(i,j)} - \eta_i(g)c$$

- Player $i$ incurs the cost of a link $ij$ only if this link is actually formed, i.e., both players announce an intention to form it.
- Setting $\delta = 0$ implies there is full decay and no flow of value.
- Setting $\delta = 1$ implies the absence of any decay.
- The payoff captures the idea that links facilitate the flow of benefits, but for $\delta \in (0, 1)$ there is some decay/delay in flow as paths get longer.
- This induces players to form links with others to shorten paths while balancing the costs of forming links.

# Nash equilibria

What is the architecture of networks that will arise?

- A link is formed iff both players want it.
- If every player announces she wants to form no links, then a best response of $i$ is to announce she also wants to form no links.
- So the empty network is a Nash equilibrium.
- More generally, for any pair $i$ and $j$, it is always a best response for both to offer to form no link.
- It seems natural to suppose that pairs of players can communicate and agree to form a link as long as they both benefit from it.

# Pairwise stability

The concept of pairwise stability formalizes the idea of two-person deviation:

**Definition**

A network $g$ is pairwise stable if

1. for every $g_{ij} = 1$, $\Pi_i(g) \geq \Pi_i(g - g_{ij})$ and $\Pi_j(g) \geq \Pi_j(g - g_{ij})$;
2. for every $g_{ij} = 0$, $\Pi_i(g + g_{ij}) > \Pi_i(g - g_{ij}) \implies \Pi_j(g + g_{ij}) < \Pi_j(g)$.

- Pairwise stability looks at the attractiveness of links in $g$ one at a time.
- The first condition requires that every present link in a stable network must be (weakly) profitable for the players involved.
- The second condition requires that for every link not present in the stable network, if one player gains from the link, then the other must be strictly worse off.

# Pairwise stability

## Improving paths

Notes:

- The concept of pairwise stability is very simple.
- For any network, it is relatively easy to check whether the two conditions are satisfied.

# Pairwise stability

## Improving paths

Notes:

- The concept of pairwise stability is very simple.
- For any network, it is relatively easy to check whether the two conditions are satisfied.

But does a pairwise stable network always exist?

### Definition

An improving path from a network $g$ to a network $g'$ is a finite sequence of networks $g = g^1, g^2, \ldots, g^k = g'$ such that, for every $l \in \{1, 2, \ldots, k-1\}$,

1. either $g^{l+1} = g^l - g_{ij}$ for some $g_{ij}^l = 1$ and $\Pi_k(g^l - g_{ij}) > \Pi_k(g^l)$ for $k \in \{i, j\}$,
2. or $g^{l+1} = g^l + g_{ij}$ for some $g_{ij}^l = 0$ and $\Pi_i(g^l + g_{ij}) > \Pi_i(g^l)$ and $\Pi_j(g^l + g_{ij}) \geq \Pi_j(g^l)$.

# Pairwise stability

## Improving paths

**Definition**

A set of networks $\hat{\mathcal{G}}$ forms a cycle if for any $g, g' \in \hat{\mathcal{G}}$ (which includes $g' = g$) there exists an improving path from $g$ to $g'$.

- A sufficient condition for the existence of pairwise stable networks is that there does not exist an improving path starting from every network.
- In a game with a finite number of players (and thus networks) a sufficient condition for this is that there are no cycles of improving paths in the network.

# Pairwise stable networks of the connections model

Recall that, in the connections model, the payoff of player $i$ in network $g$ is

$$\Pi_i(g) = 1 + \sum_{j \in N_i(g), j \neq i} \delta^{d(i,j)} - \eta_i(g)c \ .$$

**Theorem**

- *A pairwise stable network has at most one nonsingleton component,*
- *For $c < \delta - \delta^2$, the unique pairwise stable network is the complete network.*
- *For $\delta - \delta^2 < c < \delta$, a star is a pairwise stable network.*
- *For $\delta < c$, the empty network is pairwise stable and any nonempty pairwise stable network is such that each player has at least two links.*

# Pairwise equilibrium

Weakness of pairwise stability:

- Pairwise stability is a useful first check for strategic stability, but only a small set of possible deviations are ruled out.
- Only the deletion of a single link or the addition of a single link is considered.
- It is easy to construct games in which the deletion or addition of a link is not possible, but the deviation in which several links are deleted together is profitable.

# Pairwise equilibrium

Example: Deleting a subset of links.

- Suppose $n = 4$.
- Assume payoffs satisfy $\Pi_i(g) = 15$ for all networks $g$ in which player $i$ has no links, and $\Pi_i(g^c) = 10$ for the complete network $g^c$.
- Also, in a network $g$ where two players, say 1 and 2, have three links each and two players, say 3 and 4, have two links each, the payoffs to players 1 and 2 are 9 while the payoffs to players 3 and 4 are 8.
- Then the complete network is pairwise stable, since no player has an incentive to delete a single link.
- However, a player would strictly profit from deleting all links.

# Pairwise equilibrium

The notion of pairwise equilibrium supplements the idea of Nash equilibrium with the requirement that no pair of players wishes to form an additional link.

**Definition**

A network $g^*$ can be sustained in a pairwise equilibrium if

1. there is a Nash equilibrium $s^*$ which yields $g^*$;
2. for any $g_{ij}^* = 0$,

$$\Pi_i(g^* + g_{ij}^*) > \Pi_i(g^*) \implies \Pi_j(g^* + g_{ij}^*) < \Pi_j(g^*) \ .$$

# Bilateral equilibrium

- Pairwise stability does not allow deviations in which a pair of players each deletes one or more links and/or adds a link in a coordinated manner.
- In some games it is possible that
  - deleting a subset of links is not profitable for any single player and
  - adding a link is not profitable for any pair of players either, but
  - it is profitable for a pair of players to simultaneously delete a subset of their links and add a link.

# Bilateral equilibrium

Example: Assume $n = 4$ and payoffs are such that:

- any isolated player earns 0;
- in a line network the two central players earn 25 while end-players earn 10 each;
- in a cycle network every player earns 20;
- in a network with a cycle and an additional link the payoffs to the players with two links each are 20 while the payoffs to the three-link players are 15 each.

Then it follows that the cycle is a pairwise equilibrium.
However, two players in the cycle can make a coordinated move:

- This move yields a line network with themselves as the central players.
- Their payoffs are thereby increased.
- Thus the cycle is not stable with respect to coordinated deviations.

# Bilateral equilibrium

The notion of bilateral equilibrium addresses coordinated deviations.

**Definition**

A network $g^*$ is a bilateral equilibrium if

- There is a Nash equilibrium strategy profile $s^*$ which yields $g^*$.
- For any pair of players $i, j \in N$ and every strategy pair $(s_i, s_j)$,

$$\Pi_i(g(s_i, s_j, s^*_{-\{ij\}})) > \Pi_i(g(s^*)) \implies \Pi_j(g(s_i, s_j, s^*_{-\{ij\}})) < \Pi_j(g(s^*)) .$$

- Thus a network can be supported in a bilateral equilibrium if no player or pair of players can deviate (unilaterally or bilaterally, respectively) and benefit from the deviation (at least one of them strictly).
- It is simply a special case of strong equilibrium in which only two-player deviations are allowed.

1 Strategic network formation: concepts

2 One-sided link formation

3 Two-sided link formation

4 Basic network creation games
- Introduction and motivation
- Trees
- Sum version
- Max version
- A model of local costs

# Basic network creation games: Motivation

Nash equilibria in network formation games studied so far are actually unrealistic:

- Each player has an exponential number of pure strategies ($2^{n-1}$).
- Computationally bounded agents cannot even tell if they are in a Nash equilibrium.
- Thus they cannot tell whether they want to change their local strategy while not changing all others.
- In a network creation game of computationally bounded agents, it is much more reasonable to assume that an agent can only weigh a constant or sublogarithmic number of edges against each other.

# Network creation games

In a network creation game, several players (the nodes) collectively attempt to build an efficient network that interconnects everyone. Each player has two (selfish) goals:

1. to minimize the cost spent building links (creation cost) and
2. to minimize the average or maximum distance to all other nodes (usage cost).

# Network creation games

In a network creation game, several players (the nodes) collectively attempt to build an efficient network that interconnects everyone.
Each player has two (selfish) goals:

1. to minimize the cost spent building links (creation cost) and
2. to minimize the average or maximum distance to all other nodes (usage cost).

Focus is on equilibria: locally stable networks in which no agent can greedily improve their situation by changing the network.

- There are many types of equilibria depending on the types of moves allowed.
- In Nash equilibria, each player can change her entire strategy, while fixing all other players' strategies.
- This notion corresponds to one node deleting and/or adding any number of incident edges owned by the node.

# Basic network creation games

A basic form of the game was proposed by (Alon et al, 2010):

- The cost of every edge is the same.
- Each player performs edge swaps: replacing an existing (incident) edge with another (incident) edge, whenever that swap improves the player's usage cost.
- As a special case, the agent can swap an edge with an already existing edge, which corresponds to deleting an (extraneous) edge.
- We require swap equilibrium, that no edge swap decreases a player's usage cost.
- Swap equilibrium is a weaker notion since it only considers single edge swaps instead of changing all incident edges; but it is stronger since it allows a node to swap any incident edge, not just those that it owns.

# Basic network creation games

Formally, we define two basic network creation games, by considering two possible definitions of the usage cost of a node.

- **Sum version:** a graph is in sum equilibrium if, for every edge $vw$ and every node $w_0$, swapping edge $vw$ with edge $vw_0$ does not decrease the total sum of distances from $v$ to all other nodes.
- **Max version:** A graph is in max equilibrium if, for every edge $vw$ and every node $w_0$, swapping edge $vw$ with edge $vw_0$ does not decrease the local diameter of $v$, and furthermore, deleting edge $vw$ strictly increases the local diameter of $v$.
  The local diameter of a vertex $v$ is the maximum distance between $v$ and any other vertex.

Note that both equilibria can be detected easily, even locally, in polynomial time: try every possible edge swap and deletion.

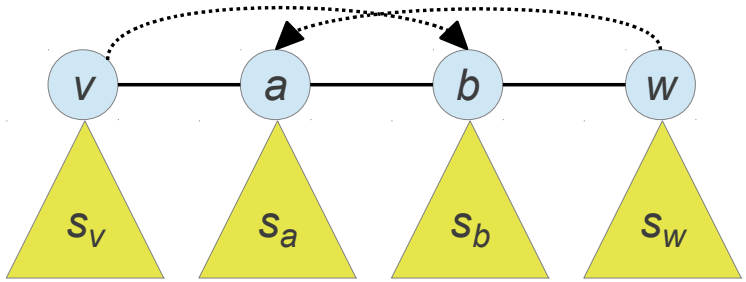
# Equilibrium trees, sum version

We start by analyzing equilibrium trees for the sum version.
There is essentially only one equilibrium tree:

**Theorem**

*If a sum equilibrium graph in the basic network creation game is a tree, then it has diameter at most 2, and thus is a star.*

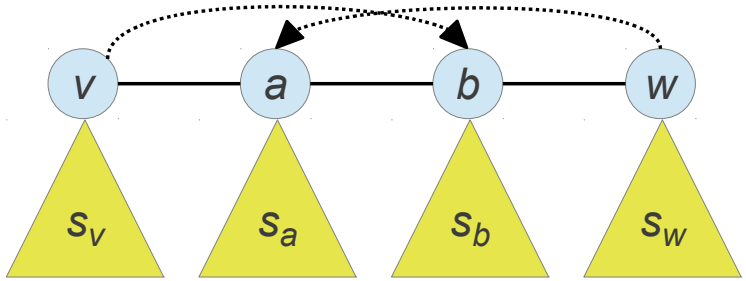
Proof. Suppose for contradiction that an equilibrium tree has diameter at least 3:




- It has two vertices $v, w$ at distance exactly 3, inducing a length 3 shortest path $v \to a \to b \to w$.

# Equilibrium trees, sum version

Proof (continued). Let $s_v, s_a, s_b, s_w$ denote the size of the subtrees rooted at $v$, $a$, $b$, $w$ (including the roots).



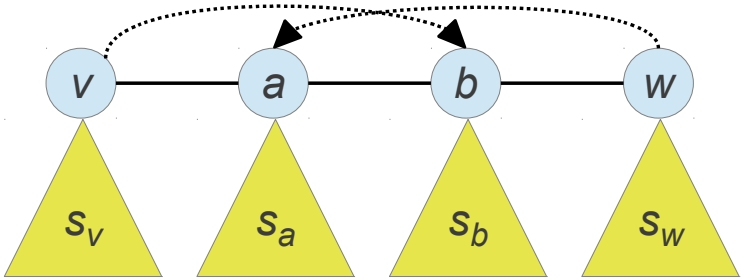

Consider two possible swaps:

1. $v$ replaces its edge to $a$ with an edge to $b$, and
2. $w$ replaces its edge to $b$ with an edge to $a$.

The first swap ($vb$ instead of $va$) improves $v$'s distance to $b$'s and $w$'s subtrees by 1 (the unique shortest path in the tree no longer having to pass through $a$), and worsens $v$'s distance to $a$'s subtrees by 1; thus, the swap is a net win unless $s_b + s_w \leq s_a$.

# Equilibrium trees, sum version

Proof (continued).

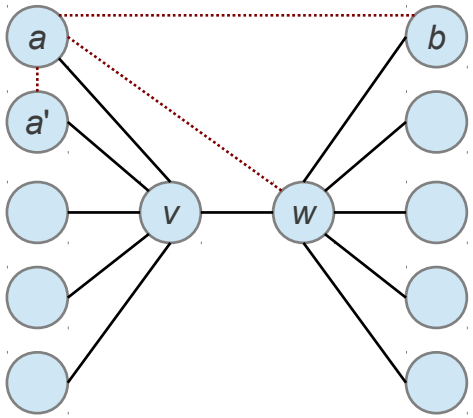



- Similarly, the second swap ($wa$ instead of $wb$) is a net win unless $s_v + s_a \leq s_b$.
- For both swaps to not be net wins, we must have both inequalities.
- Summing these inequalities, we obtain that $s_v + s_a + s_b + s_w \leq s_a + s_b$, i.e., $s_v + s_w \leq 0$, contradicting that $s_v + s_w \geq 2$ (because in particular they count $v$ and $w$). □

Obviously, diameter 2 can also be achieved (and is optimal), as evidenced by the star.

# Equilibrium trees, max version

In contrast, max-equilibrium trees can have diameter as high as 3.
Example: A tree of diameter 3 that is in max equilibrium.




- There are three types of edges we might try to add: from a leaf $a$ to a leaf $a'$ or to a leaf $b$ or to the other root $w$.
- The only option that decreases some local diameter is adding $aw$ which decreases the local diameter of $a$ (but not $w$) by 1.
- In any swap around $a$, however, this addition must be combined with the deletion of edge $av$, which restores the original local diameter of $a$.

# Equilibrium trees, max version

We will show that this diameter of 3 is the maximum possible. We first need the following lemma:

**Lemma**

*In any max-equilibrium graph, the local diameters of any two nodes differ by at most 1.*

Proof.

- Suppose $v$ has local diameter $d$ while $w$ has local diameter $\geq d + 2$.
- Let $T$ be a breadth-first search tree from $v$.
- $w$ prefers to swap its edge to its parent in $T$ with an edge to $v$ (the root of $T$): this swap only decreases the depths of nodes in $T$, so the local diameter of $v$ remains at most $d$.
- $w$'s local diameter decreases to at most $d + 1$, because $w$ can take a unit step to reach $v$ and then follow $v$'s path to any other node.
- This swap contradicts being in max-equilibrium. □

# Equilibrium trees, max version

A vertex $v$ of a connected graph is a cut vertex if $G - v$ (the resulting graph after deleting $v$) is disconnected.

**Lemma**

*If a max equilibrium graph has a cut vertex $v$, then only one connected component of $G - v$ can have a vertex of distance more than 1 from $v$.*

Proof.

- Let $d$ be the local diameter of $v$ and let $w$ be a vertex at distance $d$ from $v$, and let $W$ be the connected component of $G - v$ that contains $w$.
- Suppose for contradiction that there is a vertex $x$ in $G - W$ of distance more than 1 from $v$.
- Then any path from $x$ to $W$ must pass through $v$, so it is at least 2 longer than the corresponding path from $v$.
- Therefore the local diameter of $w$ is at least $d + 2$, contradicting the previous lemma.

# Equilibrium trees, max version

**Theorem**

*If a max equilibrium graph in the basic network-creation game is a tree, then it has diameter at most 3.*

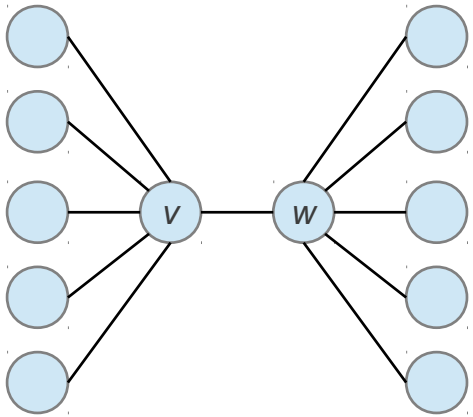
Proof.

- Suppose for contradiction that an equilibrium tree has diameter at least 4.
- Thus it has two vertices $v, w$ at distance exactly 4, inducing a length-4 shortest path $v \to a \to b \to c \to w$.
- But then $b$ is a cut vertex and two connected components of $G - b$ have vertices $v, w$ of distance more than 1 from $b$, contradicting the previous lemma. □

# Equilibrium trees, max version

Therefore, there are two families of max equilibrium trees:

1. stars (of diameter 2), and
2. "double-stars" (of diameter 3), as below:




To be in max equilibrium, this second type must have at least two leaves attached to each star root ($v$ and $w$).

# Sum version: a lower bound

We now consider the case of general networks in the sum version.
The following facts are easy to observe:

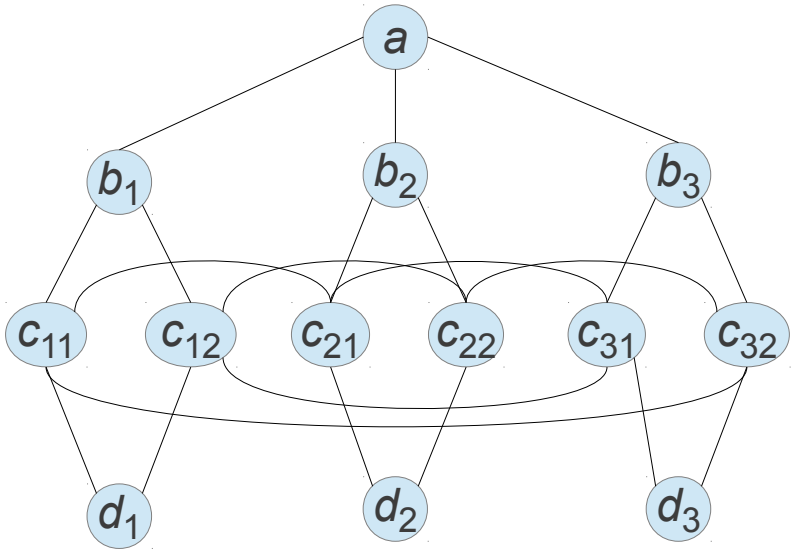
1. For a vertex $v$ of local diameter 2, swapping an incident edge does not improve the sum of distances from $v$.
2. Consider a vertex $v$ of local diameter 3. Adding an edge from $v$ to a vertex $w$ of distance $r$ decreases the sum of the distances from $v$ by at most $r-1$ for $w$ and by at most 1 for any neighbors of $w$ whose distance to $v$ was 3.
3. In any graph of girth 4, swapping an edge $vw$ with edge $vw'$ increases the distance from $v$ to $w$ by at least 2, unless $w'$ is a neighbor of $w$, in which case it increases by at least 1.
   (The girth of a graph is the length of a shortest cycle contained in the graph.)

# Sum version: a lower bound

**Theorem**

*There is a diameter-3 sum equilibrium graph.*

Proof. Consider the graph:




By considering all possible edge swaps around each vertex, and using the three facts stated previously, we can show that this is a diameter-3 sum equilibrium graph. □

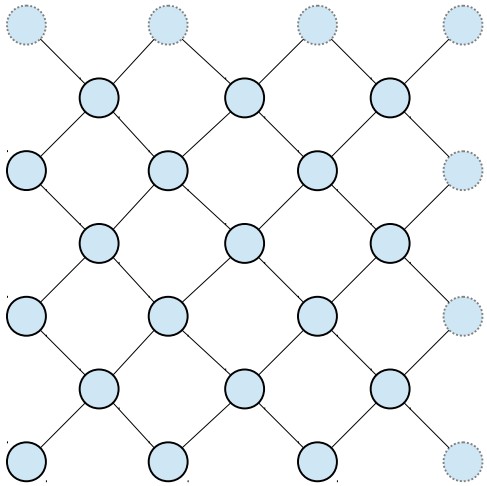
# Max version: a lower bound

We now consider the case of general networks in the max version.

**Theorem**

*There is a max equilibrium graph of diameter $\Theta(\sqrt{n})$.*

Proof. The $\Theta(\sqrt{n})$-diameter graph establishing the result is:



(The rightmost and topmost columns are actually duplicates of the leftmost and bottommost columns, respectively, and hence dotted.) □

# Local cost network creation game

A simpler, local cost network creation game, which admits an exact potential, was defined by (Nikoletseas et al, 2013).

- Let $G = (V, E)$ be any undirected graph with $n$ nodes.
- The players can be identified as the set of vertices of the graph.
- Any player $u \in V$ can swap one of its incident edges (which defines the set of available actions for each player).
- The payoff of a vertex depends only on the structure of its immediate neighborhood and not on the entire network.
- In particular, the profit of $u \in V$ in $G$ is $\gamma_G(u) = \sum_{v \in N_G(u)} \deg_G(v)$, i.e., the profit of $u$ is the sum of the degrees of its neighbors.

# Local cost network creation game

- A profitable swap is an edge swap that improves (increases) the profit of the vertex that performs it.
- An arbitrary sequence of profitable swaps (by nodes $v_1, v_2, \ldots$) actually transforms the initial graph through a sequence of configuration graphs $G_0, G_1, G_2, \ldots$.
- We write $G_i \xrightarrow{v_i} G_{i+1}$ and mean that configuration $G_i$ produces configuration $G_{i+1}$ by a selfish swap by vertex $v_i$.
- Vertex $v_i$ is called deviator in configuration $G_i$.
- A graph $G$ is a local cost swap equilibrium configuration if no vertex can perform a selfish (improving) swap.

# Local cost network creation game

**Theorem**

*If $G$ is a local cost swap equilibrium configuration, then it contains a star as a spanning subgraph.*

Proof.

- A vertex $u$ does not have a profitable swap in $G$ when $\deg_G(v) > \deg_G(w)$, for any $v \in N_G(u)$ and $w \notin N_G(u)$.
- This means that $u$ connects to all vertices of maximum degree.
- Moreover, either all vertices are connected to all vertices of maximum degree and the graph contains a star as a spanning subgraph,
- or there is some vertex $u$ not connected to at least one vertex $w$ of maximum degree, in which case $u$ can benefit from swapping one of its edges to connect to $w$ and thus increase the maximum degree.

# An exact potential

We now show that the local cost network creation game admits an exact potential function.

**Theorem**

*The function* $\Phi(G) = \frac{1}{2}\sum_{v \in V(G)} \deg_G(v)^2$ *is an exact potential.*

Proof. Consider a profitable swap performed by vertex $u$, which swaps edge $\{u, v\} \in E(G)$ with edge $\{u, w\} \notin E(G)$ and let $G'$ be the resulting graph. Then:

1. $\deg_{G'}(u) = \deg_G(u)$,
2. $\deg_{G'}(v) = \deg_G(v) - 1$,
3. $\deg_{G'}(w) = \deg_G(w) + 1$, and
4. the degrees of any other vertex remain unchanged.

# An exact potential

**Theorem**

*The function* $\Phi(G) = \frac{1}{2}\sum_{v \in V(G)} \deg_G(v)^2$ *is an exact potential.*

Proof (continued). Therefore

$$\begin{aligned} \gamma_{G'}(u) - \gamma_G(u) &= \sum_{z \in N_{G'}(u)} \deg_{G'}(z) - \sum_{z \in N_G(u)} \deg_G(z) \\ &= \deg_G(w) + 1 - \deg_G(v). \end{aligned}$$

The corresponding change in the value of the function $\Phi(\cdot)$ is then

$$\begin{aligned} \Phi(G') - \Phi(G) &= \frac{1}{2}\left(\deg_{G'}(v)^2 + \deg_{G'}(w)^2\right) - \frac{1}{2}\left(\deg_G(v)^2 - \deg_G(w)^2\right) \\ &= \deg_G(w) - \deg_G(v) + 1 = \gamma_{G'}(u) - \gamma_G(u) \end{aligned}$$

which proves that $\Phi(\cdot)$ is an exact potential. □

# Reaching equilibrium using a limited number of queries

- The potential function implies that an equilibrium graph can be found in at most $O(n^3)$ time steps (egde swaps), starting from any initial graph.
- In order for a vertex $u$ to compute a better response (i.e., a profitable swap), it requires information about the degree from all non-adjacent vertices in the graph, i.e., all $v \in V \setminus N_G(u)$.
- In many cases though, especially for large-scale networks like the Internet, it is inefficient to acquire such information about all the nodes in the network.
- On the other hand, we can assume that any vertex $u$ can get such information for a limited (e.g., constant) number of non-neighboring nodes by asking an oracle.

# Reaching equilibrium using a limited number of queries

**Theorem**

*If any vertex $u$ can obtain information about the degree of $c \geq 1$ randomly chosen non-neighboring vertices, then the local cost network formation game can converge in an equilibrium graph in a polynomial expected number of steps.*

Proof. Let $G_0 = (V_0, E_0)$ be the current graph. Consider the following procedure:

- At any time $t \geq 1$, select a vertex $u$ uniformly at random from $V$ and then ask the oracle to reveal the degrees of $c$ randomly chosen non-neighbors of $u$ (we assume that $u$ knows the degrees of its neighbors), namely $v_1, \ldots v_c \in V_t - N_{G_t}(u)$.
- If one of the vertices among $v_1, \ldots, v_c$ has degree equal or larger than the degree of some neighboring vertex of $u$ in $G_t$, then $u$ performs a profitable swap.
- Otherwise, it does nothing.

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Proof (continued).

- The resulting graph will be denoted by $G_t = (V_t, E_t)$.
- If at any time step $t$, $G_t$ is not an equilibrium graph, then there must be at least one profitable swap.
- In particular, there exist vertices $u, v, w$ such that $w \in N_{G_t}(u)$, $v \notin N_{G_t}(u)$ and $\deg_{G_t}(w) \leq \deg_{G_t}(v)$.
- The probability that $u$ is selected and also $v$ is among the randomly chosen non-neighbors of $u$ is at least $\frac{c}{n^2}$.

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Proof (continued).

- Consider now the stochastic process $\{X_t\}_{t\geq 0}$, where

$$X_t = \frac{1}{2}\sum_{z\in V_t} \deg_{G_t}(z)^2 \ .$$

- Provided $G_t$ is not an equilibrium graph,

$$\Pr(X_{t+1} \geq X_t + 1) \geq \frac{c}{n^2}$$

and also

$$\Pr(X_{t+1} = X_t) = 1 - \Pr(X_{t+1} \geq X_t + 1) \ .$$

- Notice also that the absorbing states of the stochastic process $\{X_t\}_{t\geq 0}$ correspond to equilibrium graphs, and we have $0 \leq X_t \leq \frac{n^3}{2}$ for any $t$ and in particular, for any equilibrium graph $G_t$.

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Proof (continued).

- We conclude that the number of steps needed for $\{X_t\}_{t\geq 0}$ to reach an absorbing state is stochastically dominated by a geometrically distributed random variable $Geom\left(\frac{n^3}{2}, \frac{c}{n^2}\right)$.
- Therefore, the mean number of steps needed for absorption is at most $\frac{n^5}{2c}$, which completes the proof. □

Note that we can decide whether the procedure in the proof has reached an equilibrium graph with high probability:

If after at least $\Omega(n^3)$ steps no swap has occurred, then by the Markov inequality, we can correctly (positively) decide whether we have reached equilibrium with probability at least $1 - O\left(\frac{1}{n}\right)$.

# Further reading


- S. Goyal: Connections: An Introduction to the Economics of Networks, **Chapters 7, 8 & 9**. Princeton University Press, 2007.
- V. Bala and S. Goyal: A Non-Cooperative Model of Network Creation. Econometrica **68**, pp. 1181–1229, 2000.
- V. Bala and S. Goyal: A Strategic Analysis of Network Reliability. Review of Economic Design **5**, pp. 205–229, 2000.
- N. Alon, E. D. Demaine, M. Hajiaghayi, and T. Leighton: Basic Network Creation Games. Proceedings of the 22nd ACM Symposium on Parallelism in Algorithms and Architectures (SPAA 2010), pp. 106–113, 2010.
- S. E. Nikoletseas, P. N. Panagopoulou, C. Raptopoulos, and P. G. Spirakis: On the Structure of Equilibria in Basic Network Formation. Proceedings of the 19th International Symposium on Fundamentals of Computation Theory (FCT 2013), pp. 259–270, 2013.