Maslov's Class K Revisited

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Abstract. This paper gives a new treatment of Maslov's class K in the framework of resolution. More specifically, we show that K and the class DK consisting of disjunction of formulae in K can be decided by a resolution refinement based on liftable orderings. We also discuss relationships to other solvable and unsolvable classes.

1 Introduction

Maslov's class K [13] is one of the most important solvable fragments of first-order logic. It contains a variety of classical solvable fragments including the Monadic class, the initially extended Skolem class, the Gödel class, and the two-variable fragment of first-order logic FO^2 [4]. It also encompasses a range of non-classical logics, like a number of extended modal logics, many description logics used in the field of knowledge representation [11,4, chap. 7], and some reducts of representable relational algebras.

For this reason practical decision procedures for the class K are of general interest. According to Maslov [13] the inverse method provides a means to decide the validity of disjunctions of formulae in the class K. Although Kuehner [12] noted in 1971 that there is a one-to-one correspondence between derivations in the inverse method and resolution, only in 1993 a decision procedure for a subclass of the dual of K based on a refinement of resolution is described by Zamov [4, chap. 6]. His techniques are based on non-liftable orderings which have limitations regarding the application of some standard simplification rules and the completeness proof relies on a π-ordering. An uncertainty regarding completeness (see [15]) is clarified by the work of de Nivelle [3].

In this paper we are concerned with the problem of deciding satisfiability for the dual class of K, which we call $\overline{\mathrm{K}}$, and also the class $\overline{\mathrm{DK}}$ consisting of conjunctions of formulae in $\overline{\mathrm{K}}$. In the context of resolution-based decision procedures these classes are of particular interest, since even ordering refinements of resolution do not prevent the growth of term and variable height. Most of the results in the literature on decision procedures based on ordered resolution consider classes where (i) every non-ground term contains all the variables of a clause and either (ii) all literals share the same variables, or (iii) it is possible to identify a literal either by polarity or maximal height that contains all the variables of a clause. Even unrestricted resolution, and factoring, preserve properties (i) and (ii). Then an atom ordering can be utilised to ensure that for all clauses in a derivation the maximal height of variable occurrences does not increase and the height of literals is bounded [3,4, chap. 4]. Consequently, any derivation terminates. In the case (i) and (iii), a selection function and an atom ordering guarantees termination and, in particular, terms do not grow [6]. A decision procedure for a subclass of $\overline{\mathrm{DK}}$ which allows term height growth, but not variable height growth, by means of a selection function is described in [11]. $\overline{\mathrm{K}}$ and $\overline{\mathrm{DK}}$ are classes where the mentioned approaches will not prevent the growth of the variable height.

This paper describes a resolution decision procedure based on liftable orderings (A-orderings) for the classes $\overline{\mathrm{K}}$ and $\overline{\mathrm{DK}}$, thereby extending the result of Zamov. Our proof is situated in the resolution framework of Bachmair and Ganzinger [1,2]. An additional renaming transformation of certain problematic clauses allows for the embedding of the classes under consideration into a class for which standard liftable orderings ensure closure under resolution and factoring as well as termination. This technique is described in Section 4. The results of Section 3 are similar to those of Zamov [4, chap. 6]. For this reason we omit the proofs. It should be noted however that our definitions of similarity and k-regularity in Section 2 are different to Zamov's, in particular, our notions allow for the presence of constants.

2 The class $\overline{\mathrm{K}}$ and quasi-regular clauses

The language is assumed to be that of first-order logic without equality and without function symbols. Let ϕ be a closed formula in negation normal form and ψ a subformula of ϕ. The *ϕ-prefix* of the formula ψ is the sequence of quantifiers of ϕ which bind the free variables of ψ. If a ϕ-prefix is of the form $\exists y_1 \ldots \exists y_m \forall x_1 Q_1 z_1 \ldots Q_n z_n$, where $m \geq 0$, $n \geq 0$, $Q_i \in \{\exists, \forall\}$ for all i, $1 \leq i \leq n$, then $\forall x_1 Q_1 z_1 \ldots Q_n z_n$ is the *terminal ϕ-prefix*. For a ϕ-prefix $\exists y_1 \ldots \exists y_m$ the terminal ϕ-prefix is the empty sequence of quantifiers.

By definition, a closed formula ϕ in negation normal form belongs to the *class* $\overline{\mathrm{K}}$ if there are k quantifiers $\forall x_1, \ldots, \forall x_k$, $k \geq 0$, in ϕ not interspersed with existential quantifiers such that for every atomic subformula ψ of ϕ the terminal ϕ-prefix of ψ

1. is either of length less than or equal to 1, or
2. ends with an existential quantifier, or
3. is of the form $\forall x_1 \forall x_2 \ldots \forall x_k$.

We say the variables $x_1, \ldots, x_k$, $k \geq 0$, are the *fixed universally quantified variables of* ϕ and ϕ is of *grade* k, indicating the number of fixed universally quantified variables. For example, the following formulae

$$\phi_1 = \exists a_1 \forall x_1 \forall x_2 \exists y_1 \forall z_1 \exists y_2\, p(a_1) \wedge p(a_1, y_1) \wedge (q(x_1, a_1, x_2) \vee r(x_1, y_2, z_1))$$
$$\phi_2 = \exists a_1 \exists a_2 \forall x_1 \forall x_2\, p(a_1, x_1, x_2) \vee p(x_2, a_2, x_2),$$

are elements of the class $\overline{\mathrm{K}}$ of grade 2: The variables x_1 and x_2 are the fixed universally quantified variables of ϕ_1 and ϕ_2. Every atomic subformula ψ satisfies the restrictions on the quantifier prefix binding the variables in ψ. The normal forms of Mortimer [14] for formulae in FO^2 are in $\overline{\mathrm{K}}$. Important properties of binary relations which belong to $\overline{\mathrm{K}}$ are reflexivity, irreflexivity, symmetry, seriality, and density. The following formulae

$$\phi_3 = \forall x_1\, \forall x_2\, \forall x_3\, \neg p(x_1, x_2, x_3) \wedge q(x_1, x_2)$$
$$\phi_4 = \forall x_1\, \exists x_2\, \forall x_3\, \neg p(x_1, x_2, x_3) \vee p(x_1, x_2, x_3)$$

do not belong to $\overline{\mathrm{K}}$. Important properties of binary relations which cannot be expressed in $\overline{\mathrm{K}}$ are transitivity, euclideanness, and confluence.

Since our intention is a resolution-based decision procedure for the class(es) $\overline{\mathrm{K}}$ (and $\overline{\mathrm{DK}}$) we are interested in the clause sets corresponding to formulae in $\overline{\mathrm{K}}$. Without loss of generality we can restrict ourselves to formulae in prenex form whose matrix is in conjunctive normal form, that is, formulae in $\overline{\mathrm{K}}$ have the form

$$\exists y_1 \ldots \exists y_m\, \forall x_1 \ldots \forall x_k\, Q_1 z_1 \ldots Q z_l \bigwedge_{i=1,\ldots,n} \bigvee_{j=1,\ldots,m_i} L_{i,j} \qquad (1)$$

where $m{\geq}0$, $k{\geq}0$, $l{\geq}0$, $n{>}0$, $m_i{>}0$, and $L_{i,j}$ are literals. We assume that outer Skolemisation is used in the process of transforming (1) to clausal form, that is, if $\forall z_1 \ldots \forall z_p$ is the subsequence of all universal quantifiers of the ϕ-prefix of subformula $\exists z\, \phi$ of ϕ, then $\phi[z/f(z_1, \ldots, z_p)]$ is the outer Skolemisation of $\exists z\, \phi$. The class of clause sets thus obtained is denoted by $\overline{\mathrm{KC}}$.

The remainder of this section is devoted to the definition of a syntactic characterisation of the clauses in $\overline{\mathrm{KC}}$.

A term t is said to *dominate* a term s, denoted by $t \succsim_Z s$, if at least one of the following conditions is satisfied:

1. $t = s$, s is a variable,
2. $t = f(t_1, \ldots, t_n)$, s is a variable and $s = t_i$ for some i, $1 \leq i \leq n$,
3. $t = f(t_1, \ldots, t_n)$, $s = g(t_1, \ldots, t_m)$, $n \geq m \geq 0$,

The relation $\succsim_Z$ is a quasi-ordering on terms and stable with respect to substitutions on compound terms. By compound terms we mean terms which are neither constants nor variables. The relation $\succsim_Z$ is extended to sets of terms and literals as follows. The set T_1 of terms *dominates* the set T_2 of terms if for every term t_2 from T_2 there exists a term t_1 from T_1 such that t_1 dominates t_2. A literal L_1 *dominates* a literal L_2, denoted by $L_1 \succsim_Z L_2$, if the set of non-constant arguments of L_1 dominates the set of non-constant arguments of L_2. Note that $\succsim_Z$ is also a quasi-ordering on literals. We let $\sim_Z = \succsim_Z \cap \succsim_Z^{-1}$ and $\succ_Z = \succsim_Z \setminus \sim_Z$. If $s \sim_Z t$ for terms s and t, then s and t are *similar*. Analogously, for literals.

For example, the literal $p(x, y)$ dominates $q(a, x, y)$, but not $q(f(a), x, y)$. The literals $p(x, y)$, $q(a, x, y)$, and $q(y, x)$ are similar. So are $p(f(a), x)$ and $q(g(a), x)$. Note that $p(f(a), x)$ is not similar to $q(g(b), x)$, nor does one dominate the other.

Based on the quasi-ordering $\succsim_Z$ we are able to characterise a subset of the set of all terms in the following way: A term is called *regular* if it dominates all its arguments. We extend this notion to sets of terms and literals as follows.

A set of terms is called *regular* if it contains no compound terms or it contains some regular compound term which dominates all terms of this set. A literal is called *regular* if the set of its arguments is regular.

The extension of regularity to clauses, which we regard as multisets of literals, is less straightforward. We need two more definitions: A literal L is *singular* if it contains no compound term and $\mathcal{V}(L)$ is a singleton,[1] otherwise it is *non-singular*. A literal containing a compound term is *deep*, otherwise it is *shallow*. A clause C is k*-regular* if the following conditions hold:

1. C contains regular literals only.
2. k is a non-negative integer not greater than the minimal arity of function symbols occurring in C. If C does not contain compound terms, then k is arbitrary.
3. C contains some literal which dominates every literal in C.
4. If L_1 and L_2 are non-singular, shallow literals in C, then $L_1 \sim_Z L_2$.
5. If L is a non-singular, shallow literal in C, then for every compound term t occurring in C

$$\arg_{set}(L) \setminus \mathsf{F}_0 \sim_Z \arg_{set}^{1\ldots k}(t) \setminus \mathsf{F}_0$$

   holds, where $\arg_{set}(L)$ is the set of arguments of L, $\arg_{set}^{1\ldots k}(t)$ is the set of the first k arguments of t, and F_0 is the set of all constants.

A clause is *regular* if it is k-regular for some $k \geq 0$. A clause is called *quasi-regular* if all of its indecomposable components are regular.

For example, the clause $\{p(a, y, z), q(f(a, y, z))\}$ is 3-regular. Note that the clause $\{p(x_1, x_2, x_3)\}$ can be considered to be a 2-regular clause, although the corresponding first-order formula $\forall x_1 \forall x_2 \forall x_3\, p(x_1, x_2, x_3)$ is of grade 3.

3 Resolution and factoring on quasi-regular clauses

Next we study closure of quasi-regular clauses under resolution and factoring.

Recall that the components in the variable partition of a clause are called split components, that is, split components do not share variables. A clause which is identical to its split component is indecomposable. The condensation $\mathrm{Cond}(C)$ of a clause C is a minimal subclause of C which is a factor of C.

Theorem 1. *Every indecomposable component of a clause in the clausal form of a formula ϕ of the form (1) is k-regular.*

Lemma 2 (Properties of regular expressions [4, pages 136–144]).

1. *If a regular term t dominates the term s and σ is a substitution such that $t\sigma$ is regular, then $t\sigma$ dominates $s\sigma$.*
2. *Let $L_1 = p(s_1, \ldots, s_n)$ and $L_2 = p(t_1, \ldots, t_n)$ be unifiable deep literals. If s_i is a dominating term of L_1, then also t_i is a dominating term of L_2.*
3. *Let L_1 and L_2 be regular literals and σ a most general unifier of L_1 and L_2. Then $L_1\sigma$ is regular.*

[1] $\mathcal{V}(E)$ denotes the set of variables of an expression E.

Theorem 3. *Let $\{A_1\} \cup C_1$ and $\{\neg A_2\} \cup C_2$ be indecomposable, k-regular clauses such that A_1 and A_2 are unifiable with most general unifier σ, and A_1 and $\neg A_2$ are dominating literals in $\{A_1\} \cup C_1$ and $\{\neg A_2\} \cup C_2$, respectively. Then every split component of $(C_1 \cup\ C_2)\sigma$ is a k-regular clause.*

It is crucial that in Theorem 3 we require both clauses to be k-regular, for the same k, where Zamov's version only requires regularity. The resolvent of the two regular clauses $\{\neg p(g(x, y, z), z), q(x, y), r(g(x, y, z))\}$ and $\{p(g(x', y', z'), f(x'))\}$ is a counterexample for Zamov's result.

Theorem 4. *Let $\{L_1, L_2\} \cup C$ be an indecomposable, k-regular clause such that L_1 and L_2 are unifiable with most general unifier σ and L_1 is a dominating literal in $\{L_1, L_2\} \cup C$. Then $(\{L_1\} \cup C)\sigma$ is a k-regular clause.*

Lemma 5. *Let N be a set of k-regular clauses. Let n_{var} be the maximal number of distinct variables in any clause in N. Then for any clause C derivable by resolution on $\succ_Z$-maximal literals or factoring, the number of distinct variables in C is less or equal to n_{var}.*

The number of distinct regular terms over a given vocabulary is considerably smaller than the number of distinct terms based on this vocabulary.

Theorem 6. *Let N be a set of k-regular clauses. Let ar_{fun}, n_{var}, and n_{fun} be the maximal arity of function symbols, the maximal number of variables in a clause in N, and the number of function symbols in N, respectively. The number of different terms in N does not exceed*

$$n_{terms} = (n_{fun} + n_{var})^{ar_{fun}+1}.$$

Furthermore, the number of condensed clauses in N does not exceed

$$n_{clauses} = 2^{(2 \times n_{pred} \times n_{terms}^{ar_{pred}})},$$

where n_{pred} is the number of predicate symbols in N and ar_{pred} the maximal arity of a predicate symbol.

4 An atom ordering for quasi-regular clauses

An atom ordering $\succ$ is an ordering on atoms which is stable under substitution, and total (and well-founded) on ground atoms. It is extended to literals by taking the multiset extension of $\succ$ and by identifying A and $\neg B$ with the multisets $\{A\}$ and $\{B, B\}$, respectively.

Since $\succ_Z$ is not stable under substitution, we cannot construct an atom ordering based on $\succ_Z$. The first problem with $\succ_Z$ occurs on the term-level. Consider the terms $f(x, y)$ and y. Obviously, $f(x, y) \succ_Z y$ holds. However, for the substitution $\sigma = \{y/g(z)\}$, we have $f(x, y)\sigma \not\succ_Z y\sigma$. Note that $f(x, y)\sigma$ is no longer regular. According to Lemma 2(1) $t \succ_Z s$ implies $t\sigma \succ_Z s\sigma$ for any substitution σ such that t and $t\sigma$ are regular. This problem is mainly caused by the dual use of $\succ_Z$: On the one hand it is used to define the structure of regular terms, literals and clauses and on the other hand it is used to determine the literals of a clause to resolve upon. The problem can be solved by using an ordering which is compatible with $\succ_Z$ on regular terms and is stable under substitution.

The second problem with $\succ_Z$ occurs on the atom-level. As a simple example consider the atoms $p(x, y, x)$ and $p(x, x, a)$. We have that $p(x, y, x) \succ_Z p(x, x, a)$. But since the atoms have a common instance $p(a, a, a)$, there exists no atom ordering $\succ$ such that $p(x, y, x) \succ p(x, x, a)$ holds.

It is important to remember that we have to restrict resolution inference steps in a clause $\{p(x, y, x), p(x, x, a)\}$ to the first literal. For otherwise, we can no longer guarantee that resolvents of k-regular clauses are still k-regular. For example, resolution with $\{p(z, x, z), \neg p(x, x, a)\}$ (on the second literal in each clause) results in $\{p(z, x, z), p(x, y, x)\}$ which is not k-regular.

As far as the selection of suitable literals to resolve upon is concerned, clauses meeting the following two conditions cause problems.

1. The clause C contains a singular literal which has constant arguments or duplicate variable arguments.
   Suppose the opposite. Then all singular literals are monadic. All predicate symbols of shallow, non-singular literals have arity greater than one. Thus, there are no common instances of singular literals and non-singular literals. It is straightforward to define an atom ordering $\succ$ such that the singular literals are not $\succ$-maximal in C.
2. The clause C contains a shallow, non-singular literal or there is a compound term t in C such that $|\mathcal{V}(\arg_{set}^{1..k}(t))| \geq 2$.[2]
   If C contains no shallow, non-singular literals and for all compound term t in C we have that $|\mathcal{V}(\arg_{set}^{1..k}(t))| \leq 1$, then all shallow literals in C contain exactly one variable.

At first glance the second condition may seem too general. In a 2-regular clause like $\{q(f(x, y, z)), p(x, a)\}$ the literals $q(f(x, y, z))$ and $p(x, a)$ have no common instances and it is straightforward to define an atom ordering $\succ$ such that $q(f(x, y, z))$ is strictly $\succ$-maximal. However, a resolution inference step with $\{\neg q(f(x, y, z)), p(x, y)\}$ results in $\{p(x, a), p(x, y)\}$ which is the prototypical example of a problematic clause.

This analysis motivates the following definitions.

A literal L is called *CDV* (containing <u>c</u>onstants or <u>d</u>uplicate <u>v</u>ariables) if L is singular, and there is an argument which is a constant or there are duplicate (variable) arguments. Otherwise a literal is *CDV-free*. A clause C is *CDV* if it contains a CDV literal and a shallow, non-singular literal, but no deep literal. Otherwise C is *CDV-free*.

The intuition of CDV-free clauses is that $\succ$-maximal literals are immediate for suitable atom orderings $\succ$. It should be noted that there are CDV-free clauses which contain CDV literals. Any CDV clause contains at least two literals.

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[2] $|N|$ denotes the cardinality of N.

An indecomposable, k-regular clause C is *strongly CDV-free* if satisfies at least one of the following conditions.

1. C contains no CDV literal, or
2. C contains no shallow, non-singular literal and for every compound term t occurring in any literal in C, $|\mathcal{V}(\arg_{set}^{1..k}(t))| = 1$.

A set of clauses N is (strongly) *CDV-free* if every clause in N is (strongly) CDV-free.

The sample clause $\{p(f(x,y)), q(x,y), r(x,a)\}$ is CDV-free, but not strongly CDV-free. It contains the literal $r(x,a)$ which is CDV and a shallow, non-singular literal $q(x,y)$. Also the 2-regular clause $\{p(f(x,y)), r(x,a)\}$ is not strongly CDV-free: Apart from the CDV literal $r(x,a)$ it contains a deep, non-singular literal $p(f(x,y))$ such that the term $f(x,y)$ contains more than one variable. There is a subtle point to note. If we consider $\{p(f(x,y), r(x,a))\}$ as a 1-regular clause, then it is strongly CDV-free. However, in a set of 1-regular clauses, no shallow, non-singular literals occur.

The clause $\{p(x,x,a), q(x,b), p(c,x,c)\}$ is strongly CDV-free, since it contains no non-singular literal. The clause $\{p(x,y,c), q(x,y), p(x,a,y)\}$ is strongly CDV-free, since it contains no singular literal. In general, CDV-freeness does not remain invariant under resolution. A simple counterexample is a resolution inference step between $C_1 = \{p(f(x,y)), r(x,y), q(x,a)\}$ and $C_2 = \{\neg p(z)\}$ with conclusion $\{r(x,y), q(x,a)\}$. This is due to the fact that the CDV literal in C_1 is shielded by the term $f(x,y)$. Since this term is no longer present in the conclusion, the CDV literal becomes unshielded. However, C_1 is not strongly CDV-free.

We will now show that ordered factoring and ordered resolution preserve strong CDV-freeness.

Lemma 7. *Let L be a singular, CDV-free literal and σ be a substitution. Then $L\sigma$ is CDV-free.*

Proof. Let $\mathcal{V}(L) = \{x\}$. If $x \notin \mathcal{D}(\sigma)$,[3] then $L\sigma = L$ is still CDV-free. Now, assume $x \in \mathcal{D}(\sigma)$. If $x\sigma$ is a variable, then L and $L\sigma$ are identical up to the renaming of variables. So, $L\sigma$ is CDV-free. If $x\sigma$ is a constant or a compound term, then $L\sigma$ is no longer singular and therefore CDV-free. □

Theorem 8. *Let $\{L_1, L_2\} \cup C$ be an indecomposable, strongly CDV-free, k-regular clause such that L_1 and L_2 are unifiable with most general unifier σ and L_1 is a dominating literal in $\{L_1, L_2\} \cup C$. Then $(\{L_1\} \cup C)\sigma$ is strongly CDV-free.*

Proof. If $(\{L_1\} \cup C)\sigma$ is a ground clause, then it is strongly CDV-free, since it contains no CDV literals.

Suppose that $(\{L_1\} \cup C)\sigma$ is non-ground. For every literal $L\sigma$ in $(\{L_1\} \cup C)\sigma$ the set $\mathcal{V}(L\sigma)$ is a subset of $\mathcal{V}(L)$ and the height of $L\sigma$ is equal to the height of L. Thus, no singular literal L in $\{L_1, L_2\} \cup C$ will become a non-singular literal

[3] $\mathcal{D}(\sigma)$ denotes the domain of σ, and $\mathcal{C}(\sigma)$ the codomain.

$L\sigma$ in $(\{L_1\} \cup C)\sigma$, nor will any deep literal L satisfying the second condition of strong CDV-freeness turn into a deep literal $L\sigma$ violating this condition.

If $\{L_1, L_2\} \cup C$ is strongly CDV-free, since it contains no non-singular literal, the factoring inference step will not produce a non-singular literal and $(\{L_1\} \cup C)\sigma$ remains strongly CDV-free.

If $\{L_1, L_2\} \cup C$ is strongly CDV-free, since it contains no CDV literal, we can argue as follows. Suppose $(\{L_1\} \cup C)\sigma$ contains a non-ground literal $L\sigma$ which is CDV. Then L is not singular, since by Lemma 7 any singular, CDV-free literal remains CDV-free after instantiation. Also, L is not deep, since deep literals remain deep after instantiation. So, L is a shallow, non-singular literal. However, all shallow, non-singular literals in a k-regular clause contain the same set of variables and for any shallow, non-singular literal L and for all compound terms t occurring in the clause, the set of non-constant arguments of L are similar to the subset of non-constant arguments of the first k arguments of t. If instantiation with σ turns L into a singular literal with variable x, then it does so with every shallow, non-singular literal in the clause. Furthermore, the subset of non-constant terms of the first k arguments of any term t will contain only one variable, namely x. Thus, $(\{L_1\} \cup C)\sigma$ is strongly CDV-free. □

Theorem 9. *Let $\{A_1\} \cup C_1$ and $\{\neg A_2\} \cup C_2$ be indecomposable, strongly CDV-free, k-regular clauses such that A_1 and A_2 are unifiable with most general unifier σ and A_1 and $\neg A_2$ are dominating literals in $\{A_1\} \cup C_1$ and $\{\neg A_2\} \cup C_2$, respectively. Then every split component of $(C_1 \cup C_2)\sigma$ is strongly CDV-free.*

Proof. The general observation that no singular or deep literal L in one of the premises will become a shallow, non-singular literal $L\sigma$ in the conclusion $(C_1 \cup C_2)\sigma$ remains true.

We distinguish the following cases:

1. Both A_1 and $\neg A_2$ are singular literals. Then neither $\{A_1\} \cup C_1$, $\{\neg A_2\} \cup C_2$, nor $(C_1 \cup C_2)\sigma$ contain a non-singular or deep literal. So, the conclusion of the resolution inference step is strongly CDV-free.
2. A_1 is a singular literal and $\neg A_2$ is a shallow, non-singular literal. C_1 and $C_1\sigma$ contain no non-singular or deep literal and C_2 contains no deep literal. The literal $\neg A_2\sigma$ is singular. Since all shallow, non-singular literals in C_2 are similar to $\neg A_2$, $C_2\sigma$ contains no non-singular or deep literal.
3. A_1 and $\neg A_2$ are shallow, non-singular literals. Neither C_1 nor C_2 contains a deep literal. If $A_1\sigma (= A_2\sigma)$ is singular, then $(C_1 \cup C_2)\sigma$ contains no non-singular literals. Suppose $A_1\sigma$ is again a shallow, non-singular literal. Let L be a CDV-free, singular literal in either C_1 or C_2. Since $\mathcal{C}(\sigma)$ contains only variables and constants, $L\sigma$ is either ground or CDV-free, by Lemma 7.
4. A_1 is a deep literal and $\neg A_2$ is singular. $\{\neg A_2\} \cup C_2$ contains only singular literals and that $\mathcal{V}(\{\neg A_2\} \cup C_2)$ is a singleton set. So, $\neg A_2$ is not necessarily CDV-free. W.l.o.g. we can assume that σ maps the only variable occurring in $\neg A_2$ to some compound term t. This means, $\neg A_2\sigma$, and likewise any literal in $C_2\sigma$, is deep. The elements of $\mathcal{C}(\sigma\restriction_{\mathcal{V}(A_1)})$ are either variables or constants. So, if L is a CDV-free literal in C_1, then $L\sigma$ is still CDV-free or ground.

Suppose C_1 contains a CDV literal. Then C_1 satisfies the second condition of strong CDV-freeness. Instantiation with σ will not introduce additional variables into compound terms and t also satisfies the requirements of the second condition. So, $(C_1 \cup C_2)\sigma$ is strongly CDV-free.

Suppose C_1 contains a shallow, non-singular literal L. Note that C_1 does not contain a CDV literal. If $L\sigma$ is still CDV-free, then $C_2\sigma$ contains no CDV literal and $(C_1 \cup C_2)\sigma$ is strongly CDV-free. If $L\sigma$ is a CDV literal, then we argue as in the proof of Theorem 8 that $(C_1 \cup C_2)\sigma$ satisfies the second condition of strong CDV-freeness.

5. A_1 is a deep literal and $\neg A_2$ is a shallow, non-singular literal. Since $C_2\sigma$ contains only deep literals and, possibly, CDV-free literals, the proof is similar to the previous case.

6. Both A_1 and $\neg A_2$ are deep literals. The important point to note in this case is the following. Let s and t be compound terms and in $(\{A_1\} \cup C_1)\sigma$ or $(\{\neg A_2\} \cup C_2)\sigma$. Let L be a shallow, non-singular literal in $\{A_1\} \cup C_1$ or $\{\neg A_2\} \cup C_2$. Then

$$\arg_{set}^{1\ldots k}(s) = \arg_{set}^{1\ldots k}(t) \quad \text{and} \quad \arg_{set}(L) \setminus \mathsf{F}_0 \sim_Z \arg_{set}^{1\ldots k}(s) \setminus \mathsf{F}_0.$$

Consequently, if A_1 is a shallow, non-singular literal in C_1 and A_2 is a shallow, non-singular literal in C_2, either both $A_1\sigma$ and $A_2\sigma$ are singular literals and $\mathcal{V}(\arg_{set}^{1\ldots k}(t))$ is a singleton set for any compound term t in $(C_1 \cup C_2)\sigma$, or neither $A_1\sigma$ nor $A_2\sigma$ are shallow, non-singular literals. It follows that $(C_1 \cup C_2)\sigma$ is strongly CDV-free. □

We have shown that the property of strong CDV-freeness is preserved under inferences by factoring and resolution on dominating literals. Since clause sets in $\overline{\text{KC}}$ are not necessarily strongly CDV-free, we define a satisfiability equivalence preserving transformation which transform any clause set N in $\overline{\text{KC}}$ into a strongly CDV-free clause set N'. The transformation is given by the rule

$$N \Rightarrow_{\mathcal{M}} N' \cup \mathrm{Def}_L^A,$$

where (i) L is an occurrence of a CDV literal in a clause $C \in N$ which is not strongly CDV-free, (ii) A is an atom of the form $p^f(x)$ where p^f is a new predicate symbol with respect to N and x is the variable occurring in L, (iii) Def_L^A is a *definitional* clause of the form $\{\neg A, L\}$, and (iv) N' is obtained from N by replacing any occurrence of L by A.

Note that A is CDV-free and that the clause $\{\neg A, L\}$ is strongly CDV-free. As each transformation step removes at least one CDV literal in one of the clauses which are not strongly CDV-free, by a sequence of transformation steps we eventually obtain a strongly CDV-free set of clauses . We denote the resulting clause set by $N{\downarrow_{\mathcal{M}}}$.

Theorem 10. *Let N be a set of clauses. Then $N{\downarrow_{\mathcal{M}}}$ can be constructed in polynomial time, and $N{\downarrow_{\mathcal{M}}}$ is satisfiable if and only if N is satisfiable.*

An atom ordering $\succ$ suitable for our purpose has to satisfy this condition:

If a literal L is $\succ$-maximal in a clause C of a given class C, then there is no literal L' in C with $L' \succ_Z L$. (2)

We have seen that no atom ordering satisfying this condition can exist if C is the class of all (indecomposable) k-regular clauses. We will now show that if C is the class of all (indecomposable) strongly CDV-free, k-regular clauses, we are able to define such an atom ordering.

Note that it is not relevant whether $\succ$ is applied *a priori*, like $\succ_Z$, or *a posteriori*: Since $\succ$ is stable under substitution, if $L\sigma$ is $\succ$-maximal in $C\sigma$ then L is $\succ$-maximal in C.

Let $\succ_\Sigma$ be a total precedence on predicate and functions symbols (function symbols are denoted by f and g, predicate symbols by p and q) such that

- $f \succ_\Sigma g$ if the arity of f is strictly greater than the arity of g,
- $f \succ_\Sigma p$ if f is not a constant symbol,
- $p \succ_\Sigma q$ if q is monadic and the arity of p is greater than or equal to 2, and
- $p \succ_\Sigma c$ if c is a constant symbol.

Every predicate symbol and function symbol has multiset status. Let $\succ_S$ be the recursive path ordering based on the precedence $\succ_\Sigma$.

Lemma 11. *Let L_1 and L_2 be literals in an indecomposable, strongly CDV-free, k-regular clause C. Then $L_1 \succ_Z L_2$ implies $L_1 \succ_S L_2$.*

Proof. We distinguish the following cases according to the type of L_1:
1. L_1 is non-singular and shallow. Then L_2 is singular. Therefore, L_2 contains exactly one variable x and x is an argument of L_2 and L_1. If x is the only argument of L_2, then the multiset of arguments of L_1 is obviously greater than the multiset of arguments of L_2 and the predicate symbol of L_1 has precedence over the predicate symbol of L_1 by $\succ_\Sigma$. So, $L_1 \succ_S L_2$ holds. If L_2 contains more than one argument, then L_2 is not CDV-free, contradicting our assumption that C is strongly CDV-free.
2. L_1 is deep. L_1 contains a compound term t_1 dominating all the arguments of L_1. Since $L_1 \succ_Z L_2$ and $\succ_Z$ is transitive, for every argument of t_2 of L_2, $t_1 \succ_Z t_2$ holds. The term t_2 is either a variable or a compound term. In the first case, according to the definition of $\succ_Z$, t_2 is an argument of t_1. In the second case, t_1 has the form $f(u_1, \dots, u_m)$ and t_2 has the form $g(u_1, \dots, u_n)$ with $m>n$. Then, by definition, $f \succ_\Sigma g$. Therefore, it is enough to show that $f(u_1, \dots, u_m) \succ_S u_j$ holds, for all j with $1 \leq j \leq n$, which is straightforward. □

5 A decision procedure for $\overline{\mathrm{KC}}$

We are now ready to present a decision procedure for the class $\overline{\mathrm{KC}}$. We adopt the resolution framework of [1,2]. The calculus is parameterised by an atom ordering $\succ$ and consists of three basic rules.

Deduce:
$$\frac{N}{N \cup \{\mathrm{Cond}(C)\}}$$
if C is either a resolvent or a factor of clauses in N.

Delete: $$\frac{N \cup \{C\}}{N}$$

if C is a tautology or N contains a clause which is a variant of C.

Split: $$\frac{N \cup \{C \cup D\}}{N \cup \{C\} \mid N \cup \{D\}}$$

if C and D are variable-disjoint.

Resolvents and factors are derived by the following rules.

Ordered Resolution: $$\frac{C \cup \{A_1\} \quad D \cup \{\neg A_2\}}{(C \cup D)\sigma}$$

where (i) σ is the most general unifier of A_1 and A_2, (ii) no literal is selected in C and $A_1\sigma$ is strictly $\succ$-maximal with respect to $C\sigma$, and (iii) $\neg A_2$ is either selected, or $\neg A_2\sigma$ is maximal in $D\sigma$ and no literal is selected in D.

Ordered Factoring: $$\frac{C \cup \{A_1, A_2\}}{(C \cup \{A_1\})\sigma}$$

where (i) σ is the most general unifier of A_1 and A_2; and (ii) no literal is selected in C and $A_1\sigma$ is $\succ$-maximal with respect to $C\sigma$.

Let R be any calculus in which (i) derivations are generated by strategies applying "Delete", "Split", and "Deduce" in this order, (ii) no application of "Deduce" with identical premises and identical consequence may occur twice on the same path in derivations, and (iii) the ordering is based on $\succ_S$ while the selection function is arbitrary.

Theorem 12. *Let N be a set of clauses in $\overline{\mathrm{KC}}$. Then,*

1. *N is unsatisfiable if and only if the R-saturation of $N{\downarrow_{\mathcal{M}}}$ contains the empty clause, and*
2. *any derivation from $N{\downarrow_{\mathcal{M}}}$ in R terminates.*

Proof. Part 1 is a direct consequence of Theorem 10 and the soundness and completeness of R [1,2].

By Theorems 3, 4, 8, and 9 the class of strongly CDV-free, k-regular clauses is closed under inference in R. Theorem 6 shows that the class of k-regular clauses is finite, provided condensation is applied eagerly. This proves part 2. □

Corollary 13. *The classes* K *and* $\overline{\mathrm{K}}$ *are decidable.*

Let us consider an example. None of the clauses in N defined below is strongly CDV-free. The transformed clause set is $N{\downarrow_{\mathcal{M}}}$.

N :
(1) $\{p(a,a,x), r(a,x,y)\}$
(2) $\{\neg r(a,a,x), p(a,x,y)\}$
(3) $\{r(a,a,x), \neg p(a,x,y)\}$
(4) $\{\neg p(a,a,x), \neg r(a,x,y)\}$

$N{\downarrow_{\mathcal{M}}}$:
(5) $\{p_1^+(x), r(a,x,y)\}$
(6) $\{r_1^-(x), p(a,x,y)\}$
(7) $\{r_1^+(x), \neg p(a,x,y)\}$
(8) $\{p_1^-(x), \neg r(a,x,y)\}$
(9) $\{\neg p_1^+(x), p(a,a,x)\}$
(10) $\{\neg r_1^-(x), \neg r(a,a,x)\}$
(11) $\{\neg r_1^+(x), r(a,a,x)\}$
(12) $\{\neg p_1^-(x), \neg r(a,a,x)\}$

The following presents one branch of the theorem proving derivation from $N{\downarrow_{\mathcal{M}}}$.

[(5)2, R, (8)2]	(13) $\{p_1^+(x), p_1^-(x)\}$	[(17)1, R, (8)2]	(18) $\{p_1^-(a)\}$
[(13)1, R, (9)1]	(14) $\{p(a,a,x), p_1^-(x)\}$	[(18)1, R, (12)2]	(19) $\{\neg p(a,a,a)\}$
[(14)1, R, (7)2]	(15) $\{r_1^+(a), p_1^-(x)\}$	[(19)1, R, (6)2]	(20) $\{r_1^-(a)\}$
[(15)1, Spt]	(16) $\{r_1^+(a)\}$	[(20)1, R, (10)2]	(21) $\{\neg r(a,a,a)\}$
[(16)1, R, (11)1]	(17) $\{r(a,a,a)\}$	[(21)1, R, (17)1]	(22) $\bot$

It is straightforward to check that we are able to derive the empty clause in all remaining branches by similar sequences of inference steps. Hence, the initial set N is unsatisfiable.

A particularly interesting variant of our decision procedure can by obtained by the utilisation of a selection function. The selection function $S_{\mathcal{KC}}$ selects the monadic literal $\neg A$ in a clause Def_L^A introduced by the transformation $\Rightarrow_{\mathcal{M}}$. Note that positive occurrences of A in $N{\downarrow_{\mathcal{M}}}$ are not $\succ_S$-maximal in their clauses. Thus, inferences with clauses in Def_L^A are prohibited. Only when a clause C with a $\succ_S$-maximal literal A and with selected counterpart $\neg A$ in $\mathrm{Def}_{\mathcal{M}}$ is produced by ordered resolution, will an inference step with Def_L^A be performed. Effectively, such an inference step reintroduces (an instance of) the original literal occurrence L into C. Now, leaving these inference steps aside, resolution inference steps based on the non-liftable ordering $\succ_Z$ correspond one-to-one to resolution inference step based on the liftable ordering $\succ_S$. Some factoring steps possible with respect to $\succ_Z$ are prevented in $N{\downarrow_{\mathcal{M}}}$, since one of the literals has been renamed by the transformation.

The picture changes if we take the different notions of redundancy underlying these calculi into account. The inference step leading to the clause $\{p_1^+(x), p_1^-(x)\}$ corresponds to the inference step

$$[\ (1)2,\ \mathrm{R},\ \ (4)2]\ (13')\ \{p(a,a,x), \neg p(a,a,x)\}$$

on the original clause set N which results in a tautologous clause. In a decision procedure based on the non-liftable ordering $\succ_Z$ such tautologous clauses are not redundant and, in general, cannot be eliminated without loosing completeness of the procedure. This is already evident in our example, since the only alternative inference step possible on N is the derivation of the tautologous clause $\{r(a,a,x), \neg r(a,a,x)\}$ from clauses (2) and (3). Thus, the only clauses derivable from the unsatisfiable clause set N based on the $\succ_Z$-refinement of resolution are tautologous.

In contrast, we can make use of the notion of redundancy described in [2]. For example, given the clauses $\{p(a,x,y), r(b,x,y)\}$ and $\{\neg p(a,a,z), \neg r(b,a,z)\}$ which are strongly CDV-free, the tautologous conclusion $\{p(a,a,x), \neg p(a,a,x)\}$ is redundant and can be eliminated.

6 An extension of $\overline{\mathrm{KC}}$: The class $\overline{\mathrm{DKC}}$

In this section we consider the class $\overline{\mathrm{DK}}$ containing all possible (finite) conjunctions of formulae of class $\overline{\mathrm{K}}$. The grade of the formulae in such a conjunction can vary. The class of clause sets obtained from formulae of class $\overline{\mathrm{DK}}$ is denoted by $\overline{\mathrm{DKC}}$.

We will prove that $\overline{\mathrm{DKC}}$ is decidable with respect to satisfiability using the procedure of the previous section.

Let ϕ be a formula of class $\overline{\mathrm{K}}$ of grade k. Let N be the corresponding set of clauses. Then we call a non-constant Skolem function f occurring in some clause in N *k-originated*. Let C be a k-regular clause such that all non-constant Skolem functions occurring in C are k-originated. Then C is *strongly k-regular*.

There is a rather subtle point to note about the definition of k-regular and strongly k-regular clauses. The value of k can be chosen almost arbitrarily if the clause does not contain compound terms. For example, if we talk about 3-regular clauses in the following, this will include clauses like $\{p(x_1, x_2), p(x_2, x_1)\}$ and $\{q(x_3, c, x_2, x_1, x_4)\}$. To distinguish clauses like these from clauses actually containing k-originated function symbols, we introduce the notion of *inhabited clauses*, that is, clauses containing at least one non-constant function symbol.

Lemma 14. *Let ϕ be a formula of class $\overline{\mathrm{K}}$ and let ϕ be of grade k. Let N be the corresponding set of clauses. Then every clause in N is strongly k-regular.*

Proof. Immediate. □

Corollary 15. *Let ϕ be a formula of class $\overline{\mathrm{DK}}$. Let N be the corresponding set of clauses. Then every clause in N is strongly k-regular for some positive k.*

Lemma 16. *Let $\{A_1\} \cup C_1$ and $\{\neg A_2\} \cup C_2$ be indecomposable, strongly k-regular clauses such that A_1 and A_2 are unifiable with most general unifier σ and A_1 and A_2 are dominating literals in $\{A_1\} \cup C_1$ and $\{\neg A_2\} \cup C_2$, respectively. Then the split components of the resolvent $(C_1 \cup C_2)\sigma$ are strongly k-regular.*

Proof. By Lemma 4 the split components of $(C_1 \cup C_2)\sigma$ are k-regular. Since all the Skolem functions of $(C_1 \cup C_2)\sigma$ already occur in one of the parent clauses, the Skolem functions are k-originated. Thus the split components of $(C_1 \cup C_2)\sigma$ 'are strongly k-regular. □

Lemma 17. *Let C_1 be an indecomposable, strongly k-regular clause. Let C_2 be a factor of C_1. Then the split components of C_2 are strongly k-regular.*

Proof. By a similar argument to the previous lemma. □

Lemma 18. *Let $C_1 = \{A_1\} \cup D_1$ be a k_1-regular clause and $C_2 = \{\neg A_2\} \cup D_2$ a k_2-regular clauses such that A_1 and A_2 are unifiable with most general unifier σ and A_1 and $\neg A_2$ are dominating literals in C_1 and C_2, respectively. Let all the non-constant Skolem functions in C_1 and C_2 be k'_1-originated and k'_2-originated, respectively. Then $k'_1 = k'_2$ and all the non-constant Skolem functions in $(D_1 \cup D_2)\sigma$ are k'_1-originated.*

Proof. If neither C_1 nor C_2 contain function symbols, the lemma is trivially true. W.l.o.g. we assume that at least one function symbol occurs in C_1. So, the clause C_1 contains a deep literal and A_1 is deep. Therefore, there exists a compound term t_1 in A_1 such that t_1 dominates every argument of every literal in C_1. According to the possible types of $\neg A_2$, we distinguish the following cases.

1. $\neg A_2$ is a singular literal with variable x. Then $\sigma = \{x/t_1\}$. The resolvent $(D_1 \cup D_2)\sigma$ will contain only Skolem functions of $\{A_1\}$. Since all Skolem functions in $\{A_1\} \cup D_1$ are k'_1-originated, this will be the case for $(D_1 \cup D_2)\sigma$ as well.
2. $\neg A_2$ is a literal of non-singular and shallow. Similar to the previous case.
3. $\neg A_2$ is a deep literal. Then there is a term t_2 in $\neg A_2$ dominating every argument of every literal in $\{\neg A_2\} \cup D_2$. According to Lemma 2(2) the terms t_1 and t_2 occur at the same argument position in A_1 and $\neg A_2$, respectively. Since A_1 and A_2 are unifiable, t_1 and t_2 are unifiable, too. Hence, the top function symbols of t_1 and t_2 are equal. This function symbol is both k'_1-originated and k'_2-originated according to our assumptions. Thus, $k'_1 = k'_2$. Consequently, all non-constant function symbols in $(D_1 \cup D_2)\sigma$ are k'_1-originated. □

Corollary 19. *Let C_1 be an inhabited, strongly k_1-regular clause and C_2 be an inhabited, strongly k_2-regular clause such that $k_1 \neq k_2$. Then C_1 and C_2 have no $\succ_S$-resolvent.*

Thus by Lemma 18 and Corollary 19:

Lemma 20. *Let C_1 be an indecomposable, strongly k_1-regular clause and C_2 an indecomposable, strongly k_2-regular clause such that C_1 and C_2 are variable-disjoint. Let C be a $\succ_S$-resolvent of C_1 and C_2. Every split component of C is strongly k-regular for some k.*

Theorem 21. *The procedure described in Section 5 is a decision procedure for $\overline{\text{DKC}}$ and $\overline{\text{DK}}$.*

7 Conclusion

The results presented in this paper together with the result in [7] concerning fragments of first-order logic including transitivity based on the ordered chaining calculus, and the result in [6] concerning the guarded fragment, show that for a wide range of first-order fragments resolution-based decision procedures can be provided in a uniform framework.

We may ask whether the limits of decidability can be pushed further than $\overline{\text{DK}}$. It follows from a recent result in [8] that allowing transitive relations leads to undecidability. Even though there are a number of weaker classes which, extended with equality are decidable, the enrichment of $\overline{\text{K}}$ with equality gives an undecidable class [10]. We could imagine that decidable extensions of $\overline{\text{K}}$ or $\overline{\text{DK}}$ with restricted forms of equality can be obtained by following [5]. However, it should be kept in mind that even equalities between constants are non-trivial in our context, since their application may destroy the regularity of a literal.

The class $\overline{\text{DK}}$ covers the relational translation of modal formulae of basic modal logic as well as the correspondence properties of many modal axiom schemata. Therefore, it is interesting to investigate the relation of $\overline{\text{DK}}$ to the guarded fragment. From the perspective of first-order logic, the two fragments are incomparable. The formula ϕ_2 of Section 2 belongs to $\overline{\text{DK}}$, but not to the guarded fragment, while it is the opposite for the formula ϕ_4. However, from a modal perspective, the class $\overline{\mathrm{DK}}$ may be regarded as a generalisation of Boolean modal logic, the multi-modal logic defined over families of binary relations closed under the Boolean operations [9].

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