Symbolic Arithmetical Reasoning with Qualified Number Restrictions

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Abstract

Many inference systems used for concept description logics are constraint systems that employ tableaux methods. These have the disadvantage that for reasoning with qualified number restrictions n new constant symbols are generated for each concept of the form $(\geq n\ R\ C)$. In this paper we present an alternative method that avoids the generation of constants and uses a restricted form of symbolic arithmetic considerably different from the tableaux method. The method we use is introduced in Ohlbach, Schmidt and Hustadt 1995 for reasoning with graded modalities. We exploit the exact correspondence between the concept description language $\mathcal{ALCN}$ and the multi-modal version of the graded modal logic $\overline{\mathbf{K}}$ and show how the method can be applied to $\mathcal{ALCN}$ as well.

This paper is a condensed version of Ohlbach *et al.* 1995. We omit proofs and much of the technical details, but we include some examples.

1 The description logic $\mathcal{ALCN}^+$

The description logic $\mathcal{ALCN}$ defined in Hollunder and Baader 1991 extends the logic $\mathcal{ALC}$ of Schmidt-Schauß and Smolka 1991 with numerical quantifier constructs. The logic $\mathcal{ALC}$ can be viewed as a restricted form of a multi-modal logic, called $\mathbf{K}_{(m)}$ [Schild,1991]. The language of $\mathcal{ALC}$ includes the operations $\sqcap$ (conjunction), $\sqcup$ (disjunction), $\neg$ (negation), $\exists$ (existential role quantification) and $\forall$ (universal role quantification), and it includes two designated primitive concepts $\top$ (top) and $\bot$ (bottom). The Boolean operations correspond to the propositional connectives in modal logic, and role quantifier expressions of the general form $\exists R\!:\!C$ and $\forall R\!:\!C$ correspond to the modal formulae $\langle R\rangle C$ and $[R]C$, respectively. The numerical quantifier constructs in the language of $\mathcal{ALCN}$ have the general form

$$(\geq n\ R\ C) \quad \text{and} \quad (\leq n\ R\ C)$$

(with $n \geq 0$) and have modal correspondents as well [van der Hoek and de Rijke,1992]. $(\geq n\ R\ C)$ and $(\leq n\ R\ C)$ define sets of elements which have, respectively, at least n and at most n successors by R in C. For example,

$$\begin{aligned} \textit{city} \;&=\; \textit{place} \sqcap \\ &\quad (\geq 100\,001\ \textit{inhabited-by people}) \end{aligned} \tag{1}$$

defines the concept *city* to be a place with more than 100 000 inhabitants.

The language we consider is slightly more expressive than $\mathcal{ALCN}$. Our version, referred to as $\mathcal{ALCN}^+$, has no restrictions on the inclusion statements. On the left hand side of inclusions arbitrary concepts may occur, whereas in $\mathcal{ALCN}$ only atomic concepts can occur on the left hand side. Also, terminological cycles are allowed. $\mathcal{ALCN}^+$ coincides with the language $\mathcal{ALCNR}$ considered in Buchheit *et al.* 1993, except that role conjunction is not provided for.

Formally, the syntax of $\mathcal{ALCN}^+$ is defined as follows. The signature of the terminological language of $\mathcal{ALCN}^+$ consists of a set Σ_R of *role names* and a disjoint set Σ_C of *concept names*. From role names $Q \in \Sigma_R$ and concept names $A \in \Sigma_C$ compound concept terms C are formed according to the following rules:

$$\begin{aligned} C, D \quad \longrightarrow \quad & A \,|\, \neg C \,|\, C \sqcap D \,|\, C \sqcup D \,| \\ & \exists R\!:\!C \,|\, \forall R\!:\!C \,| \\ & (\geq n\ R\ C) \,|\, (\leq n\ R\ C) \,| \\ & C \sqsubseteq D \,|\, C = D. \end{aligned}$$

n is a non-negative integer. Most authors define the symbols $\sqsubseteq$ and $=$ to be sentential symbols. We define them to be connectives just as $\sqcap$ and $\sqcup$ are. Note, we consider terminological sentences of the form $C \sqsubseteq D$ and $C = D$ to be concept terms. In $\mathcal{ALCN}$ terminological sentences are constrained to be of the form $A \sqsubseteq C$ and $A = C$, where A are concept names. A *T-Box* is defined as a set of concept terms.

The semantics of $\mathcal{ALCN}^+$ is specified by an interpretation $\mathcal{I} = (U, V)$ with U a non-empty set U (the domain of interpretation) and a signature assignment V. The signature assignment maps role names to binary relations on U and it maps concept names to subsets of U. The interpretation of concept terms C and D specified by:

$$\begin{aligned} C^{\mathcal{I}} &= V(C) \quad \text{if } C \text{ is a concept name} \\ (C \sqsubseteq D)^{\mathcal{I}} &= (U \setminus C^{\mathcal{I}}) \cup D^{\mathcal{I}} \\ (C = D)^{\mathcal{I}} &= (U \setminus (C^{\mathcal{I}} \cup D^{\mathcal{I}})) \cup (C^{\mathcal{I}} \cap D^{\mathcal{I}}) \end{aligned}$$

and as usual for the remaining operations. Atomic concept names in a T-Box T are interpreted as the entire domain and are all equivalent to the top concept $\top$. $\top$ is the largest element in the subsumption ordering. The complement of $\top$ is $\bot$ and represents the empty set.

An interpretation $\mathcal{I} = (U, V)$ with $C^{\mathcal{I}} = U$ for all concept terms C in the T-Box T is a *model* of T. A concept term C is *universal* iff $C^{\mathcal{I}} = U$ for *all* interpretations $\mathcal{I}$. C is *empty* or *incoherent* iff $C^{\mathcal{I}} = \emptyset$ for *all* interpretations $\mathcal{I}$. The entailment relation $\models$ between concept terms is defined by: $C \models D$ iff $D^{\mathcal{I}} = U$ for every interpretation $\mathcal{I}$ of C. Then $C \models D$ iff $C \sqsubseteq D$ is universal iff $C \sqcap \neg D$ is empty.

We treat sets $\{C_1, \ldots, C_n\}$ of concept terms in the same way as the conjunction $C_1 \sqcap \ldots \sqcap C_n$. Thus, a given T-Box T will be treated as the conjunction of its elements.

In contrast to other terminological languages the language $\mathcal{ALCN}^+$ includes no role-forming operators. Roles that occur are all atomic. To simplify our presentation, without loss of generality we assume there is one atomic role R.

2 The graded modal logic $\overline{\mathbf{K}}$

The corresponding modal logic of $\mathcal{ALCN}^+$ is the multi-modal version of the graded modal logic $\overline{\mathbf{K}}$ (defined for example in van der Hoek 1992). Graded modalities are modal operators indexed with cardinal numbers which fix the number of worlds in which a formula is true. The formula $\Diamond_n \varphi$ (with n a non-negative integer) is true in a world iff there are more than n accessible worlds in which the formula φ is also true. The dual formula $\Box_n \varphi$, given by $\neg\Diamond_n \neg\varphi$, is then true in a world iff there are at most n accessible worlds in which $\neg\varphi$ is true. Another form of graded modal formula is $\Diamond!_n \varphi$. $\Diamond!_0 \varphi$ abbreviates $\Box_0 \neg\varphi$ and $\Diamond!_n \varphi$ abbreviates $\Diamond_{n-1}\varphi \wedge \neg\Diamond_n \varphi$ for $n > 0$. $\Diamond!_n \varphi$ is true in a word iff φ is true in exactly n accessible worlds. More formally, the semantics of the graded modal operators is defined in terms of one accessibility relation, say R, by:

$$\begin{aligned} &\mathcal{M}, x \models_{\overline{\mathrm{K}}} \Diamond_n \varphi \\ &\quad \text{iff} \quad |\{y \mid R(x,y) \;\&\; \mathcal{M}, y \models_{\overline{\mathrm{K}}} \varphi\}| > n \\ &\mathcal{M}, x \models_{\overline{\mathrm{K}}} \Box_n \varphi \\ &\quad \text{iff} \quad |\{y \mid R(x,y) \;\&\; \mathcal{M}, y \models_{\overline{\mathrm{K}}} \neg\varphi\}| \leq n \\ &\mathcal{M}, x \models_{\overline{\mathrm{K}}} \Diamond!_n \varphi \\ &\quad \text{iff} \quad |\{y \mid R(x,y) \;\&\; \mathcal{M}, y \models_{\overline{\mathrm{K}}} \varphi\}| = n. \end{aligned}$$

$\mathcal{M}$ denotes a model and x, y denote possible worlds. For any set X, $|X|$ denotes the cardinality of X. The modal version of $(\geq n\ R\ C)$ is $\Diamond_{n-1} C$ and the modal version of $(\leq n\ R\ C)$ is $\Box_n \neg C$. We think of the diamond and box operators being associated with the accessibility relation that defines the role R.

The logic $\overline{\mathbf{K}}$ has a Hilbert-style presentation $\Gamma_{\overline{\mathbf{K}}}$ that is sound and complete with respect to its natural possible worlds semantics [Fine,1972; de Caro,1988]. $\Gamma_{\overline{\mathbf{K}}}$ is defined by the following axioms

A1 the axioms of propositional logic

A2 $\vdash_{\overline{\mathrm{K}}} \Diamond_{n+1}\varphi \rightarrow \Diamond_n \varphi$

A3 $\vdash_{\overline{\mathrm{K}}} \Box_0(\varphi \rightarrow \psi) \rightarrow (\Diamond_n \varphi \rightarrow \Diamond_n \psi)$

A4 $\vdash_{\overline{\mathrm{K}}} \Box_0 \neg(\varphi \wedge \psi) \rightarrow$
$((\Diamond!_n \varphi \wedge \Diamond!_m \psi) \rightarrow \Diamond!_{n+m}(\varphi \vee \psi))$

together with the uniform substitution rule, Modus Ponens, and the necessitation rule for $\Box_0$:

US if φ is a theorem so is every substitution instance of φ

MP if $\vdash_{\overline{\mathrm{K}}} \varphi$ and $\vdash_{\overline{\mathrm{K}}} \varphi \rightarrow \psi$ then $\vdash_{\overline{\mathrm{K}}} \psi$

N if $\vdash_{\overline{\mathrm{K}}} \varphi$ then $\vdash_{\overline{\mathrm{K}}} \Box_0 \varphi$.

Van der Hoek 1992 shows $\overline{\mathbf{K}}$ is decidable.

3 $\mathcal{ALCN}^+$ and $\overline{\mathbf{K}}$

We now show that $\mathcal{ALCN}^+$ can be embedded in $\overline{\mathbf{K}}$. Define a mapping τ from $\mathcal{ALCN}^+$ to $\overline{\mathbf{K}}$ by:

$$\begin{aligned} \tau(C) &= C \quad \text{if } C \text{ is an atomic concept} \\ \tau(\neg C) &= \neg\tau(C) \\ \tau(C \sqcap D) &= \tau(C) \wedge \tau(D) \\ \tau(C \sqcup D) &= \tau(C) \vee \tau(D) \\ \tau(C \sqsubseteq D) &= \tau(C) \rightarrow \tau(D) \\ \tau(C = D) &= \tau(C) \leftrightarrow \tau(D) \\ \tau(\exists R\!:\!C) &= \Diamond_0 \tau(C) \\ \tau(\forall R\!:\!C) &= \Box_0 \tau(C) \\ \tau(\geq n\ R\ C) &= \Diamond_{n-1}\tau(C) \\ \tau(\leq n\ R\ C) &= \Box_n \neg\tau(C) \end{aligned}$$

Theorem 1 *The translation τ is sound and complete: For any concept C, C is universal iff $\tau(C)$ is a tautology, i.e. $\models_{\overline{\mathrm{K}}} \tau(C)$.*

Proof. Let $\mathcal{I} = (U, V)$ be any interpretation of a T-Box of $\mathcal{ALCN}^+$. Let $\mathcal{M}$ be the modal model $(U, R^{\mathcal{I}}, V)$. By induction on the structure of C prove, for every $x \in U$: $x \in C^{\mathcal{I}}$ iff $\mathcal{M}, x \models_{\overline{\mathrm{K}}} \tau(C)$. The proof is routine and we omit the details. $\Box$

Since $\overline{\mathbf{K}}$ is sound and complete, $\Gamma_{\overline{\mathbf{K}}}$ provides an axiomatization for $\mathcal{ALCN}^+$ with one role (that is both sound and complete). We have:

Theorem 2 *For any concept C, C is universal iff its translation $\tau(C)$ is provable in $\Gamma_{\overline{\mathbf{K}}}$.*

A sound and complete axiomatisation for $\mathcal{ALCN}^+$ with arbitrarily many roles is given by the multi-modal version of $\Gamma_{\overline{\mathbf{K}}}$ in which each modal operator is indexed with an R.

4 Reasoning for $\mathcal{ALCN}^+$

Tableaux systems like the constraint system described in Hollunder and Baader 1991, for description logics with numerical quantifier constructs, are based on the set-theoretic semantics and generate for each numerical quantifier $\geq n$ (or $\Diamond_{n-1}$) n new constant symbols. For large n as in the sample definition (1) this is infeasible. On the other hand, the axiomatization of $\overline{\mathbf{K}}$ is formulated with arithmetical terms, and in principle, this allows for invoking arithmetical computations, thus, avoiding the manipulation of explicitly generated constants. However, Hilbert systems have other disadvantages that makes them unsuitable to form the basis for automated reasoning.

Ohlbach *et al.* 1995 show that the axiomatization $\Gamma_{\overline{\mathbf{K}}}$ of $\overline{\mathbf{K}}$ can be transformed into a first-order theory that exactly captures $\overline{\mathbf{K}}$. The first-order theory is defined by the set of clauses P1–P12 given in Figure 1 below.

Theorem 3 *Any graded modal formula φ is provable in $\overline{\mathbf{K}}$ iff its first-order translation* FO(φ) *is a logical consequence of the first-order theory iff* $\neg$FO(φ) *in clause form is refutable from* P1–P12.

P1–P12 are obtained in two reduction steps.

The first reduction embeds $\overline{\mathbf{K}}$ in an intermediary multi-modal logic, called $\overline{\mathbf{K}}_E$, with a standard Kripke semantics.

The second reduction uses an optimization of the functional translation method of multi-modal logics into sorted predicate logic [Ohlbach and Schmidt,1995].

The first reduction is necessary, because the functional translation method is formulated for multi-modal logics and $\overline{\mathbf{K}}$ is not a multi-modal logic in the usual sense that each modality is associated with a different binary relation (in $\overline{\mathbf{K}}$ every modality $\Diamond_n$ is associated with the same relation R).

5 The logics $\overline{\mathbf{K}}_E$ and $\overline{\mathbf{K}}$

$\overline{\mathbf{K}}_E$ has two kinds of modalities:

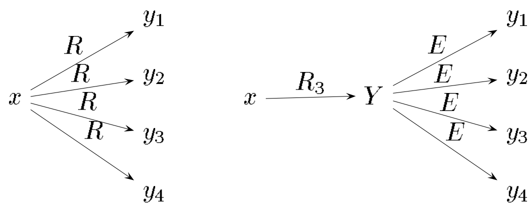
(i) $\langle n\rangle$ and $[n]$, which are characterized in the Kripke frames by infinitely but countably many different relations R_n ($n \in \mathrm{N}_0$), and

(ii) $\Diamond$ and $\Box$, which are characterized by a designated relation E.

Ohlbach et al. translate formulae of $\overline{\mathbf{K}}$ to formulae of $\overline{\mathbf{K}}_E$ according to the following rules (for modal operators):

$$\begin{aligned} \Pi(\Diamond_n\varphi) &= \langle n\rangle\Box\Pi(\varphi) \\ \Pi(\Box_n\varphi) &= [n]\Diamond\Pi(\varphi). \end{aligned} \quad (2)$$

The intuitive idea underlying the translation of $\Diamond_n\varphi$ is this: If φ is true in a set Y of worlds with more than n elements then we introduce an accessibility relation R_n that connects the actual world and a world w_Y which we can think of as being a representative for the set Y. This defines the $\langle n\rangle$ operator. $\Box\varphi$ and its associated accessibility relation E expresses that φ is true in all the worlds of the set Y. E can be thought of as the reverse membership relation. Thus, $\langle n\rangle\Box\varphi$ encodes 'there is a set with more than n elements (encoded by $\langle n\rangle$) and φ is true for all the elements of this set (encoded by $\Box$)'. For example, consider the formula $\Diamond_3\varphi$. According to the semantic definition $\Diamond_3\varphi$ is true in a world x iff there are at least 4 worlds to which x is R-related. This definition is depicted in the first picture below. The second picture depicts our new alternative view.



The relation R is replaced by the relational composition of the two new relations R_3 and E. In the process we have introduced a new world which we labelled Y as it is meant to represent the set of worlds y_1, y_2, y_3 and y_4.

$\overline{\mathbf{K}}_E$ has a Hilbert-style axiomatisation $\Gamma_{\overline{\mathbf{K}}_E}$. The axioms and rules are:

N1 the axioms of propositional logic and Modus Ponens

N2 the K-axioms for $[n]$ and $\Box$:
$\vdash_{\overline{\mathrm{K}}_E} [n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$
$\vdash_{\overline{\mathrm{K}}_E} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

N3 the necessitation rules for $[n]$ and $\Box$:
if $\vdash_{\overline{\mathrm{K}}_E} \varphi$ then $\vdash_{\overline{\mathrm{K}}_E} [n]\varphi$
if $\vdash_{\overline{\mathrm{K}}_E} \varphi$ then $\vdash_{\overline{\mathrm{K}}_E} \Box\varphi$

N4 $\vdash_{\overline{\mathrm{K}}_E} [0]\Diamond\varphi \to [n]\Box\varphi$

N5 $\vdash_{\overline{\mathrm{K}}_E} \langle n\rangle\Box\varphi \to \langle n\rangle\Diamond\varphi$

N6 $\vdash_{\overline{\mathrm{K}}_E} [n]\varphi \to [n+1]\varphi$

N7 $\vdash_{\overline{\mathrm{K}}_E} \langle n+m\rangle\Box(\varphi \vee \psi) \to (\langle n\rangle\Box\varphi \vee \langle m\rangle\Box\psi)$

N8 $\vdash_{\overline{\mathrm{K}}_E} \langle n\rangle\Box\varphi \wedge \langle m\rangle\Box\psi \wedge [j]\Diamond\neg(\varphi \wedge \psi)$
$\to \langle n+m+1-j\rangle\Box(\varphi \vee \psi)$.

Here, N8 is in a more general form than that of Ohlbach *et al.* 1995 without changing $\overline{\mathbf{K}}_E$. The resulting set of first-order clauses obtained by the reduction to come is larger and better suited for our purposes, though.

Theorem 4 *The transformation of $\overline{\mathbf{K}}$ to $\overline{\mathbf{K}}_E$ is sound and complete: for any graded modal formula φ, φ is provable from $\Gamma_{\overline{\mathbf{K}}}$ iff its translation Π (defined in (2) for modal formulae) is provable from $\Gamma_{\overline{\mathbf{K}}_E}$.*

6 $\overline{\mathbf{K}}_E$ and first-order logic

One of the axioms of $\overline{\mathbf{K}}_E$ is not first-order definable, if we use, in the second reduction step, the standard relational translation of multi-modal logics to first-order logic. Using the functional translation we can avoid this problem [Ohlbach and Schmidt,1995]. Furthermore, the

P1 $de_n(w_*),\ [w_* x_n z] = [w_* f_0^n(w_*, x_n, z) y]$

P2 $AF_m \sqsubseteq AF_n$ for all $m > n$

P3 $\neg de_n(w_*),\ de_m(w_*)$ for all $m > n$

P4 $de_{n+m}(w_*),\ [w_* x_{n+m} h1^{nm}(w_*, x_{n+m}, y)] = [w_* h2_n^{nm}(w_*, x_{n+m}, y, z) y]$

P5 $de_{n+m}(w_*),\ [w_* x_{n+m} h1^{nm}(w_*, x_{n+m}, y)] = [w_* h3_n^{nm}(w_*, x_{n+m}, y) z]$

P6 $de_{\max(n,m)}(w_*),\ \neg de_{n+m+1}(w_*),\ [w_* x_n k1^{nm}(w_*, x_n, y_m)] = [w_* y_m k2^{nm}(w_*, x_n, y_m)]$

P7 $de_{\max(n,m)}(w_*),\ [w_* x_n k3^{nm}(w_*, x_n, y_m)] = [w_* y_m k4^{nm}(w_*, x_n, y_m)],$
$[w_* x_n k5^{nm}(w_*, x_n, z)] = [w_* k_{n+m+1}^{nm}(w_*, x_n, y_m) z],$
$[w_* y_m k6^{nm}(w_*, y_m, z)] = [w_* k_{n+m+1}^{nm}(w_*, x_n, y_m) z].$

P8 $de_{\max(n,m)}(w_*),\ \neg de_{n+m+1-j}(w_*),\ [w_* f7_j^{nmj}(w_*, x_n, x_m) u] = [w_* x_n f5^{nmj}(w_*, x_n, u)]$

P9 $de_{\max(n,m)}(w_*),\ \neg de_{n+m+1-j}(w_*),\ [w_* f7_j^{nmj}(w_*, x_n, x_m) u] = [w_* x_m f6^{nmj}(w_*, x_m, u)]$

P10 $de_{\max(n,m)}(w_*),\ \neg de_j(w_*),\ [w_* f1_{n+m+1-j}^{nmj}(w_*, v, x_n) v] = [w_* x_n f2^{nmj}(w_*, v, x_n)],$
$[w_* f3_{n+m+1-j}^{nmj}(w_*, v, x_m) v] = [w_* x_m f4^{nmj}(w_*, v, x_m)]$

P11 $de_{\max(n,m)}(w_*),\ [w_* f1_{n+m+1-j}^{nmj}(w_*, v, x_n) v] = [w_* x_n f2^{nmj}(w_*, v, x_n)],$
$[w_* f3_{n+m+1-j}^{nmj}(w_*, v, x_m) v] = [w_* x_m f4^{nmj}(w_*, v, x_m)],$
$[w_* f7_j^{nmj}(w_*, x_n, x_m) u] = [w_* x_n f5^{nmj}(w_*, x_n, u)]$

P12 $de_{\max(n,m)}(w_*),\ [w_* f1_{n+m+1-j}^{nmj}(w_*, v, x_n) v] = [w_* x_n f2^{nmj}(w_*, v, x_n)],$
$[w_* f3_{n+m+1-j}^{nmj}(w_*, v, x_m) v] = [w_* x_m f4^{nmj}(w_*, v, x_m)],$
$[w_* f7_j^{nmj}(w_*, x_n, x_m) u] = [w_* x_m f6^{nmj}(w_*, x_m, u)]$

Figure 1: The first-order theory that captures $\overline{\mathbf{K}}_E$, $\overline{\mathbf{K}}$ and $\mathcal{ALCN}^+$ with one role.

functional approach has computational advantages over the standard relational translation method.

Recall, the relational translation of modal logics is commonly denoted by ST and uses the Kripke semantics definition, see e.g. van Benthem 1984. For (multi-) modal operators, ST is defined by:

$$\begin{aligned} \mathrm{ST}(\langle R\rangle\varphi, x) &= \exists y\ R(x,y) \wedge \mathrm{ST}(\varphi, y) \\ \mathrm{ST}([R]\varphi, x) &= \forall y\ R(x,y) \rightarrow \mathrm{ST}(\varphi, y) \end{aligned}$$

The functional translation method [Ohlbach,1991] is based on the fact that any binary relation R can be defined by a set AF_R of partial functions, namely:

$$R(x,y) \quad \leftrightarrow \quad \exists \gamma \in AF \ \ y = \gamma(x).$$

The functional translation π_f for (multi-) modal formulae is:

$$\begin{aligned} \pi_f(\langle R\rangle\varphi, x) &= \neg de_R(x) \wedge \exists\gamma{:}AF_R\ \pi_f(\varphi, \downarrow(\gamma, x)) \\ \pi_f([R]\varphi, x) &= \neg de_R(x) \rightarrow \forall\gamma{:}AF_R\ \pi_f(\varphi, \downarrow(\gamma, x)). \end{aligned}$$

The term $\neg de_R(x)$ is meant to capture that x is not a dead-end in the relation R, i.e. R is defined for x. $\downarrow$ is the 'apply' function, so, we can think of the term $\downarrow(\gamma, x)$ as representing $\gamma(x)$. Total (serial) relations can be defined by a set of total functions and have a simpler functional formulation not involving dead-end predicates. The relation E in the semantics of $\overline{\mathbf{K}}_E$ is a total relation. For modal operators, like $\Box$ of $\overline{\mathbf{K}}_E$, which are determined by a total relation the functional translation π_f is given by:

$$\begin{aligned} \pi_f(\langle R\rangle\varphi, x) &= \exists\gamma{:}AF_R\ \pi_f(\varphi, \downarrow(\gamma, x)) \\ \pi_f([R]\varphi, x) &= \forall\gamma{:}AF_R\ \pi_f(\varphi, \downarrow(\gamma, x)). \end{aligned}$$

Take for example this concept

$$(\geq 1\ R\ (\geq 1\ R\ (\geq 4\ R\ \top))).$$

Its modal translation by τ is $\Diamond_0\Diamond_0\Diamond_3\top$. The first reduction by Π (in (2)) yields its $\overline{\mathbf{K}}_E$-version: $\langle 0\rangle\Box\langle 0\rangle\Box\langle 3\rangle\Box\top$. According to the functional translation the second reduction yields, for a world w (in a simplified form):

$$\begin{aligned} &\neg de_0(w) \\ &\wedge \exists\alpha{:}AF_0\ \forall\gamma{:}AF_E\ \neg de_0([\alpha\gamma]w) \\ &\wedge \exists\beta{:}AF_0\ \forall\delta{:}AF_E\ \neg de_3([\alpha\gamma\beta\delta]w). \end{aligned} \tag{3}$$

de_n is an abbreviated notation for de_{R_n}. Note, since E is a total relation it is defined by a set AF_E of total functions, which implies the dead-end predicates de_E are superfluous.

Likewise the axioms $\Gamma_{\overline{\mathbf{K}}_E}$ of $\overline{\mathbf{K}}_E$ can be systematically translated into their functional representation. We omit the details, but see Ohlbach *et al.* 1995. This translation does not yet give us P1–P12 of Figure 1. It yields a set of second-order formulae. To compute the first-order equivalents, we use the elimination algorithm of second-order quantifiers of Gabbay and Ohlbach 1992 together with an optimization step developed in Ohlbach and Schmidt 1995. The resulting clauses are P1–P12. (Again, we omit the details.)

Ohlbach and Schmidt 1995 prove:

Theorem 5 *The functional translation is sound and complete relative to the completeness of the relational translation.*

The theorem applies to extensions of the normal multi-modal logic $\mathbf{K}_{(m)}$. Let Φ be the additional axioms that define the extension. The theorem says, more formally, *provided the second-order relational translation of the extension Φ is complete with respect to a first-order class of frames, φ is provable in Φ iff $\Pi_f(\Phi) \rightarrow \Pi_f(\varphi)$ is a predicate logic theorem.* (Π_f is the functional translation mapping for axioms and rules that uses the translation mapping π_f for modal formulae.)

Ohlbach and Schmidt 1995 also prove a stronger theorem for the functional translation together with an optimization that allows for functional existential and universal quantifiers to be swapped arbitrarily. This rule exploits that one relational frame in general corresponds to many 'functional frames', and there is always one which is rich enough to allow for moving existential quantifiers over universal quantifiers.

Theorem 6 *The functional translation with the quantifier exchange rule is sound and complete relative to the completeness of the relational translation.*

Combining Theorems 4 to 6 it is easy to see that the sequence of translations which we described in Sections 5 and 6 is sound and complete. *For any $\overline{\mathbf{K}}$-formula φ,* we have *φ is a $\overline{\mathbf{K}}$-theorem iff* (P1–P12) $\rightarrow \Upsilon(\Pi_f(\varphi))$ *is a predicate logic theorem.* This is the main theorem which we stated earlier in Theorem 3. The operation FO of Theorem 3 is the combination of the $\overline{\mathbf{K}}$ to $\overline{\mathbf{K}}_E$ translation Π and the $\overline{\mathbf{K}}_E$ to predicate logic translation Π_f followed by Υ, the optimized functional translation. Υ is the operation that moves existential functional quantifiers inwards over universal quantifiers. This operation is not mandatory. It is useful for replacing in complete modal logics, modal axioms that alone (without the interaction with other axioms and rules) have no first-order (relational) correspondence property, by weaker (functional) correspondence properties.

Note, the notation used in Figure 1 is very much abbreviated. It is known as the world path notation. Subscripts indicate the sort of a symbol. For example, x_n is a variable of sort AF_n. f_0^n is a Skolem function of sort AF_0. The absence of subscripts as for z and $h1^{nm}$ indicates being of sort AF_E. The variable w_* is of sort AF_R^*, which is the supersort of all sorts AF_n.

P1 has the following reading: if w_* is an arbitrary (path to a) world from which a path leads to a set of worlds with more than n worlds ($de_n(w_*), \ldots$) then for any path x_n to a set X of worlds with $|X| > n$ and any world z in this set there exists a path $f_0^n(w_*, x_n, z)$ to a non-empty set B and any y in this set is also in X. This implies $\{z\} \subseteq X$.

7 Symbolic arithmetical reasoning for $\mathcal{ALCN}^+$ and graded modal logic

Evidently, because of the exact correspondence between $\mathcal{ALCN}^+$ and the graded modal logic $\overline{\mathbf{K}}$, Theorem 3 remains true if we replace 'graded modal formula' and φ by 'concept' and C, respectively. Consequently we may apply the inference method of Ohlbach *et al.* 1995 for $\mathcal{ALCN}^+$ too. The method uses theory resolution based on P1–P12. This has the advantage that instead of counting constants we use traditional resolution together with symbolic arithmetical reasoning. We demonstrate this by way of two examples.

Example 1 Consider the set of concepts

$$\begin{array}{c}(\geq 1\ R\ (\geq 1\ R\ (\geq 4\ R\ \top))),\\ (\geq 1\ R\ (\geq 1\ R\ (\leq 3\ R\ \top))),\\ (\geq 1\ R\ (\leq 1\ R\ \top)),\\ (\leq 1\ R\ \top).\end{array} \tag{4}$$

The corresponding set of $\overline{\mathbf{K}}$-formulae are

$$\Diamond_0\Diamond_0\Diamond_3\top, \quad \Diamond_0\Diamond_0\Box_3\bot, \quad \Diamond_0\Box_1\bot, \quad \Box_1\bot. \tag{5}$$

The conjunction of (4) is incoherent (or unsatisfiable). This amounts to the same thing as saying, the set (5) is inconsistent.

Above we derived the functional translation (3) for the first concept in (4). The functional translation for the remaining concepts is obtained analogously. The sets reduce to the following set of clauses.

C_1	$\neg de_0([])$	C_5	$de_3([c_0x'd_0y'])$
C_2	$\neg de_0([a_0x])$	C_6	$de_1([e_0x''])$
C_3	$\neg de_3([a_0xb_0y])$	C_7	$de_1([])$
C_4	$\neg de_0([c_0x'])$		

The clauses C_1, C_2 and C_3 represent (3). The empty path [] replaces the world variable w. a_0 and b_0 are the Skolem constant associated with α and β. The variables x and y are associated with γ and δ.

Now, we demonstrate how we can use P1–P12 and refutational theorem proving together with limited symbolic arithmetic to verify the claim that the conjunction of (4) is incoherent. For the refutation we use P1 with $n = 0$ and P6 with $n = 0$ and $m = 0$. P1 can immediately be simplified with clause C_1. The instances are:

P1′ $[f_0^0([], x_0, z)y] = [x_0z]$

P6′ $de_0(w_*)$, $\neg de_1(w_*)$, $[w_*x_0k1^{00}(w_*, x_0, y_0)]$
$= [w_*y_0k2^{00}(w_*, x_0, y_0)]$.

The result of simultaneously resolving P6′, C_1, and C_7 using the unifier $\{w_* \mapsto []\}$ is

$$C_8 \quad [x_0k1^{00}([], x_0, y_0)] = [y_0k2^{00}([], x_0, y_0)].$$

Paramodulating with C_8 and unifier

$$\{x_0 \mapsto a_0,\ x \mapsto k1^{00}([], a_0, y_0)\},$$

C_3 becomes (this means we do equality replacement with unification in C_3 using the equation C_8)

$$C_9 \quad \neg de_3([y_0 k2^{00}([], a_0, y_0) b_0 y]).$$

This becomes

$$C_{10} \quad \neg de_3([x_0 z b_0 y])$$

when paramodulating with P1$'$ using the unifier

$$\{y_0 \mapsto f_0^0([], x_0, z),\ y \mapsto k2^{00}([], a_0, y_0)\}.$$

We resolve P6$'$ and C_4 using unifier

$$\{w_* \mapsto [c_0 x],\ x' \mapsto x\}$$

to get

$$C_{11} \quad \neg de_1([c_0 x]),\ [c_0 x x_0' k1^{00}([c_0 x], x_0', y_0')] = [c_0 x y_0' k2^{00}([c_0 x], x_0', y_0')].$$

Now, use the unifier

$$\{x_0 \mapsto c_0,\ x' \mapsto z \mapsto x, x_0' \mapsto d_0,\ y_0' \mapsto b_0,\ y' \mapsto k1^{00}([c_0 x], d_0, b_0),\ y \mapsto k2^{00}([c_0 x], d_0, b_0)\}$$

and apply E-resolution to C_5, C_{10} and C_{11} and get

$$C_{12} \quad \neg de_1([c_0 x]).$$

(This means we resolve between C_5 and C_{10} using an equation in C_{11}.) Resolving this with C_6 using E-resolution with C_8 yields the empty clause. The unifier is

$$\{x_0 \mapsto e_0,\ x'' \mapsto k1^{00}([], e_0 c_0),\ y_0 \mapsto c_0,\ x'' \mapsto k2^{00}([], e_0 c_0)\}.$$

Note, the computational effort does not depend on the numbers we use. The proof of this example is not significantly different for other (possibly large) values.

Example 2 Suppose the universe consists of at most thirty horses. If there are at least twenty horses that are white and there are at least twenty horses that are black, then there are at least ten zebras. Let W denote the set of white horses and B the set of black horses. Then $W \cap B$ denotes the set of zebras.

A standard tableaux system for the number operators would generate twenty witnesses for W, twenty witnesses for B and then it would need to identify ten of them in order not to exceed the limit of thirty. But there are combinatorically many ways for identifying ten of them.

In our system we prove the conjecture by showing the following set of $\mathcal{ALCN}^+$ concepts is incoherent:

$$(\geq 20\ R\ W),\quad (\geq 20\ R\ B),\quad (\leq 30\ R\ \bot),\quad (\leq 9\ R\ \neg(W \sqcap B)).$$

The corresponding set of $\overline{\mathbf{K}}$-formulae are:

$$\Diamond_{19} W,\quad \Diamond_{19} B,\quad \Box_{30}\bot,\quad \Box_9 \neg(W \wedge B).$$

We could choose any other suitable combination of numbers. This would not change the structure of the proof at all. The translation into first-order clause form is:

$$\neg de_{19}([]) \wedge W([a_{19} x]),\quad \neg de_{19}([]) \wedge B([b_{19} x]),\quad de_{30}([]),\quad de_9([]) \vee \neg W([y_9 c]) \vee \neg B([y_9 c]).$$

The corresponding set of clauses consists of:

$$\begin{array}{ll} C_1 & \neg de_{19}([]) \\ C_2 & W([a_{19} x]) \\ C_3 & B([b_{19} y]) \\ C_4 & de_{30}([]) \\ C_5 & de_9([]),\ \neg W([y_9 c]),\ \neg B([y_9 c]) \end{array}$$

Resolve C_5 with P3 and C_1 and eliminate the $de_9([])$ literal from C_5 leaving:

$$C_5' \quad \neg W([y_9 c]),\ \neg B([y_9 c])$$

We resolve the instance of P9 with $n = m = 19$, $j = 9$, namely

$$\text{P9}' \quad de_{19}([]),\ \neg de_{30}([]),\ [f7_9^{19\,19\,9}([], x_{19}, x_{19}')u] = [x_{19} f6^{19\,19\,9}([], x_{19}', u)],$$

with C_1 and C_4 and obtain

$$C_6 \quad [f7_9^{19\,19\,9}([], x_{19}, x_{19}')u] = [x_{19}' f6^{19\,19\,9}([], x_{19}', u)].$$

Applying the unifier

$$\{y_9 \mapsto f7_9^{19\,19\,9}([], x_{19}, x_{19}'), u \mapsto c\},$$

we can use this in a paramodulation step with C_5' resulting in

$$C_7 \quad \neg W([x_{19}' f6^{19\,19\,9}([], x_{19}', c)]),\ \neg B([f7_9^{19\,19\,9}([], x_{19}, x_{19}')c])$$

Unify in C_2 and C_7 with

$$\{x_{19}' \mapsto a_{19}, x \mapsto f6^{19\,19\,9}([], x_{19}', c)\}.$$

Resolving C_2 and C_7 yields

$$C_8 \quad \neg B([f7_9^{19\,19\,9}([], x_{19}, a_{19})c]).$$

Now we use the following instance of P8:

$$\text{P8}' \quad de_{19}(w_*),\ \neg de_{30}(w_*),\ [w_* f7_9^{19\,19\,9}(w_*, x_{19}, x_{19}')u] = [w_* x_{19} f5^{19\,19\,9}(w_*, x_{19}, u)]$$

This can be reduced with C_1 and C_4 to the equation

$$C_9 \quad [f7_9^{19\,19\,9}([], x_{19}, x_{19}')u] = [x_{19} f5^{19\,19\,9}([], x_{19}, u)],$$

which we can now use in a paramodulation step with C_8. We get

$$C_{10} \quad \neg B([x_{19} f5^{19\,19\,9}([], x_{19}, c)]).$$

The empty clause is obtained if we resolve C_{10} with C_3 using the appropriate unifier.

8 Conclusion

In description logics with qualified number restrictions it is possible to express properties of finite sets. The usual constraint inference algorithms similar to those described in Schmidt-Schauß and Smolka 1991 and Hollunder and Baader 1991 generate for all sets used in the proof at least as many constants (witnesses) as the cardinality of each set. Even for moderate values a vast number of witnesses are generated which are processed by case distinctions in the proof. For weaker description languages considered by Donini *et al.* 1994 and Calvanese *et al.* 1994 reasoning with unqualified number restrictions by case distinction can be avoided.

In this paper we have considered an alternative method for reasoning with both qualified and unqualified number restriction which does not have the overhead of evaluating case distinctions. Instead the method uses limited arithmetical reasoning. The method was developed for graded modal logics and can be readily applied to all description languages that are included in $\mathcal{ALCN}^+$. We showed that there is a close correspondence between $\mathcal{ALCN}$ and the graded modal logic $\overline{\mathbf{K}}$. In fact, there is an exact correspondence between the terminological operators and the modal operators.

We conclude with two remarks. (i) In the examples only one role R was used. The approach can be applied to description logics with multiple roles (without role definitions), too. (ii) We have restricted our attention to TBox reasoning. This corresponds directly to that in modal logic. We haven't accounted for A-Box reasoning about concrete instantiations of concepts/sets and roles/relations. The functional translation applied to A-Box terms generates many equations. It is not immediate how these can be treated efficiently. It remains to be investigated how the method can be extended to handle ABox reasoning as well.

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