Knowledge Representation & Reasoning

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Module Delivery

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Module notes can be found at
http://www.csc.liv.ac.uk/~wiebe/Teaching/COMP521/

Module Delivery

- Lectures are:

Monday	12:00 - 13:00	
Wednesday	11:00 - 12:00	CTH-LTD
Friday	13:00 - 14:00	JHERD-LT

- Tutorials are:

	Monday	10:00 - 11:00
OR	Friday	16:00 - 17:00

- Module notes can be found at

  `http://www.csc.liv.ac.uk/~wiebe/Teaching/COMP521/`

Module Aims

- To introduce Knowledge Representation as a research area.
- To give a complete and critical understanding of the notion of representation languages and logics.
- To study modal logics and their use;
- To study description logic and its use;
- To study epistemic logic and its use
- To study methods for reasoning under uncertainty

Learning Outcomes

- be able to explain and discuss the need for formal approaches to knowledge representation in artificial intelligence, and in particular the value of logic as such an approach;
- be able to demonstrate knowledge of the basics of propositional logic
- be able to determine the truth/satisfiability of modal formula;
- be able to perform modal logic model checking on simple examples
- be able to perform inference tasks in description logic
- be able to model problems concenring agents' knowledge using epistemic logic;
- be able to indicate how updates and other epistemic actions determine changes on epistemic models;
- have sufficient knowledge to build "interpreted systems" from a specification, and to verify the "knowledge" properties of such systems;
- be familiar with the axioms of a logic for knowledge of multiple agents;

Example: Russian Cards

> *Three players, say A, B and C hold 7 cards, say the deal d is such that A holds 0, 1 2, B holds 3, 4, 5 and C has 6. Each player knows its own cards, and it is common knowledge how many cards everybody has and how many cards there are.*

Now, design a protocol P (means here: an exchange of publicly announced messages) after which it is common knowledge that

1. A and B both know d
2. C knows only of card 6 who owns it

Example: Muddy Children

n children have a party, $k \leq n$ of them get muddy. Father calls the children in a circle around him: every child can see the other children, no child sees itself. Let φ be: "at least of you is muddy. If you know that you are, please step forward"

1 Father says: φ! nothing happens

2 Father says: φ! nothing happens

3 . . . . nothing happens

k Father says: φ!
textcolorred the k muddy children step forward!

Module objectives

At the end of the module you should

- understand the need for formal approaches to knowledge representation
- understand the value of logic as a formal approach
- understand the basics of modal and description logics and how they are used
- be able to model epistemic problems using Kripke models
- be able to indicate how updates and other epistemic actions determine changes on these models
- be able to determine the truth of epistemic formulae in a given state
- be able to decide whether a given epistemic formula is satisfiable in a given class of models

Module structure (1)

Two parts:

Part 1: Logics for KR& R

- Knowledge representation and reasoning: introduction, logical approach
- Modal Logics: syntax, semantics (Kripke models), model checking, theorem proving
- Description Logics: syntax, semantics, satisfiability checking

Module structure (2)

Two parts:

Part 2: Applications of modal logic: Epistemic logic

- one agent: S5 models, specific properties
- multiple agents: modeling epistemic puzzles, reasoning about other's konwledge and ignorance, alternating bit protocol
- group notions of knowledge: distributed knowledge, common knowledge, muddy children example
- computational models: distributed systems

Module assessment

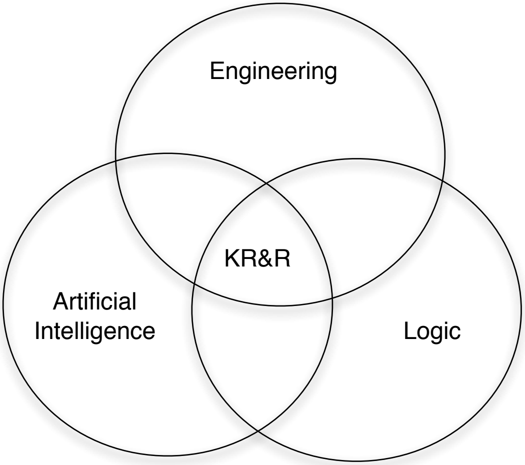
- There is no coursework for this module
  (This means you have more time for self study,
  not more spare time!)
- This module is assessed by exam and two continuous assessment exercises!
- The exam will take 2.5 hours and will assess the module objectives

Optional textbooks

Part 1: M. R. A. Huth and Mark D. Ryan
Logic in Computer Science: Modelling and reasoning about systems
Cambridge University Press (2000)
ISBN 0-521-65602-8.

Part 1 & 2: J.-J. Ch. Meyer and W. van der Hoek
Epistemic Logic for Computer Science and Artificial Intelligence
Cambridge Tracts in Theoretical Computer Science 41
Cambridge University Press (1995)
ISBN 0-521-46014.

Knowledge representation and reasoning (1)



Knowledge representation and reasoning (2)

Knowledge representation and reasoning is at the intersection of AI, Logic, and Engineering.

AI The science of understanding intelligent entities and the engineering of intelligent entities

In symbolic AI, intelligent entities have an explicit model of the world, and are able to reason about it

Consequently, we have to find out what these explicit models consist of and how it is possible to reason about them

Logic The science of reasoning

In particular, logic studies formalisms which can describe partial models of the world and calculi which allow to reason with them

Engineering The science of reasoning

the application of mathematics, scientific and practical knowledge in order to invent, design, build,

Knowledge representation and reasoning (4)

The designation 'Knowledge representation and reasoning' suggests that we are interested in two related problems:

- The representation of knowledge
- The reasoning about knowledge based on its representation

Historically, the relative importance of the two problems has been subject to a long-lasting debate.
We look at two contributions to this debate:

- Newell and Simon: The physical symbol system hypothesis
- Feigenbaum et al.: The knowledge principle

Physical symbol systems

A physical symbol system consist of

- a set of symbols
- a set of expressions (also called symbol structures)
- a set of procedures that operate on expressions to produce other expressions: Create, Modify, Reproduce, Destroy.

Physical symbol systems

- The symbol structures form a low-level representation of our memory
- The procedures form a low-level realisation of our reasoning processes

The physical symbol system hypothesis

Newell and Simon (1976)

> *A physical symbol system has the necessary and sufficient means for general intelligent action.*
>
> *By necessary we mean that any system that exhibits intelligence will prove upon analysis to be a physical symbol system.*
>
> *By sufficient we mean that any physical symbol system of sufficient size can be organized further to exhibit general intelligence.*
>
> *By general intelligent action we wish to indicate the same scope of intelligence as we see in human action.*

Maybe appropriate for high level tasks ('chess') but less so for lower level tasks such as vision

The knowledge level hypothesis

Newell and Simon

> *There exists a distinct computer systems level which is characterized by knowledge as the medium and the principle of rationality as the law of behaviour.*

Principle of rationality: if an agent has knowledge that one of its actions will lead to one of its goals then the agent will select that action.

Knowledge: Whatever can be ascribed to an agent such that its behaviour can be computed according to the principle of rationality.

The physical symbol system hypothesis: Critique

- The emphasis of physical symbol systems is on the procedures, therefore, on reasoning and ignores the importance of knowledge
- There is no claim that there is one fundamental physical symbol system but each system/entity showing intelligent behaviour could be a different physical symbol system
- This makes it difficult to falsify the hypothesis:
  - Suppose you show me a physical symbol system of which you claim that it is intelligent
  - I show you an example of ‘unintelligent’ behaviour of this system
  - Then you simply amend your system to avoid this particular behaviour

The knowledge principle (1)

Feigenbaum (1994)

> *The power of AI programs to perform at high levels of competence is primarily a function of the program's knowledge of its task domain, and not of the program's reasoning processes.*

Lenat and Feigenbaum (1989)

> *A system exhibits intelligent understanding and action at a high level of competence primarily because of the specific knowledge that it can bring to bear: the concepts, facts, representations, methods, models, metaphors, and heuristics about its domain of endeavor.*

The knowledge principle (2)

Feigenbaum (1994)

> *Physicians, not logicians, treat sick patients.*

Underlying assumptions here:

- Logicians are the better reasoners but have little medical knowledge
- Physicians are not as good at reasoning but have the relevant medical knowledge

Obviously, the knowledge principle puts the emphasis on knowledge (and its representation), instead of reasoning

The knowledge principle (3)

> *Knowledge is power, and computers that amplify that knowledge will amplify every dimension of power (Feigenbaum)*
>
> *The power of an intelligent program is to perform its task well depends primarily on the quantity and quality of knowledge it has about that task (Buchanan and Feigenbaum (1982))*

Feigenbaum started to work on the first expert system in 1962. Still, Douglas Lenat's Cyc project, builds upon this principle

The knowledge principle: Critique (1)

- Dichotomy between knowledge and reasoning is not clear cut
  - Knowing how to reason correctly is in itself knowledge
  - In addition, ‘rules of reasoning’ or ‘reasoning processes’ may themselves be domain-specific

Example: “if Liverpool is the capital of Britain then grass is green” is true in propositional logic, since “ grass is green” is true

However, ‘ Liverpool is the capital of Britain’ has no relevance to ‘ grass is green’.
So, it might seem counterintuitive that the implication is true.
In relevance logic it would be false.

The knowledge principle: Critique (2)

- The knowledge principle also underestimates the complexity of the reasoning problem
  - For first-order logic (even restricted to 'rules') complete 'reasoning processes' may not terminate
  - For sufficiently expressive, decidable logics complete 'reasoning processes' may not terminate within a reasonable amount of time (e.g. your life span)
- Nevertheless, expert systems (i.e. system based on the knowledge principle) were built around a single basic reasoning procedure, which contributed to their failure
- Consequently, we now often use logics which are specifically tailored for an application domain or single application

The knowledge principle: Critique (3)

- Finally, the knowledge principle shifts the focus from general problem solving ability (which we commonly equate to 'intelligence') to knowing the right answer and knowing the right approach
- It abandons the original aim of AI and shifts the focus to producing useful tools instead of producing intelligent entities
- Consequently, systems based on the knowledge principle which are using an explicit representation of knowledge (as rules) are in direct competition to systems which embody knowledge implicitly in a program (as algorithms)

Synthesis

- The physical symbol system hypothesis and the knowledge principle can be seen as representing two extreme positions concerning KR&R
- A more moderate position can be characterised by saying that

  > *There are problems and problem domains where an explicit representation of knowledge using a formal language and reasoning about this knowledge using a logical calculus as the primary means of applying this knowledge to a problem, is the best possible approach*

- Of course, another extreme position would be that the whole of KR&R is obsolete

Knowledge Representation

Question: How do we *represent* knowledge in a form amenable to computer manipulation?

Desirable features:

- representational adequacy
- inferential adequacy
- inferential efficiency
- well-defined syntax and semantics
- naturalness

Summary

- Module overview
- Knowledge representation: Overview
- The physical symbol system hypothesis
- The physical symbol system hypothesis: Critique
- The knowledge hypothesis
- The knowledge hypothesis: Critique
- Synthesis

What is Logic?

- determines whether it is justified to reason from given assumptions to a conclusion
- note: a logician cannot determine whether it rains
- he can conclude it rains from the assumptions if I hear drips on the roof, then it rains and I hear drips on the roof
- formally: $\varphi \to \psi, \varphi \vdash \psi$

There exist many many logics!

A Formal Approach

Any Logic comes in three parts:

- **syntax** what are the well-formed formulas (wffs)?
- **semantics** what do formulas mean, how do we interpret them?
- **deduction** how to mechanically formulate formulas, giving us for instance the valid ones?

We do the enterprise for Propositional Logic

Syntax

Let $\mathcal{P}$ be a set of atoms $p, q, p_1, p_2, \ldots$. Then $\mathcal{L}(\mathcal{P})$ or $\mathcal{L}_0$ is the smallest set closed unter the following rules:

- $\top, \bot \in \mathcal{L}_0$
- $\mathcal{P} \subseteq \mathcal{L}_0$
- if $\varphi, \psi \in \mathcal{L}_0$, then $(\varphi \wedge \psi)$, $(\varphi \to \psi), (\varphi \leftrightarrow \psi), (\varphi \vee \psi)$ and $\neg\varphi \in \mathcal{L}_0$

Note: φ, ψ are not formulas, just variables over them.

symbol	name	capital
φ	*phi*	Φ
ψ	*psi*	Ψ
χ	*chi*	
γ	*gamma*	Γ
α	*alpha*	
β	*beta*	

Exercise

(1) *Which of the following are fomulas of $\mathcal{L}_0$, which are*

Syntax, ctd

Sometimes a more economical set is chosen:

- $\mathcal{P} \subseteq \mathcal{L}_0$
- if $\varphi, \psi \in \mathcal{L}_I$, then $(\varphi \wedge \psi)$, and $\neg\varphi \in \mathcal{L}_0$

And then define:

$$
\begin{array}{lcl}
\top & \stackrel{\text{def}}{=} & (p \vee \neg p) \\
\bot & \stackrel{\text{def}}{=} & \neg\top \\
(\varphi \vee \psi) & \stackrel{\text{def}}{=} & \neg(\neg\varphi \wedge \neg\psi) \\
(\varphi \rightarrow \psi) & \stackrel{\text{def}}{=} & (\neg\varphi \vee \psi) \\
(\varphi \leftrightarrow \psi) & \stackrel{\text{def}}{=} & ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))
\end{array}
$$

Syntax: Conventions

- In order to minimise the number of brackets, a precedence is assigned to the logical operators and it is assumed that they are left associative. Outermost brackets are omitted. Starting from the highest to lowest precedence we have:

$$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow$$

- Thus,

$p \rightarrow q$	stands for	$(p \rightarrow q)$
$p \vee q \vee r$	stands for	$((p \vee q) \vee r)$
$p \vee q \wedge r$	stands for	$(p \vee (q \wedge r))$
$p \wedge q \leftrightarrow r$	stands for	$((p \wedge q) \leftrightarrow r)$
$\neg p \rightarrow q$	stands for	$(\neg p \rightarrow q)$
$\neg(p \rightarrow q)$	stands for	$\neg(p \rightarrow q)$

Semantics

We could say: $\wedge$ means ‘and’.....
But is:

> *I woke up and took a shower*

the same as

> *I took a shower and woke up*

???

Semantics

We specify the semantics of propositional logic in truth tables

$\wedge$	p	q	$(p \wedge q)$
	0	0	0
	0	1	0
	1	0	0
	1	1	1

and

$\neg$	p	$\neg p$
	0	1
	1	0

Exercise

Make truth-tables for the other connectives $\vee$ and $\leftrightarrow$, using the definition given above.

Implication

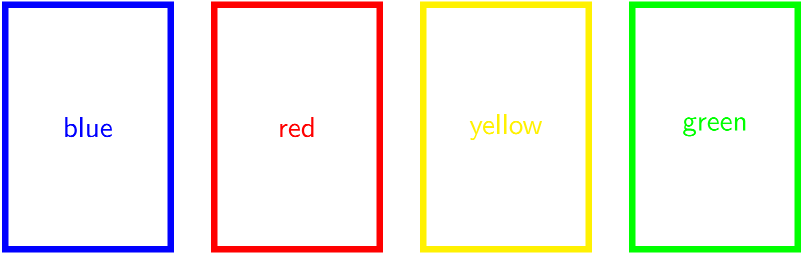
Consider the following:
If x is greater than 7, it is also bigger than 4
This is true, no matter what x is!

If y is greater than 7, it is also bigger than 11

$\rightarrow$	p	q	$(p \rightarrow q)$	
	0	0	? 1	$x = 3$
	0	1	? 1	$x = 5$
	1	0	? 0	$y = 9$
	1	1	? 1	$x = 8$

Implication

Every card has a color at each side. Four are displayed on a table:



Claim: *if a card has a red side, it also has a blue side*
Question: how many cards minimally to turn over to verify claim?

Exercise

Explain the outcome of the four colours card puzzle using the truth-table of $\rightarrow$.

Semantics

Notation: $\models \varphi$ for: "φ is always true"
Such a φ is also called a tautology.
Procedure: check that in the table, we only have 1's

Example

$p \vee \neg p$, $p \to p$, $(p \to q) \to ((p \to r) \to (p \to (q \wedge r)))$

Example

p	q	r	$(p \to q)$	$\to$	$((p \to r)$	$\to$	$(p$	$\to$	$(q \land r)))$
0	0	0	1	1	1	1	1	0	
0	0	1	1	1	1	1	1	0	
0	1	0	1	1	1	1	1	0	
0	1	1	1	1	1	1	1	1	
1	0	0	0	1	0	1	0	0	
1	0	1	0	1	1	0	0	0	
1	1	0	1	1	0	1	0	0	
1	1	1	1	1	1	1	1	1	

Counterexamples

We have: $\not\models (p \vee q) \rightarrow q$

p	q	$(p \vee q)$	$\rightarrow$	q
0	0	0	1	0
0	1	1	1	1
1	0	1	0	0
1	1	1	1	1

p	q	$(p \vee q)$	$\rightarrow$	q
0	0	0	1	0
0	1	1	1	1
1	0	1	0	0
1	1	1	1	1

A situation in which p is true, and q is false, is a counterexample.

Semantics

Exercise

Check which of the following formulas are a tautology. If not, give a counterexample.

(1) $p \to (q \to p)$

(2) $p \to (p \to q)$

(3) $(p \to q) \vee (q \to p)$

(4) $(p \to (q \vee r)) \to ((p \to q) \vee (p \to r))$

Semantics: Consequence

For Γ a set of formulas, and φ a formula,
$\Gamma \models \varphi$ means: if all formulas in Γ is true, φ is also true.

Example

$\{p, p \to q\} \models q$

p	q	p	$(p \to q)$	q
0	0	0	1	0
0	1	0	1	1
1	0	1	0	0
1	1	1	1	1

Semantics: Consequence

Exercise

Check whether the following are true:

(1) $\{\neg q, p \to q\} \models \neg p$

(2) $\{\neg p, p \to q\} \models \neg q$

(3) $\{p, p \to q, (\neg r \to \neg q)\} \models r$

(4) $\{p \lor q, p \to r, q \to r\} \models r$

Deduction

$A1 \quad \varphi \to (\psi \to \varphi)$
$A2 \quad (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
$A3 \quad (\neg\varphi \to \psi) \to ((\neg\varphi \to \neg\psi) \to \varphi)$

Define $\vdash \varphi$ iff there exists a sequence of formulas $\alpha_1, \ldots \alpha_n$ such that $\alpha_n = \varphi$ and for every $\alpha_i (i \leq k)$:

(1) α_i is an instantiation of $A1, A2$ or $A3$; or
(2) there are $j, k < n$, $\alpha_j = \alpha_k \to \alpha_i$

(2) says that $\vdash \alpha_k, \vdash \alpha_k \to \alpha_i \Rightarrow \vdash \alpha_i$ is a derivation rule: it is called *Modus Ponens*. $A1 - A3$ are called *axioms*.
Define $\Gamma \vdash \varphi$ iff there exists a sequence of formulas $\alpha_1, \ldots \alpha_n$ such that $\alpha_n = \varphi$, and for every $\alpha_i (i \leq k)$:

(1) α_i is an instantiation of $A1, A2$ or $A3$; or
(2) there are $j, k < n$, $\alpha_j = \alpha_k \to \alpha_i$; or
(3) $\alpha_i \in \Gamma$

(2) says that $\vdash \alpha_k, \vdash \alpha_k \to \alpha_i \Rightarrow \vdash \alpha_i$ is a derivation rule: it is called *Modus Ponens*. $A1 - A3$ are called *axioms*.

Deduction: Example

$A1$	$\varphi \to (\psi \to \varphi)$
$A2$	$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
$A3$	$(\neg\varphi \to \psi) \to ((\neg\varphi \to \neg\psi) \to \varphi)$
MP	$\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$

Example

$\vdash p \to p$

1	$(p \to ((p \to p) \to p)) \to$ $((p \to (p \to p)) \to (p \to p))$	$A2$
2	$p \to ((p \to p) \to p)$	$A1$
3	$(p \to (p \to p)) \to (p \to p)$	1,2, MP
4	$p \to (p \to p)$	$A1$
5	$p \to p$	3,4, MP

Connecting $\models$ and $\vdash$

An inference system is sound wrt a semantics if for all φ,
$\vdash \varphi \Rightarrow \models \varphi$

Exercise

(1) *Show that the inference system defined on slide 45 is sound wrt the semantics of slide 36.*

(2) *A sound inference system is easily found, in a trivial way. How?*

An inference system is complete wrt a semantics if for all φ,
$\models \varphi \Rightarrow \vdash \varphi$.

Exercise

(1) *A complete inference system is easily found, in a trivial way. How?*

Completeness

Theorem

We have, for all Γ and φ:

$$\Gamma \vdash \varphi \Leftrightarrow \Gamma \models \varphi$$

This is the adequateness theorem for propositional logic, stating that our deduction formalism is both sound and complete with respect to the semantics of truth tables.

Further Remarks

- The problem, for given φ whether $\vdash \varphi$, is NP-complete
- A well-known extension of propositional logic is predicate logic, with quantifiers $\forall, \exists$.
  - This is basis for a very popular KR formalism
  - *all students are bright* $\forall x(Sx \rightarrow Bx)$
  - *some students are lazy* $\exists x(Sx \wedge Lx)$
  - This logic is also known as first order logic
  - For this logic, the question wether $\vdash \varphi$ is undecidable
- The theory of predicate logic $+$ arithmetic is not even axiomatisable!

KR&R and formal logic

- Our KR&R doctrine:

  There are problems and problem domains where an explicit representation of knowledge using a formal language and reasoning about this knowledge using a logical calculus as the primary means of applying this knowledge to a problem, is the best possible approach

- In the following the combination of formal language, well-defined semantics, and deductive system will be called a formal logic
- Question: Why do we want to use formal logics?

Formal languages (3)

- Important is the distinction between object language and meta language.
- An object language is the formal language of some logic.

  In an object language the meaning of extralogical constants will be given by an interpretation.
- The meta language is the language we use to talk about an object language.

Semantics (1)

- One of the most important features of a formal logic is that its expressions have a well-defined meaning (semantics)
- This allows for a common understanding of the represented information while also allowing concise descriptions
- In constrast to standard logics, natural language does not have a well-defined meaning. In particular, its handling of both logical connectives and quantifiers is often ambiguous

Semantics (2)

Example

All dogs hate a cat

First-order logic:

1. $\forall x(dog(x) \rightarrow \exists y(hate(x, y) \wedge cat(y)))$
2. $\exists y(cat(y) \wedge \forall x(dog(x) \rightarrow hate(x, y)))$
3. $\forall x \forall y((dog(x) \wedge cat(y)) \rightarrow hate(x, y))$
4. $\forall x \exists y(dog(x) \rightarrow (hate(x, y) \wedge cat(y)))$

Semantics (3)

- In all the cases we will consider, the semantics of a formal logic is based on the notion of an interpretation
- An interpretation will be given by
  - a mathematical structure

  together with
  - an interpretation function that maps the extralogical constants of the logic to elements of the mathematical structure, and
  - an inductive definition of the meaning of logical connectives and quantifiers, which allows us to define the notion of a true formula

Semantics (4)

- The class of all interpretations can then be used to distinguish between
  - formulae which are not true in any interpretation (unsatisfiable formulae)
  - formulae which are true in some interpretations but not necessarily in all interpretations (satisfiable formulae)
  - formulae which are true in every interpretation (valid formulae)

Semantics (5)

It is important to note that identical formulae can fall into different categories depending on which class of interpretations we are using.

Example

- Consider a first-order language with constants John and Robin and unary predicate symbol male.
- Suppose that all we know is that

  $$\text{male}(\text{John})$$

  is valid (true in all interpretations)
- Does this mean that

  $$\neg\text{male}(\text{Robin})$$

  is valid?

Semantics (6)

Example (continued):

- Does this mean that

$$\neg \text{male}(\text{Robin})$$

  is valid?
- In first-order logic, the answer is negative, since in the absence of any additional information, there are two interpretations:
  - one interpretation where male(Robin) is true,
  - one interpretation where male(Robin) is false.
- In a database system, the answer is positive, since it will only consider the initial model, that is,
  - the interpretation where male(Robin) is false.

Deductive systems (1)

- For most non-trivial logics, the class of interpretations contains an infinite number of interpretations.
- Suppose we would like to check whether a formula φ is valid. We cannot simply go through all interpretations checking whether φ is true in all of them, since this process would never terminate.
- Deductive systems provide the means to derive valid formulae without the need to inspect interpretations.
- Deductive systems can be viewed and understood as games where one or more players move according to a given set of rules.

Deductive systems: Example (1)

The following is a game we play on strings consisting only of the letters **M**, **U**, and **I**.
We always start with the string **MI**
The four rules of the game are:

1. If we have a string of the form $x\mathbf{I}$, we can replace it by $x\mathbf{IU}$
2. If we have a string of the form $\mathbf{M}x$, we can replace it by $\mathbf{M}xx$
3. If we have a string of the form $x\mathbf{III}y$, we can replace it by $x\mathbf{U}y$
4. If we have a string of the form $x\mathbf{UU}y$, we can replace it by xy

where x and y are arbitrary (possibly empty) strings

Deductive systems: Example (2)

The following is a sequence of moves using these rules:

(a)	**MI**	start
(b)	**MII**	from (a) using rule 2
(c)	**MIIII**	from (b) using rule 2
(d)	**MIIIIU**	from (c) using rule 1
(e)	**MUIU**	from (d) using rule 3
(f)	**MUIUUIU**	from (e) using rule 2
(g)	**MUIIU**	from (f) using rule 4

Deductive systems: Example (3)

- At the moment, we don't have a definition of what it means to win our example game.
- In some deductive systems no such definition is required, because every situation is a winning situation.
- In other deductive system we will have some indicator which signifies a winning situation, e.g. if we can reach the string **MU** then we have won.

Aside:

- This game is a simple physical symbol system.

Meta Reasoning about **MIU**

Is **MU** derivable in the system **MIU**?
NO! We have

Theorem

Let the **I**-count *of a string be the number of* **I***symbols in it. Then: every string* x *that is derivable from* **MI** *has an* **I**-count *which is never a multiple of 3.*

Axiom:		**MI**
Rules:	1	if x**I** is a theorem, then so is x**IU**
	2	if **M**x is a theorem, then so is **M**xx
	3	if x**III**y is a theorem, then so is x**U**y
	4	if x**UU**y is a theorem, then so is xy

Summary

- Formal logics allow for a concise description of a problem which disregards irrelevant peculiarities of natural language.
- Formal logics have a well-defined semantics which provides the basis for a common, unambiguous understanding of a problem description.
- Formal logics are accompanied by deductive systems which allow us to derive information which is implied by a problem description, but not necessarily explicitly stated in it.

Exercises (1)

Exercise

Give translations of the following statements in a first order language. First indicate which symbols you choose for which terms.

(1) *students are humans*

(2) *everybody who loves somebody, is blessed*

(3) *nobody has read all books, but everybody has read some*

(4) *two courses are loved by everyone*

(5) *everybody loves two courses*

Exercises (2)

Exercise

Is the following reasoning sound (m and i are constants)

$\forall x H(x, m), (H(m, i) \wedge \forall x(H(m, x) \rightarrow x = i)) \vdash m = i$

Now, take the following translations for the constants and predicates:

$H(x, y)$	x *loves* y
m	*Madonna*
i	*me*

The reasoning would represent:

If everybody loves Madonna, and Madonna loves only me, then I am Madonna.

What is unsatisfactory in this anyalysis?

Substitution

$[\alpha/\beta]\varphi$ means 'φ, with subformula β replaced by α'.

Example

$$
\begin{array}{lcl}
[q/p]\ ((p \wedge r) \rightarrow \neg p) & = & ((q \wedge r) \rightarrow \neg q) \\
[(p \wedge q)/\neg p]\ ((p \wedge q) \rightarrow \neg p) & = & ((p \wedge q) \rightarrow (p \wedge q)) \\
[(p \wedge q)/\neg p]\ ((p \wedge q) \rightarrow \neg p) & = & ((q \wedge q) \rightarrow (p \wedge q)) \\
[q/p]\ ((q \wedge r) \rightarrow \neg t) & = & ((q \wedge r) \rightarrow \neg t)
\end{array}
$$

Extensionality

A property of propositional and predicate logic:

$$\models (\alpha \leftrightarrow \beta) \rightarrow (\varphi \leftrightarrow [\alpha/\beta]\varphi)$$

In words: if α and β are equivalent, then you may freely substitute one for the other in any formula without altering its meaning.
The meaning of a formula is also called extensionality
Thus: the meaning of a formula only depends on the extensionality of its subformulas, not on their form.

Extensionality: Examples

$\models (\alpha \leftrightarrow \beta) \rightarrow (\varphi \leftrightarrow [\alpha/\beta]\varphi)$

α: I am happy
β: I study in Liverpool

We are now in situation in which $(\alpha \leftrightarrow \beta)$ is true!

today it rains and I am happy
is equivalent to
today it rains and I study in Liverpool

I give you a hundred pound or I am happy
is equivalent to
I give you a hundred pound or I study in Liverpool

I am happy because
I follow Knowledge Representation
TRUE!
is equivalent ??? to
I study in Liverpool because
I follow Knowledge Representation

Upshot

- propositional logic is extensional;
- for knowledge, time, desires, because ... we do not want extensionality
- Conclusion: propositional logic is not suitable if we want to deal with knowledge, time, desires, because ...
- modal logic will help us out: it is an example of an intensional logic

Modal Logic: Syntax

Let $\mathcal{P}$ be a set of atoms $p, q, p_1, p_2, \ldots$. Then $\mathcal{L}_m(\mathcal{P})$ or $\mathcal{L}$ is smallest set:

- $\top, \bot \in \mathcal{L}$
- $\mathcal{P} \subseteq \mathcal{L}$
- if $\varphi, \psi \in \mathcal{L}$, then $(\varphi \wedge \psi)$, $(\varphi \to \psi), (\varphi \leftrightarrow \psi), (\varphi \vee \psi)$, $\neg\varphi$ and $\Box\varphi \in \mathcal{L}$

What does $\Box\varphi$ mean? Well, that is a matter of semantics....

Informal meaning of $\Box$

$\Box\varphi$ has many possible readings:

reading	meaning
alethic	φ is necessarily the case
epistemic	φ is known to be the case
doxastic	φ is believed to be the case
temporal	φ is always the case
dynamic	φ is caused by a program
provabilistic	φ is provably the case
motivational	φ is desired to be the case
deontic	φ ought to be the case

Informal meaning of $\Diamond$

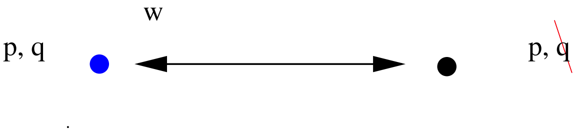
Define $\Diamond\varphi \overset{\text{def}}{=} \neg\Box\neg\varphi$

reading	meaning $\Box$	notation	meaning $\Diamond$
alethic	φ is necessarily the case	$\Box\varphi$	possibly
epistemic	φ is known to be the case	$K\varphi$	held possible
doxastic	φ is believed to be the case	$B\varphi$	held possible
temporal	φ is always the case	$\Box\varphi$	sometime
dynamic	φ is caused by a program	$[\alpha]\varphi$	possible result
provabilistic	φ is provably the case	$\Box\varphi$	consistent
motivational	φ is desired to be the case	$D\varphi$	acceptable
deontic	φ ought to be the case	$\bigcirc\varphi$	permitted

Modal Semantics

- Let p be: it is sunny in Liverpool, and q the same for Manchester.
- An agent (1) in Liverpool knows the weather there, but not in Manchester.
- He considers two alternatives: p, q and $p, \neg q$.
- We call such alternatives worlds, with names $w, v, w', \ldots$.
- An outsider might be able to distinguish what the real world is (designated in blue)

Modal Semantics



If an agent cannot distinguish between two worlds w and w', we draw an arrow between them.

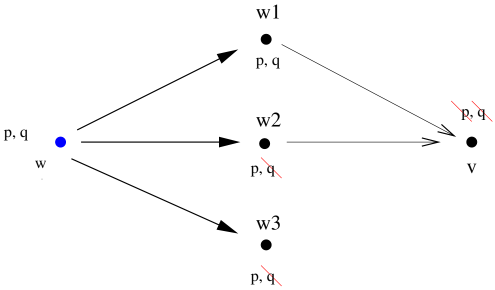
Modal Semantics

If an agent cannot distinguish between two worlds w and w', we draw an arrow between them..

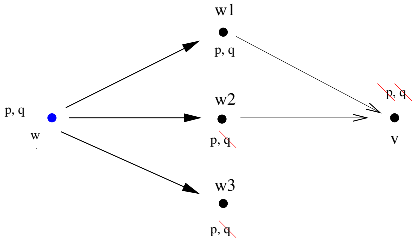
Of course, he cannot distinguish any world from itself. We now want to express that, given w, agent 1 knows p, but he does not know q

$$M, w \models (K_1 p \wedge \neg K_1 q \wedge \neg K_1 \neg q)$$

Kripke Models



Modal Semantics



$$
\begin{array}{ll}
W = & \{w, w1, w2, w3, v\} \\
R = & \{(w, w1), (w, w2), (w, w3), \\
 & (w1, v), (w2, v), (w3, v)\} \\
\pi : & \pi(w)(p) = \pi(w)(q) = \mathit{true} \\
 & \pi(v)(p) = \pi(v)(q) = \mathit{false} \\
 & \pi(w2)(p) = \pi(w3)(p) = \mathit{true} \\
 & \pi(w2)(q) = \pi(w3)(q) = \mathit{false} \\
 & \pi(w1) = \pi(w)
\end{array}
$$

Definition

A Kripke Model $M = \langle W, R, \pi \rangle$ where

- W is a set of worlds
- $R \subseteq W \times W$ is a binary relation
- $\pi : W \to \mathcal{P} \to \{\mathit{true}, \mathit{false}\}$

Truth Definition

Definition

A Kripke Model $M = \langle W, R, \pi \rangle$ where

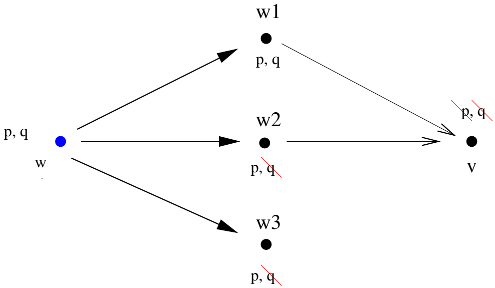
- W is a set of worlds
- $R \subseteq W \times W$ is a binary relation
- $\pi : W \to \mathcal{P} \to \{true, false\}$

Definition

We define what it means that $M, w \models \varphi$:

$$
\begin{array}{lll}
M, w \models p & \text{iff} & \pi(w)(p) = true \\
M, w \models \varphi \wedge \psi & \text{iff} & M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models \neg\varphi & \text{iff} & \text{not: } M, w \models \varphi \\
M, w \models \Box\varphi & \text{iff} & \text{for all } v : (Rwv \Rightarrow M, v \models \varphi)
\end{array}
$$

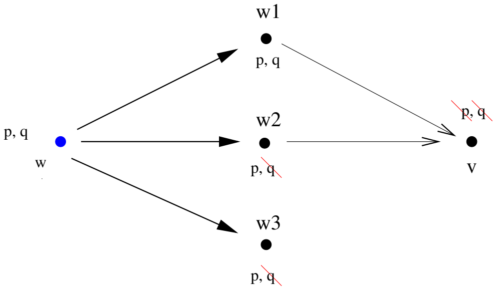
Modal Semantics: Example



$$M, w \models \quad (p \leftrightarrow q)$$
$$M, w \models \quad \neg(\Box p \leftrightarrow \Box q)$$

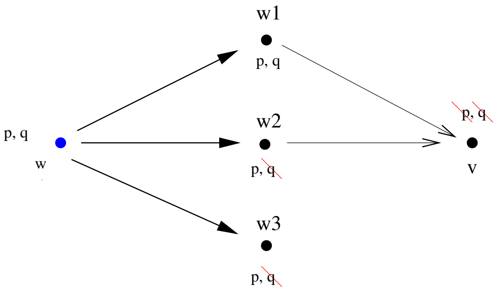
$$\left.\begin{aligned} M, w &\models \quad (p \leftrightarrow q) \\ M, w &\models \quad \neg(\Box p \leftrightarrow \Box q) \end{aligned}\right\} \Rightarrow \text{got rid of extensionality!}$$

Modal Semantics: Example



$$M, w \models \quad (p \leftrightarrow q)$$
$$M, w \models \quad \neg(\Box p \leftrightarrow \Box q)$$
$$M, w \models \quad \Box(p \rightarrow q)? \quad \textit{NO}!$$
$$M, w \models \quad (\Box p \rightarrow \Box q)? \quad \textit{NO}!$$
$$M, w \models \quad \Box(q \vee \neg q)? \quad \textit{YES}!$$
$$M, w \models \quad \neg\Box\neg(p \wedge q)? \quad \textit{YES}!$$
$$M, w \models \quad \Diamond(p \wedge q)? \quad \textit{YES}!$$

Modal Semantics



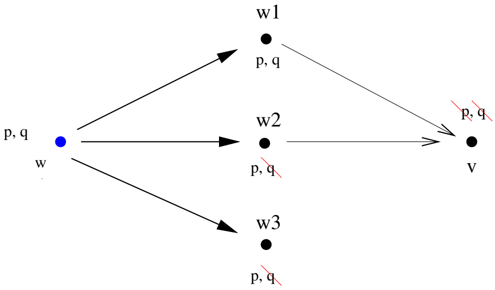
$$
\begin{aligned}
M, w1 &\models \Box(\neg p \wedge \neg q) \\
M, w2 &\models \Box(\neg p \wedge \neg q) \\
M, w3 &\models \Box(\neg p \wedge \neg q)! \\
M, w &\models \Box\Box(\neg p \wedge \neg q)! \\
M, w &\models \Box\Box(\neg p \wedge \neg q)!
\end{aligned}
$$

Modal Semantics

Exercise

Recall that the definition of $\Diamond\varphi \stackrel{\text{def}}{=} \neg\Box\neg\varphi$ Show that:
$M, w \models \Diamond\varphi$ iff there is a v: Rwv and $M, v \models \varphi$

Modal Semantics



Exercise

Verify of the model M above whether:

$M, w \models \quad (p \leftrightarrow q)$

$M, w \models \quad \Box\Diamond p$

$M, w \models \quad \Diamond\Box(p \leftrightarrow q)$

$M, w \models \quad \Box(\Box p \rightarrow \neg\Diamond q)$