# A Modal Interpretation of Nash-Equilibria and Related Concepts

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**Abstract**

Multi-agent environments comprise decision makers whose deliberations involve reasoning about the expected behaviour of other agents. Apposite concepts of rational choice have been studied and formalized in game theory and our particular interest is with their integration in a logical specification language for multi-agent systems. This paper concerns the logical analysis of the game-theoretical notions of a (subgame perfect) Nash equilibrium and that of a (subgame perfect) best response strategy. Extensive forms of games are conceived of as Kripke frames and a version of Propositional Dynamic Logic ($PDL$) is employed to describe them. We show how formula schemes of our language characterize those classes of frames in which the strategic choices of the agents can be said to be Nash-optimal. Our analysis is focused on extensive games of perfect information without repetition.

## 1 Introduction

Agents can be thought of as systems that are capable of reasoning about their own and other agents' knowledge, preferences, future and past actions. As an agent may be confronted with several, mutually exclusive, ways how to act, decision making is imperative. Which action an agent eventually performs may very well depend on his beliefs concerning the other agents' actions and their responses to his actions. Since game theory is devoted to the study of such reasoning mechanisms and the

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associated notion of strategic rationality, many of its concepts are more than just relevant to the study of multi-agent systems.

The emphasis of this paper is on the incorporation of some game-theoretical notions in Propositional Dynamic Logic (cf. Pratt [1976], Harel [1984], Goldblatt [1992]) as a step in the direction of the development of a comprehensive logical framework in which multi-agent systems can be described, specified and reasoned about. Along with the closely related concept of a best response strategy our investigations focus on the both celebrated and criticized solution concept of a Nash equilibrium. We also deal with their subgame perfect varieties.

The game theoretical notions are introduced in the next section. The third section concerns the logical language and its semantics. Games in extensive form are linked up to the models of our logical framework in the fourth section. Subsequently, we present the main results of this paper: a logical characterisation of strategy profiles in (subgame perfect) Nash-equilibrium and those comprising best response strategies (section 5). The final section deals with related and future research.

# 2 Some Game Theoretical Notions

## 2.1 Strategic Considerations

The investigations of this paper concern finite games in extensive form with perfect information. Before going into the mathematical technicalities, however, we would like to draw the reader's attention to the following informal considerations concerning games and strategies.

A (pure) *strategy* for a game, $\sigma$, consists of a complete plan for a player $i$ how to play that game. Focusing on extensive games (games in tree-form), a strategy for a player $i$ can be conceived of as a function from the nodes at which $i$ is to move to succeeding nodes. *Strategy profiles*, denoted by $\bar{\sigma}$, combine strategies, one for each player, by means of set theoretical union. In virtue of the the rules of the game, a strategy profile determines for each node a unique outcome, though not necessarily for each node the same one.

The following example of a game in extensive form will be employed to illustrate matters throughout this paper.

**Fact 2.1** Consider the two-person game in extensive form as depicted in Figure 1. Let $RL$ denote the strategy for player $i_1$ that consists in his going right at $v_0$ and going left at $v_3$. $RL$ can be conceived of as the function that maps $v_0$ onto

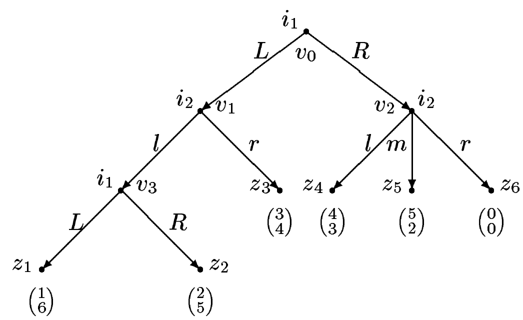



*Figure 2.1*

Figure 1: An example of a game in extensive form

$v_2$ and $v_3$ onto $z_1$. The pair of strategies $\langle RL, ll\rangle$ — where $ll$ is the strategy for player $i_2$ which prescribes her to go left at both $v_1$ and $v_2$ — denotes a strategy profile and determines the outcome $z_4$, and granting a payoffs of 4 and 3 to $i_1$ and $i_2$, respectively.

Whether a strategy $\sigma_k$ is a best response for a player $i_k$ is relative to the strategies the other players adopt, i.e., to a strategy profile. Assuming that play commences at the root node, a strategy profile $\bar{\sigma}$ is said to contain a *best response for player* $i_k$, if $i_k$ cannot increase her payoff by playing another strategy available to her when the other players stick to their strategies as specified in $\bar{\sigma}$. A strategy profile $\bar{\sigma}$ is a *Nash-equilibrium* if none of the players can increase her payoff by unilaterally playing another strategy. Equivalently, a Nash equilibrium $\bar{\sigma}$ could be characterized as a strategy profile which contains a best response strategy for each players (cf. Osborne and Rubinstein [1994], p. 98).

It has been argued that Nash equilibria do not in general do justice to the sequential structure of an extensive game. In our example, $\langle LR, lr\rangle$ is, along with $\langle RL, ll\rangle$, $\langle RL, rl\rangle$, $\langle RR, ll\rangle$ and $\langle RR, rl\rangle$, a Nash equilibrium. This is, however, dependent on the fact that $i_2$ going right at $v_2$ minimizes $i_1$'s pay-off rather than that it maximizes that of $i_2$. Player $i_2$, as it were, threatens to go right at $v_2$ if $i_1$ goes right at $v_0$. Player $i_1$, however, need not take this threat seriously if the sequentiality of the game is taken into account. The node $v_2$ will be reached only if $i_1$ moved right at $v_0$ at a previous state of the game. Once in $v_2$, strategic ratio-

| $\bar{\sigma}$ *contains a best response* | *for one player i* | *for all players* |
|---|---|---|
| *at the root node* | $\bar{\sigma}$ *contains a best response strategy for i* | *Nash-equilibrium* |
| *at all internal nodes (i.e. in all subgames)* | $\bar{\sigma}$ *contains a subgame perfect best response strategy for i* | *subgame perfect Nash-equilibrium* |

Figure 2: Nash-equilibrium and its interrelationships with some related concepts.

nality prescribes $i_2$ to move to $z_4$ rather then go right to $z_6$. As there is nothing in the description of the game committing $i_2$ to move to $z_6$ in $v_2$, the strategy profile $\langle LR, lr \rangle$ should be ruled out as a rational alternative. This is a manifestation of the more general phenomenon that a strategy profile contains instructions for the players how to act in nodes that it itself precludes ever to be reached in the course of the game and in some cases allows for "irrational" moves off the equilibrium path.

A refinement of the solution concepts of Nash equilibrium that meets this objection can be achieved by requiring Nash equilibria to be *subgame perfect*. In extensive form, a subgame can be conceived of as a cutting of the game tree, which results in another game in extensive form. A strategy $\sigma$ is a *subgame perfect* best response strategy for a player relative to some strategy profile $\bar{\sigma}$ in a game if it is a best response strategy with respect to $\bar{\sigma}$ in all its subgames. A subgame perfect Nash equilibrium $\bar{\sigma}$ can duly be understood as a union of strategies each of which a subgame perfect best response strategy with respect to $\bar{\sigma}$. Figure 2 summarizes the above concepts and how they relate to one another.

A strategy profile determines a unique outcome. By deviating unilaterally, a player can force several outcomes to come about by choosing her strategy. The one guaranteeing her the highest outcome is her best response strategy with respect to the respective strategy profile. These outcomes can be represented graphically by the leaf nodes of the game tree from which are removed all edges that do not comply with the strategies of the other players as laid down in the strategy profile. Such a reduced tree we shall call a player's *strategy search space* with respect to a strategy profile. In our example the strategy search space for $i_1$, given a strategy profile containing $i_2$'s strategy $ll$, can be depicted as in Figure 3.

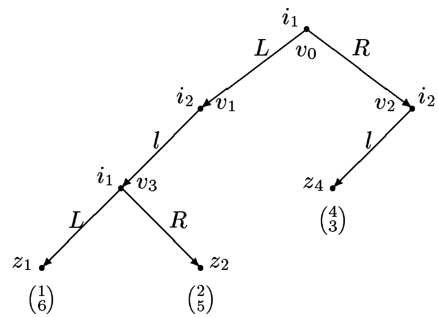



*Figure 2.2*

Figure 3: Player $i_1$'s strategy search space given that player $i_2$'s plays strategy $ll$.

Game trees, being graphs, correspond to Kripke structures and as such they can be described by means of the language of Propositional Dynamic Logic ($PDL$). The nodes of the game tree represent the states of the frame and the edges define the accessibility relation. A strategy for a player $i$ is identified with the graph of a function from the nodes at which $i$ can move to successor nodes. A strategy profile combines strategies of the individual players and as such it is the graph of a function on the internal nodes of the game tree. In this manner, strategies, strategy profiles and strategy search spaces can be represented by programs of our dynamic logic.

Fundamental to the present analysis is that frames in which the program representing a strategy profile $\bar{\sigma}$ contains a (subgame perfect) best response strategy for some player or is a (subgame perfect) Nash-equilibrium, possess certain structural properties which are expressible in $PDL$. The objective of this paper is to specify formally which constraints a frame satisfies if the strategy program corresponds to a strategy profile that is a (subgame perfect) Nash-equilibrium or incorporates a (subgame perfect) best response strategy. Another, rather more tendentious way of putting it would be that it is our aim to unearth the formal conditions under which the strategy program reflects the choices of a community of (omniscient) agents that employ subgame perfect Nash-equilibrium as a solution concept. We show how formula schemes of $PDL$ characterize the frames satisfying these structural properties. As such this study could be taken as an exercise in modal correspondence theory.

## 2.2 Games & Nash-equilibria

So far the concepts of game theory relevant to this paper have only been presented in a rather informal fashion. In this section we give a formal account in which we go a long way in following Bonanno's (cf. Bonanno [1998]). A game in extensive form with perfect information without repetition is identified with a a tuple $\langle\langle V, \prec, Z, N, \iota\rangle, u\rangle$, where $V$ is a finite set of vertices and $\prec$ a relation on $V$, representing the possible moves at each vertex. The pair $\langle V, \prec\rangle$ is a non-trivial, irreflexive, finite, and hence finitely branching, tree. Furthermore, $Z$ is the set of leaves of the tree , and $N$ is the set of players. The function $\iota$ assigns a player to each internal node of the game tree and is supposed to be surjective (onto). Finally, $u$ specifies the payoffs to the players at each of the vertices.

**Definition 2.2** (*(Extensive Forms and Games)*)

- A *finite extensive form with perfect information* $EF$ is a tuple $\langle V, \prec, Z, N, \iota\rangle$, where $\langle V, \prec\rangle$ is a finite, irreflexive, non-trivial tree, $Z := \{z \in V \mid \forall v \in V,\ z \not\prec v\}$ is the set of leaves, $N$ is a finite set of players, and $\iota : V\backslash Z \to N$ is a surjective active player assignment. As a notational convention, we use $v_0$ to denote the root and we let, for each $i$, $V_i := \{v \in V \mid \iota(v) = i\}$. Let further for each $v \in V$, $V_v := \{v' \in V \mid v \prec^* v'\}$, where $\prec^*$ is the reflexive, transitive closure of $\prec$.
- A *game* $G$ on an extensive form $EF$ is a pair $\langle EF, u\rangle$ with $u : N \to V \to \mathbb{N}$.[1]

When establishing formal properties of finite games on an extensive form a well-chosen induction measure is often more than just serviceable. The height of a vertex in a tree turns out to be of particular convenience. Note that it is because we are dealing with finite trees, that such a notion can suitably be defined.

**Definition 2.3** For each game $G = \langle V, \prec, Z, N, \iota, u\rangle$, define for each $v \in V$, the *height of* $v$, $h(v)$ as:

$$h(v) := \begin{cases} 0 & \text{if } v \in Z \\ \mathtt{max}(\{h(v') \mid v \prec v'\}) + 1 & \text{otherwise} \end{cases}$$

Let further $V_n := \{v \in V \mid h(v) \leq n\}$ and note that $V_0 = Z$.

[1] For technical reasons we define the utility function $u$ for each player on all vertices rather than on the leaves only, as is customary. This, however, does not affect the game-theoretical features we deal with in this paper.

For each game the notions of a strategy for a player and that of a strategy profile can now be defined much as one would expect:

**Definition 2.4** (*(Strategies and Strategy Profiles)*) Let $G = \langle V, \prec, Z, N, \iota, u\rangle$, with $N = \{i_1, \ldots, i_n\}$. Define:

- A *strategy* $\sigma$ for a player $i \in N$ is a *total* function $\sigma : V_i \to V$ such that for all $v \in V_i$ , $v \prec \sigma(v)$. Let furthermore $\Sigma_G(i)$ denote the set of all strategies for player $i$.
- A *strategy profile* is a function $\bar{\sigma} : V \backslash Z \to V$ such that there are strategies $\sigma_{i_1} \in \Sigma_G(i_1), \ldots, \sigma_{i_n} \in \Sigma_G(i_n)$ and $\bar{\sigma} = \bigcup_{1 \leq k \leq n} \sigma_{i_k}$.
  Let further $\overline{\Sigma}_G$ be the set of all strategy profiles of $G$.
  Note that $\bar{\sigma} \in \overline{\Sigma}_G$ is a *total* function $\bar{\sigma} : V \backslash Z \to V$ such that for all $v \in V \backslash Z$, $v \prec \bar{\sigma}(v)$. Each $\bar{\sigma} \in \overline{\Sigma}_G$ is well-defined as a function, since $\bigcap_{i \in N} V_{i_k} = \varnothing$.

In the sequel, the subscript $G$ in $\Sigma_G$ and $\overline{\Sigma}_G$ will be omitted when no confusion is likely. Define further for all $\bar{\sigma}, \bar{\sigma}' \in \overline{\Sigma}$, and each $W \subseteq V$:

- $\bar{\sigma} \sim_W \bar{\sigma}' \quad :\Longleftrightarrow \quad \forall v \in W : \ \bar{\sigma}(v) = \bar{\sigma}'(v)$
- $\bar{\sigma}[W]\bar{\sigma}' \quad :\Longleftrightarrow \quad \bar{\sigma} \sim_{V \backslash W} \bar{\sigma}'$

Hence $\bar{\sigma} \sim_W \bar{\sigma}'$ denotes that $\bar{\sigma}$ and $\bar{\sigma}'$ coincide on their values for $W$, whereas $\bar{\sigma}[W]\bar{\sigma}'$ signifies that $\bar{\sigma}$ and $\bar{\sigma}'$ differ *at most* in their values for $W$.

Each strategy profile $\bar{\sigma}$ determines a unique outcome in the sense that if all players stick throughout the game to the strategies in $\bar{\sigma}$, the game terminates in precisely one final stage. As such, a strategy profile $\bar{\sigma}$ gives rise to a function that maps each internal node $v$ to the leaf node $\bar{\sigma}$ determines as its outcome when play is commenced at $v$. To capture this notion we define for each $\bar{\sigma} \in \overline{\Sigma}$, the function $\bar{\bar{\sigma}}$ as follows:

**Definition 2.5** For each game $G = \langle V, \prec, Z, N, \iota, u\rangle$, and each $\bar{\sigma} \in \overline{\Sigma}_G$, define $\bar{\bar{\sigma}} : V \to Z$ by induction on $h(v)$, as:

$$\bar{\bar{\sigma}}(v) := \begin{cases} v & \text{if } h(v) = 0 \\ \bar{\bar{\sigma}}(\bar{\sigma}(v)) & \text{otherwise} \end{cases}$$

The outcome , $\bar{\bar{\sigma}}(v)$, a strategy profile $\bar{\sigma}$ determines for a particular node $v$ only depends on the moves it prescribes for nodes that can still be reached. This is exactly what the following fact says:

**Fact 2.6** For all games $G = \langle V, \prec, Z, N, \iota, u\rangle$ and all $\bar{\sigma}, \bar{\sigma}' \in \overline{\Sigma}$:

$$\bar{\sigma} \sim_{V_v \backslash Z} \bar{\sigma}' \Longrightarrow \bar{\bar{\sigma}}(v) = \bar{\bar{\sigma}}'(v).$$

**Proof:** By an easy induction on $h(v)$. $\dashv$

The ground has now been cleared to give formal definitions of the game theoretical notions of a best response strategy relative to a strategy profile and a Nash-equilibrium, as well as their subgame perfect (sometimes abbreviated to *s.p.*) variations:

**Definition 2.7** Let $G = \langle V, \prec, Z, N, \iota, u\rangle$ be a game on an extensive form and $\bar\sigma \in \overline{\Sigma}_G$. Let further $v_0$ be the root of $G$ and $i \in N$. Then define:

$(1)$ $\bar\sigma$ comprises a *best response strategy for* $i$
$:\Longleftrightarrow\ \forall\bar\sigma' \in \overline{\Sigma} : \bar\sigma[V_i]\bar\sigma' \Longrightarrow u(i)(\bar{\bar\sigma}'(v_0)) \leq u(i)(\bar{\bar\sigma}(v_0))$

$(2)$ $\bar\sigma$ comprises a *subgame perfect (s.p.) best response strategy for* $i$
$:\Longleftrightarrow\ \forall v \in V, \forall\bar\sigma' \in \overline{\Sigma} : \bar\sigma[V_i]\bar\sigma' \Longrightarrow u(i)(\bar{\bar\sigma}'(v)) \leq u(i)(\bar{\bar\sigma}(v))$

$(3)$ $\bar\sigma$ comprises a *Nash-equilibrium*
$:\Longleftrightarrow\ \forall i \in N, \forall\bar\sigma' \in \overline{\Sigma}_G : \bar\sigma[V_i]\bar\sigma' \Longrightarrow u(i)(\bar{\bar\sigma}'(v_0)) \leq u(i)(\bar{\bar\sigma}(v_0))$

$(4)$ $\bar\sigma$ comprises a *subgame perfect (s.p.) Nash-equilibrium*
$:\Longleftrightarrow\ \forall i \in N, \forall v \in V, \forall\bar\sigma' \in \overline{\Sigma}_G : \bar\sigma[V_i]\bar\sigma' \Longrightarrow u(i)(\bar{\bar\sigma}'(v)) \leq u(i)(\bar{\bar\sigma}(v))$.

# 3 Logical Appliances: Syntax & Semantics

## 3.1 Models and Frames

Being graphs, game trees can be correlated with Kripke structures in a straightforward manner and modal languages can be deployed to describe them. The formalism by means of which the analyses of this paper are conducted is a language for $PDL$ augmented by a set of modal operators $\{\Box_i\}_{i\in N}$. Reinforced thus, the language gains expressive power with respect to the players' preference orderings on the possible outcomes as they are determined by the payoff structure of the corresponding game. The correspondence between games and frames is relative to a strategy profile. The latter is represented in the language by the so-called *strategy program* $c$, which is syntactically atomic. The language also contains an atomic program for each player. Semantically, each of these is interpreted as the possible moves the respective player can make at the nodes assigned to them.

The resulting logical language is a multi-modal dynamic language, with the set of players as atomic programs, a special $c$-program and a model operator "$\Box_i$" for each $i \in N$. For formulae we have furthermore the usual Boolean operations "$\bot$" (*falsum*) and "$\rightarrow$" (material implication) and as program connectives, ";" (sequentialization), "$\cup$" (non-deterministic choice) and "$*$" (iteration) as well as a program forming operation on formulae "?" (test).

**Definition 3.1** (*(Syntax of $L$)*) Let $\Phi_0$ be a countable set of propostional variables, with typical element $A$. Let $\Pi_0$ be a set of atomic programs, with typical element $a$ and which includes a *finite* set $N = \{i_1, \ldots, i_n\}$ representing a set of players as well as the strategy program $c$. The set of formulae of $L$, $\Phi$, with typical element $\varphi$ and the set of programs $\Pi$, with typical element $\alpha$ are generated by the following grammar:

- $\varphi ::= A \mid \perp \mid \varphi_1 \to \varphi_2 \mid [\alpha]\varphi \mid \Box_i\varphi$
- $\alpha ::= a \mid \alpha_1; \alpha_2 \mid \alpha_1 \cup \alpha_2 \mid \alpha^* \mid \varphi?$

Each modal operator $\Box_i$, as carefully to be distinguished from $[\,i\,]$, runs over the relative preference relation, $\leq_i$, that is going to be defined over the states for each player. Hence, $\Box_i\varphi$ intuitively means that in all worlds preferred by player $i$ to the local one, $\varphi$ holds. Negation ($\neg\varphi$), conjunction ($\varphi \wedge \psi$) and disjunction ($\varphi \vee \psi$) are, furthermore, introduced as the respective abbreviations of $\varphi \to \perp$, $\neg(\varphi \to \neg\psi)$ and $\neg(\neg\varphi \wedge \neg\psi)$, as usual. Let further $\langle\alpha\rangle\varphi$, $\Diamond_i\varphi$, and $\ulcorner\texttt{while}\ \varphi\ \texttt{do}\ \alpha\urcorner$ be short for $\neg[\alpha]\neg\varphi$, $\neg\Box_i\neg\varphi$, and $\ulcorner(\varphi?;\alpha)^*;\neg\varphi?\urcorner$, respectively. [2]

The models for the language $L$ are Kripke structures with the additional feature of a preference relation on the states being specified for all players.

**Definition 3.2** (*(Frames and Models for $L$)*)

- A *frame* $F$ for the language $L$ is a triple $\langle S, \{\xrightarrow{a}\}_{a\in\Pi_0}, \{\leq_i\}_{i\in N}\rangle$, where $S$ is a set of states, for each $a \in \Pi_0$ and each $i \in N$ $\xrightarrow{a}$ and $\leq_i$ are binary relations on $S$, i.e. $\xrightarrow{a} \subseteq S \times S$ and $\leq_i \subseteq S \times S$.
- A *model* $M$ on frame $F$ is a tuple $\langle F, I\rangle$ with $I$ being a function that assigns subsets of $S$ to the propositional variables, i.e., $I : \Phi_0 \to 2^S$.

We are now in a position to interprete the language $L$ on the models as they have been specified above and, subsequently, a notion of logical validity:

**Definition 3.3** (*(Semantics for $L$)*) Define for each program $\alpha \in \Pi$ the accessibility relation $R_\alpha$ for a model $M = \langle F, I\rangle$ as a subset of $S \times S$ recursively as:

- $R_a := \xrightarrow{a}$
- $R_{\alpha_1;\alpha_2} := \{\langle s, s'\rangle \mid \exists s'' \in S :\ sR_{\alpha_1}s''\ \&\ s''R_{\alpha_2}s'\}$
- $R_{\alpha_1\cup\alpha_2} := R_{\alpha_1} \cup R_{\alpha_2}$
- $R_{\alpha^*} := R_\alpha^*$, i.e., the ancestral, or reflexive and transitive closure, of $R_\alpha$
- $R_{\varphi?} := \{\langle s, s\rangle \mid M, s \models \varphi\}$

[2] Throughout this paper we will use Quine quotes, “$\ulcorner$” and “$\urcorner$”, sparingly and only if they enhance readability.

Define simultaneously satisfaction of a formula $\varphi$ in a model $M = \langle F, I\rangle$ as:

- $M, s \models A \quad :\Longleftrightarrow \quad s \in I(A)$
- $M, s \not\models \bot$
- $M, s \models \varphi \to \psi \quad :\Longleftrightarrow \quad M, s \not\models \varphi$ or $M, s \models \psi$
- $M, s \models [\alpha]\varphi \quad :\Longleftrightarrow \quad \forall s' \in S$: $sR_\alpha s' \Longrightarrow M, s' \models \varphi$
- $M, s \models \Box_i \varphi \quad :\Longleftrightarrow \quad \forall s' \in S : s \leq_i s' \Longrightarrow M, s' \models \varphi$

We will use $sR_\alpha$ as an abbreviation for $\{s' \in S \mid sR_\alpha s'\}$.

**Definition 3.4** (*(Logical validity)*) Define for all $\varphi \in \Phi$, and for all frames $F = \langle S, \{\xrightarrow{a}\}_{a\in\Pi_0}, \{\leq_i\}_{i\in N}\rangle$ and all models $M$:

- $M \models \varphi :\Longleftrightarrow$ for all $s \in S$: $M, s \models \varphi$
- $F, s \models \varphi :\Longleftrightarrow$ for all models $M$ on $F$: $M, s \models \varphi$
- $F \models \varphi :\Longleftrightarrow$ for all models $M$ on $F$: $M \models \varphi$
- $\models \varphi :\Longleftrightarrow$ for all frames $F$: $F \models \varphi$

# 4 Games as Frames

## 4.1 A Class of Frames

The investigations of this paper will be restricted to a particular class of frames. In the next subsection we will establish a correspondence between games and frames and show that each frame corresponding to a game belongs to this class. The properties a frame has to satisfy if the strategy profile of the corresponding game is a Nash equilibrium can be characterized by formulae (schemes) of $L$ with respect to this class.

The class of frames we are going to consider satisfies certain properties that reflect its interpretation as a set of games and which are axiomatized by the schemes $T_i$, $4_i$ and $G1$–$G4_i^\alpha$:

| | | |
|---|---|---|
| $T_i$ | $\Box_i\varphi \to \varphi$ | (reflexivity of $\leq_i$) |
| $4_i$ | $\Box_i\varphi \to \Box_i\Box_i\varphi$ | (transitivity of $\leq_i$) |
| $D^c$ | $\langle c\rangle\varphi \to [c]\varphi$ | (determinacy of $c$) |
| $G1$ | $\langle c\rangle\varphi \to \bigvee_{i\in N}\langle i\rangle\varphi$ | |
| $G2$ | $\bigvee_{i\in N}\langle i\rangle\top \to \langle c\rangle\top$ | |
| $G3_i$ | $\langle i\rangle\top \to \bigwedge_{j\in N\setminus\{i\}}[j]\bot$ | |
| $G4_i^\alpha$ | $(\langle\alpha\rangle\varphi \wedge \langle\alpha\rangle\psi) \to (\langle\alpha\rangle(\varphi \wedge \Diamond_i\psi) \vee \langle\alpha\rangle(\psi \wedge \Diamond_i\varphi))$ | |

The preference relation for each player on the final states is thought of as being induced by the payoff structure of a game; the higher the payoff awarded to a player in a state, the higher the respective player values that state. Any such preference relation will induce a total preorder on the states. Hence, $T_i$ and $4_i$, which reflect reflexivity and transitivity of $\leq_i$. The axiom scheme $G4_i^\alpha$ captures, for each program $\alpha$, the comparability with respect to $\leq_i$ of any two states in which $\alpha$ terminates. We also assume determinacy of the strategy program ($D^c$) as a strategy profile induces a path through the game tree and determines a unique outcome. $G1$ assures that the strategy profile only prescribes moves the players can perform. Moreover, $G2$ makes certain that, whenever a player program is enabled so is the strategy program (a player cannot adopt the strategy not to move at all at any of his nodes). Finally, $G3_i$ guarantees that no two players can move at the same stage of the game.

**Proposition 4.1** For all frames $F = \langle S, \{\xrightarrow{a}\}_{a\in\Pi_0}, \{\leq_i\}_{i\in N}\rangle$, $\varphi \in \Phi$ and $i \in N$:

$(i)$ $F \models G1 \iff R_c \subseteq \bigcup_{i\in N} R_i$

$(ii)$ $F \models G2 \iff \forall s, s' \in S, \forall i \in N:\ i \in N, sR_is' \Longrightarrow \exists s'' \in S, sR_cs''$

$(iii)$ $F \models G3_i \iff \forall s, s' \in S, \forall j \in N:\ j \neq i\ \&\ sR_is' \Longrightarrow \forall s'' \in S$: *not* $sR_js''$

$(iv)$ $F \models G4_i^\alpha \iff \forall s, s', s'' \in S: (sR_\alpha s'\ \&\ sR_\alpha s'') \Longrightarrow s' \leq_i s''$ or $s'' \leq_i s'$

**Proof:** All proofs are straightforward. ⊣

Let $\mathcal{C}$ be the class of frames $F$ for which $\leq_i$ is reflexive and transitive for all $i \in N$ and which satisfy the conditions on the right-hand side of the equivalences in proposition 4.1. Let $KTD4G$ be the smallest normal logic containing the schemata $T_i$, $4_i$, $G1$, $G2$, $G3_i$ and $G4_i^\alpha$ for all $i \in N$ and all $\alpha \in \Pi$.

**Conjecture 4.2** (Soundness and Completeness) $KTD4G$ is sound and complete with respect to the class of frames $\mathcal{C}$.

**Proof:** Check whether the Fisher & Ladner filtration of the canonical model for $KTD4G$ is based on a frame that belongs to the class $\mathcal{C}$. ⊣

## 4.2 Linking up Games and Frames

Games in extensive form are defined as trees and as such can be correlated to the frames that serve as semantical entities of our logic. The players of a game are identified with the actions they can perform at the nodes at which they are to make a move. The program $c$ is interpreted as the functional relation a strategy profile defines on the tree, here denoted by $R_c$. Accordingly, $vR_cv' \iff \bar{\sigma}(v) = v'$.

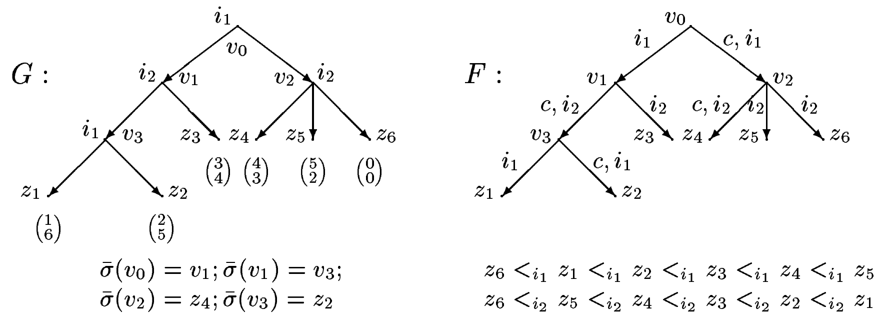



*Figure 4.1*

Figure 4: Correspondence between games and frames: $G \simeq_{\bar{\sigma}} F$ where $\bar{\sigma}$ is such that $v_0 \mapsto v_1, v_1 \mapsto v_2, v_2 \mapsto z_4$ and $v_3 \mapsto z_2$.

This makes that the correspondence between the games and frames is relative to a strategy profile $\bar{\sigma}$. The payoff structure straightforwardly induces for each player a preference order on the final states of the frame. In this manner, each game in extensive form is associated with a frame for $L$.

**Definition 4.3** Let $G = \langle V, \prec, Z, M, \iota, u \rangle$ a game, $\bar{\sigma} \in \overline{\Sigma}_G$ a strategy profile and $F = \langle S, \{\xrightarrow{a}\}_{a \in \Pi_0}, \{\leq_i\}_{i \in N} \rangle$ a frame for language $L$ with $N \cup \{c\} \subseteq \Pi_0$, then define $G \simeq_{\bar{\sigma}} F$ as:

$$G \simeq_{\bar{\sigma}} F :\Longleftrightarrow \begin{cases} M & = & N \\ V & = & S \\ u(i)(s) \leq u(i)(s') & \Longleftrightarrow & s \leq_i s' \\ i = \iota(v) \;\&\; v \prec v' & \Longleftrightarrow & v \xrightarrow{i} v' \\ \bar{\sigma}(v) = v' & \Longleftrightarrow & v \xrightarrow{c} v'. \end{cases}$$

To illustrate this definition, the frame $F$ corresponding to the game of example 2.1, $G$, given a strategy profile such that $\bar{\sigma}(v_0) = v_1; \bar{\sigma}(v_1) = v_3; \bar{\sigma}(v_2) = z_4; \bar{\sigma}(v_3) = z_2$, is depicted alongside with $G$ itself in Figure 4.

As a rule, $i$ is a non-deterministic program, because a player in a game has several options how to act when it is his turn to move. In contrast, the program $c$, which, linked to strategy profile $\bar{\sigma}$ as it is, will be a deterministic program, defined on each of the internal nodes. We also use $\iota$ as a notational device in the context of frames. Accordingly, for any frame $F = \langle S, \{\xrightarrow{a}\}_{a\in\Pi_0}, \{\leq_i\}_{i\in N}\rangle$ for which there is a game $G$ and a strategy profile $\bar{\sigma} \in \overline{\Sigma}$ such that $G \simeq_{\bar{\sigma}} F$, $\iota(s)$ denotes the 'agent program' that can be executed in state $s$.

As a final result of this section the following fact is obtained.

**Fact 4.4** For all games $G$, $\bar{\sigma} \in \overline{\Sigma}$, frames $F$ such that $G \simeq_{\bar{\sigma}} F$: $F \in \mathcal{C}$.

**Proof:** (*Sketch*.) Consider arbitrary $G$ and $F$ such that $G \simeq_{\bar{\sigma}} F$ for some $\bar{\sigma} \in \overline{\Sigma}$. It suffices to show that $F$ satisfies the $KTD4G$-axioms. Since $u$ defines a total preorder on the set of states $S$ in $G$ for each $i \in N$, $\leq_i$ satisfies reflexivity and transitivity. For the same reason $\leq_i$ is defined for $i \in N$ on all nodes. Hence, $G4_i^\alpha$ also holds in $F$. The functionality of $\bar{\sigma}$ makes that $R_c$ is deterministic and so $D^c$ is validated in $F$. The functionality of $\iota$ warrants the validity of $G3_i$ in $F$. Finally, each edge of the game tree corresponds to a possible move by one of the players. This makes that there are no edges on which $c$ could be defined that are not labelled by one of the players. So, finally, $F \models G1$ and $F \models G2$. $\dashv$

In our treatment of Nash equilibria we restricted our attention to *finite* games. As $PDL$ cannot distinguish in general between such frames and infinite ones, the class of frames for which there are games $G$ and $\bar{\sigma} \in \overline{\Sigma}$ such that $G \simeq_{\bar{\sigma}} F$ cannot be characterized within our logical system.

# 5 Characterizing Nash Equilibria

## 5.1 The $\alpha(M)$-Program

Having introduced the logical symbolism and the correspondence that obtains between frames and games, properties of the $c$-program that reflect the corresponding strategy profile comprising a (*s.p.*) best response strategy or a (*s.p.*) Nash equilibrium still remain to be defined. To this end we introduce, for each subset of players $M \subseteq N$, a complex non-deterministic program, $\alpha(M)$, as an auxiliary notion.

**Definition 5.1** For each $M = \{i_0, \ldots, i_k\} \subseteq N$, let $\bigcup M$ be the program $i_0 \cup \ldots \cup i_k$. Define for any $M \subseteq N$ the program $\alpha(M)$ as:

$\alpha(M) :\equiv \texttt{while } \langle c\rangle\top \texttt{ do } \bigcup M \cup c$

Usually we will abbreviate $\alpha(\{i\})$ to $\alpha(i)$. Note further that:

$\alpha(\varnothing) \equiv \texttt{while } \langle c\rangle\top \texttt{ do } c.$

The intuition behind this definition becomes clear when we concentrate on frames in the class $\mathcal{C}$. The program $\alpha(M)$ is non-deterministic if any of the $i \in M$ is. For each $M \subseteq N$, $\alpha(M)$ executes any of the atomic programs $i \in M$ if enabled in a state, the strategy program $c$ otherwise. The program terminates when $c$ is no longer enabled. In any frame satisfying $G2$ no $i \in N$ will then be enabled either. In contradistinction, $\alpha(\varnothing)$ reduces to a deterministic program that repeats $c$ until it terminates.

In any frame $F \in \mathcal{C}$, $R_c \subseteq R_{\bigcup N}$, and so the program $\alpha(N)$, when executed in state $s$, terminates exactly those states $s'$ that are reachable by a path $s = s_0 \xrightarrow{i_1} \dots \xrightarrow{i_k} s_k = s'$ ($i_m \in N$ for $1 \leq m \leq k$) such that for all $j \in N$ and states $s'' \in S$, $s' \stackrel{j}{\not\longrightarrow} s''$. The larger the set $M$, the more non-determinism is brought into the program $\alpha(M)$, with the deterministic $\alpha(\varnothing)$ on the one end of the spectrum and $\alpha(N)$ on the other.

**Fact 5.2** For all $F = \langle S, \{\xrightarrow{a}\}_{a \in \Pi_0}, \{\leq_i\}_{i \in N}\rangle \in \mathcal{C}$ and all $M, M' \subseteq N$:
$M \subseteq M' \Longrightarrow R_{\alpha(M)} \subseteq R_{\alpha(M')}$

**Proof:** Consider an arbitrary frame $F \in \mathcal{C}$ as well as $M \subseteq N$. Assume for arbitrary $s, s' \in S$ that $sR_{\alpha(M)}s'$. Hence there is a sequence of states such that such that $s = s_0 R_{\bigcup M \cup c}\, s_1 \dots s_{n-1} R_{\bigcup M \cup c} s_n = s'$ and $s' R_{\bigcup M \cup c} = \varnothing$. By definition of $\cup$ also $s = s_0 R_{\bigcup M' \cup c} s_1 \dots s_{n-1} R_{\bigcup M' \cup c} s_n = s'$. Moreover, since $s' R_{\bigcup M \cup c} = \varnothing$ certainly also $s' R_c = \varnothing$. For the same reason and because $F \models G2$ for all $i \in N$, $s' R_i = \varnothing$. Hence $s' R_{\bigcup M' \cup c} = \varnothing$, which concludes the proof. $\dashv$

If $G \simeq_{\bar{\sigma}} F$, for some game $G$ and strategy profile $\bar{\sigma}$, when executed in $s$, $\alpha(N)$ will exactly terminate in the leaf nodes still reachable from $s$. With the program $c$ encoding a strategy profile $\bar{\sigma}$, commencing in $s$, $\alpha(\varnothing)$ terminates precisely in that node which $\bar{\sigma}$ determines as its unique outcome, i.e. $sR_{\alpha(\varnothing)}s' \iff s' = \bar{\bar{\sigma}}(s)$ (cf. lemma 5.8$(iv)$, below). Moreover, as the program $i$ is interpreted as the moves available to player $i$, the possible runs of the program $\alpha(i)$ terminate in exactly the leaf nodes which, by choosing her strategy, $i$ can guarantee the game to end if the other players stick to their respective strategies as specified in $\bar{\sigma}$. As such, $\alpha(i)$ represents the strategy search space of player $i$ given fixed strategies of the other players (cf. page 5). In our example, $R_{\alpha(i_1)}$ is the set $\{\langle v_0, z_1\rangle, \langle v_0, z_2\rangle, \langle v_0, z_4\rangle, \langle v_1, z_1\rangle, \langle v_1, z_2\rangle, \langle v_2, z_4\rangle, \langle v_3, z_1\rangle, \langle v_3, z_2\rangle\}$, and $\alpha(i_1)$ can duly be pictured as in Figure 5. The reader compare it to Figure 3!

It is precisely this insight that is exploited in the next subsection to characterize frames for which the strategy program $c$ matches the Nash optimal strategy profile

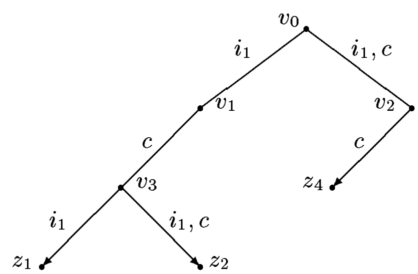



*Figure 5.1*

Figure 5: The program $\alpha(i)$ if $v_0 R_c v_2 R_c z_4$ and $v_1 R_c v_3 R_c z_1$.

of the corresponding games.

## 5.2 Player Preference

In order to obtain expressive power with respect to the game theoretical notions we set out to model, the syntax of $L$ includes a set of modal operators $\{\Box_i\}_{i \in N}$. At each stage of the game the preference order of the players with respect to the still reachable outcomes are relevant to establishing whether the strategy profile under consideration comprises a (subgame-perfect) Nash-equilibrium or best response strategy for a player. Hence, we would like to have some device in our logic to refer to these still possible outcome states and the player's preferences with respect to them. To this end we introduce the following abbreviation:

**Definition 5.3** $\quad \boxminus_i \varphi :\equiv [\alpha(N)](\varphi \to \Box_i \varphi)$

Intuitively, $\boxminus_i \varphi$ holds in a state if and only if the player $i$ prefers any of the still $\alpha(N)$-reachable outcome states in which $\varphi$ holds to any in which the latter is not the case. The following lemma shows that this informal interpretation is warranted for any frames $F \in \mathcal{C}$.

**Lemma 5.4** $\quad$ For all frames $F = \langle S, \{\xrightarrow{a}\}_{a \in \Pi_0}, \{\leq_i\}_{i \in N}\rangle \in \mathcal{C}$, all models $M$ on $F$ and $s \in S$:

If $\quad M, s \models \boxminus_i \varphi \quad$ then

$\forall s', s'' \in S$: $(sR_{\alpha(N)}s'$ & $sR_{\alpha(N)}s''$ & $M, s' \models \varphi$ & $M, s'' \not\models \varphi) \Longrightarrow s'' <_i s'$

**Proof:** Consider arbitrary frame $F \in \mathcal{C}$, $s \in S$ and $i \in N$. Assume for contraposition that for some $s', s'' \in S$ we have:

$(i)\ sR_{\alpha(N)}s' \quad (ii)\ sR_{\alpha(N)}s'' \quad (iii)\ s' \models \varphi \quad (iv)\ s'' \not\models \varphi \quad (v)\ s'' \not\leq_i s'$.

Since $F \models G4_i^{\alpha(N)}$, $s' \leq_i s''$ or $s'' \leq_i s'$ and so with $(v)$, $s' \leq_i s''$. Having assumed $(iv)$, $s' \not\models \Box_i\varphi$ and as both $s' \models \varphi$ and $sR_{\alpha(N)}s'$, finally $s \not\models [\alpha(N)](\varphi \to \Box_i\varphi) \quad (\equiv \boxminus_i\varphi)$. $\dashv$

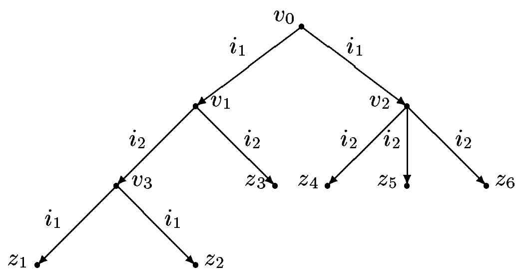
Note that the opposite direction of this lemma does not hold. It is perfectly well possible that the antecedent holds for a model and a state but that there is another state more preferred than any of the $\alpha(N)$-reachable states in which $\varphi$ does not hold. Such a state could rightly be described as utopian. This shows that in modelling Nash equilibria we abstract from preferences with respect to unatainable states.

## 5.3 Some Properties of Frames and Their Characterization

The program $\alpha(\varnothing)$, as it boils down to an iterated execution of the $c$ program until a final state is reached, combines the strategies of the players as encoded in the strategy profile concerned. Different choices in this respect by the players, some of which may be Nash-optimal, will give rise to different $\alpha(\varnothing)$ programs. The question that is addressed in this section is which structural properties a frame should comply to, if the program $\alpha(\varnothing)$ is to mirror a strategy profile that contains a (subgame perfect) best response strategy for a player or one that is in a (subgame perfect) Nash-equilibrium. Eventually, we show that each class of frames that satisfies one of these structural properties can be characterized by means of a formula scheme in $L$.

If a player $i$ acts in accordance with his own interest, one would expect $i$ to choose that strategy in his strategy search space which guarantees him the highest payoff. In terms of frames, this would render $c$ to be such that, if from some state both a final state $z$ is reachable by the $\alpha(\varnothing)$ program and another final state $z'$ by $\alpha(i)$, $i$ either prefers $z$ to $z'$ or is indifferent between them. Otherwise, $i$ could alter his strategy in such a way that $\alpha(\varnothing)$ terminates in $z$. These considerations give rise to the following properties, which some frames satisfy and others do not. In the next subsection we will demonstrate that they are the model theoretic counterparts of the game theoretical notions elaborated upon above, *viz.* (subgame perfect) best response strategies and (subgame perfect) Nash-equilibria.

**Definition 5.5** For all frames $F = \langle S, \{\stackrel{a}{\longrightarrow}\}_{a \in \Pi_0}, \{\leq_i\}_{i \in N}\rangle$, define:




$z_6 <_{i_1} z_1 <_{i_1} z_2 <_{i_1} z_3 <_{i_1} z_4 <_{i_1} z_5$
$z_6 <_{i_2} z_5 <_{i_2} z_4 <_{i_2} z_3 <_{i_2} z_2 <_{i_2} z_1$

*Figure 5.2*

Figure 6: The frame corresponding to the game of Figure 2.1

(1) $R_c$ is $i$ *beneficial in* $s$ $:\Longleftrightarrow$ $\forall s', s'' \in S$: $sR_{\alpha(\varnothing)}s'$ & $sR_{\alpha(i)}s'' \Longrightarrow s'' \leq_i s'$
(2) $R_c$ is *totally* $i$ *beneficial* $:\Longleftrightarrow$ $\forall s \in S$, $R_c$ is $i$ beneficial in $s$
(3) $R_c$ is *Nash induced in* $s$ $:\Longleftrightarrow$ $\forall i \in N$, $R_c$ is $i$ beneficial in $s$
(4) $R_c$ is *totally Nash induced* $:\Longleftrightarrow$ $\forall i \in N$, $R_c$ is totally $i$ beneficial

By way of illustration, the reader consider once more our example (cf. Figure 6). If $R_c = \{\langle v_0, v_1\rangle, \langle v_1, z_3\rangle, \langle v_3, z_1\rangle, \langle v_2, z_5\rangle\}$, $R_c$ is not totally Nash induced. For a counterexample, observe that $v_0 R_{\alpha(\varnothing)} z_3$ and $v_0 R_{\alpha(i_1)} v_5$, but $z_3 <_{i_1} z_5$. However, if $R_c = \{\langle v_0, v_2\rangle, \langle v_2, z_4\rangle, \langle v_1, v_3\rangle, \langle v_3, z_2\rangle\}$, $R_c$ is totally Nash induced, as can easily be established. Moreover, it is exactly in these circumstances that $R_c$ coincides with the graph of a strategy profile that is in a subgame perfect Nash-equilibrium. In the sequel we prove that this is no coincidence.

The formula scheme $\ulcorner(\boxminus_i\varphi \wedge \langle\alpha(i)\rangle\varphi) \to [\alpha(\varnothing)]\varphi\urcorner$ turns out to characterize frames for which $R_c$ is totally $i$ beneficial. This is established in theorem 5.6. In spite of its apparent inscrutability, an intuitive interpretation can be attached to the formula scheme. In any model satisfying $\ulcorner(\boxminus_i\varphi \wedge \langle\alpha(i)\rangle\varphi) \to [\alpha(\varnothing)]\varphi\urcorner$, at each state $s$ of the frame, $i$ prefers any $\alpha(N)$ reachable final state in which $\varphi$ holds to any in which $\varphi$ does not ($\boxminus_i\varphi$). Moreover, if a final state in which $\varphi$ holds is in $i$'s search space below $s$ ($\langle\alpha(i)\rangle\varphi$), then a final state in which $\varphi$ holds will be reached if $i$ adheres to his strategy as it is encoded in the $c$-program ($[\alpha(\varnothing)]\varphi$). Since this should hold for any formulae $\varphi$, it means that given the strategies of the other players, $i$'s strategy as it is incorporated in $c$, serves $i$'s interests best. If $R_c$ is totally Nash induced, not surprisingly, $\ulcorner(\boxminus_i\varphi \wedge \langle\alpha(i)\rangle\varphi) \rightarrow [\alpha(\varnothing)]\varphi\urcorner$ should hold for each player $i \in N$. Since the set $N$ is assumed to be finite, this 'quantification' over all players can be achieved by conjunction. The formula scheme obtained thus, $\ulcorner\bigwedge_{i\in N}((\boxminus_i\varphi \wedge \langle\alpha(i)\rangle\varphi) \rightarrow [\alpha(\varnothing)]\varphi)\urcorner$, can, in point of fact, be proved to characterize frames with $R_c$ totally Nash induced. By requiring the respective formula schemes to hold at the root node only, one obtains the 'partial' versions of these results.

**Theorem 5.6** Let $\vartheta_i \equiv (\boxminus_i\varphi \wedge \langle\alpha(i)\rangle\varphi) \rightarrow [\alpha(\varnothing)]\varphi$. For all frames $F \in \mathcal{C}$ and all players $i \in N$:

| | | | |
|---|---|---|---|
| $(i)$ | $R_c$ is $i$ beneficial in $v$ | $\iff$ | $F, v \models \vartheta_i$ |
| $(ii)$ | $R_c$ is totally $i$ beneficial | $\iff$ | $F \models \vartheta_i$ |
| $(iii)$ | $R_c$ is Nash induced in $v$ | $\iff$ | $F, v \models \bigwedge_{i\in N}\vartheta_i$ |
| $(iv)$ | $R_c$ is totally Nash induced | $\iff$ | $F \models \bigwedge_{i\in N}\vartheta_i$ |

**Proof:** Consider an arbitrary $F = \langle S, \{\xrightarrow{a}\}_{a\in\Pi_0}, \{\leq_i\}_{i\in N}\rangle \in \mathcal{C}$ and an equally arbitrary player $i \in N$. The proofs for $(i)$–$(iv)$ are all analogous. Here we confine ourselves to demonstrating $(iv)$.

$\Rightarrow$: (Contraposition) Let $M$ be a model on $F$. Assume that for some $s \in S$ and $i \in N$ :

$(i)$ $M, s \models \boxminus_i\varphi$, $(ii)$ $M, s \models \langle\alpha(i)\rangle\varphi$, and $(iii)$ $M, s \not\models [\alpha(\varnothing)]\varphi$

From $(ii)$ we obtain that for some $z \in S$, $sR_{\alpha(i)}z$ and $M, z \models \varphi$. By $(iii)$, however, there is also an $z' \in S$ such that $sR_{\alpha(\varnothing)}z'$ and $M, z' \not\models \varphi$. Fact 5.2 gives us $sR_{\alpha(N)}z$ and $sR_{\alpha(N)}z'$. By $(i)$ and lemma 5.4 we are entitled to conclude that $z' <_i z$ and, ultimately, that $R_c$ is not totally Nash induced.

$\Leftarrow$: Assume for an arbitrary model on $F$ and for some $i \in N$, $s, s', s'' \in S$:

$(i)$ $sR_{\alpha(\varnothing)}s'$, $(ii)$ $sR_{\alpha(i)}s''$, and $(iii)$ $s'' \not\leq_i s'$.

Set $I$ in such a way that for $A \in Prop$ and each $t \in S$:

$t \in I(A) :\iff s'' \leq_i t$

Let $M = \langle F, I\rangle$. We are now in a position to establish subsequently that:

$(a)$ $M, s \not\models [\alpha(\varnothing)]A$, $(b)$ $M, s \models \langle\alpha(i)\rangle A$, and $(c)$ $M, s \models \boxminus_i A$.

$(a)$ holds because of $(i)$, $(iii)$ and the definition of the interpretation function $I$. Invoking $(ii)$ instead of $(i)$, much the same applies to $(b)$. For $(c)$ consider an arbitrary $t \in S$ such that $sR_{\alpha(N)}t$. As, by fact 5.2 and $(ii)$, $sR_{\alpha(N)}s''$. With $F \models G4_i^{\alpha(N)}$ we may assume that either $s'' <_i t$ or $t \leq_i s''$. In either case $M, t \models A \to \Box_i A$ (note that $\leq_i$ can be assumed to be transitive). Finally, from $(a)$–$(c)$ together we have $F \not\models \bigwedge_{i \in N} \vartheta_i$. $\dashv$

## 5.4 Characterizing Nash Equilibria

So far we have not ventured far outside the bounds of modal correspondence theory. In this section, however, we prove that a game-theoretic interpretation of the notions of a strategy profile being (totally) $i$ beneficial and that of a strategy profile being (totally) Nash induced is justified. The graph of the choice program in a frame turns out to be $i$ beneficial in $v_0$ if and only if the corresponding strategy profile $\bar{\sigma}$ comprises a best response strategy for $i$ in the corresponding game. In a similar manner, totality can be linked to subgame perfection and $R_c$ being Nash induced to $\bar{\sigma}$ comprising a Nash-equilibrium.

Before these results can be presented, however, some logical handiwork has still to be carried out. The following fact and lemma clear the ground in this respect.

**Fact 5.7** For all frames $F = \langle S, \{\xrightarrow{a}\}_{a \in \Pi_0}, \{\leq_i\}_{i \in N}\rangle$, all finite games on an extensive form $G = \langle V, \prec, Z, N, \iota, u\rangle$ and strategy profiles $\bar{\sigma} \in \overline{\Sigma}_G$ such that $G \simeq_{\bar{\sigma}} F$, and all $v \in V \backslash Z,\ z \in Z$:

$$vR_{\alpha(M)}z \iff \begin{cases} vR_c z & \text{if } \iota(v) \notin M \ \&\ h(v) = 1 \\ vR_{\iota(v)} z & \text{if } \iota(v) \in M \ \&\ h(v) = 1 \\ \exists v' \in V:\ vR_c v' R_{\alpha(M)} z & \text{if } \iota(v) \notin M \ \&\ h(v) > 1 \\ \exists v' \in V:\ vR_{\iota(v)} v' R_{\alpha(M)} z & \text{if } \iota(v) \in M \ \&\ h(v) > 1 \end{cases}$$

**Proof:** (*Sketch.*) Consider an arbitrary frame $F = \langle S, \{\xrightarrow{a}\}_{a \in \Pi_0}, \{\leq_i\}_{i \in N}\rangle$, and an arbitrary game on a finite extensive form $G = \langle V, \prec, Z, N, \iota, u\rangle$ such that for some strategy profile $\bar{\sigma} \in \overline{\Sigma}_G$, $G \simeq_{\bar{\sigma}} F$. It is sufficient to observe that in virtue of $G \simeq_{\bar{\sigma}} F$, for each $v \in V \backslash Z$: $\bigcup_{i \in N} vR_i = vR_{\iota(v)}$, and that in general for each $v \in V$: $vR_c \subseteq vR_{\iota(v)}$. Hence, at each $v \in V \backslash Z$ we have $vR_{\bigcup M \cup c} = vR_c$, if $\iota(v) \notin M$, and $vR_{\bigcup M \cup c} = vR_{\iota(v)}$, if $\iota(v) \in M$. $\dashv$

**Lemma 5.8** Let $G = \langle V, \prec, Z, N, \iota, u\rangle$ be a finite game, $\bar{\sigma} \in \overline{\Sigma}$ and $F = \langle S, \{\xrightarrow{a}\}_{a \in \Pi_0}, \{\leq_i\}_{i \in N}\rangle$, a frame such that $G \simeq_{\bar{\sigma}} F$. Then for all $v, v' \in V$, $M \cup \{i\} \subseteq N$ and $\bar{\sigma}', \bar{\sigma}'' \in \overline{\Sigma}$:

| | | | |
|---|---|---|---|
| $(i)$ | $vR_{\iota(v)}v'$ | $\iff$ | $\exists \bar{\sigma}' \in \overline{\Sigma}_G:\ \bar{\sigma}'(v) = v'$ |
| $(ii)$ | $vR_{\alpha(M)}v'$ | $\iff$ | $\exists \bar{\sigma}' \in \overline{\Sigma}$: $\bar{\sigma}[\bigcup_{i \in M} V_i]\bar{\sigma}'$ & $\bar{\bar{\sigma}}'(v) = v'$ |
| $(iii)$ | $vR_{\alpha(\varnothing)}v'$ | $\iff$ | $\bar{\bar{\sigma}}(v) = v'$ |
| $(iv)$ | $vR_{\alpha(i)}v'$ | $\iff$ | $\exists \bar{\sigma}' \in \overline{\Sigma}$: $\bar{\sigma}[V_i]\bar{\sigma}'$ & $\bar{\bar{\sigma}}'(v) = v'$ |

**Proof:** Item $(i)$ is straightforward. $(iii)$ and $(iv)$, as can easily be recognized, are the special cases of $(ii)$. The proof of the latter is by an induction on $h(v)$ of which we here only present the $\Rightarrow$-direction of the induction step, $h(v) = n \Rightarrow h(v) = n + 1$.

Consider an arbitrary game $G$ and frame $F$ such that $G \simeq_{\bar{\sigma}} F$ for some $\bar{\sigma} \in \overline{\Sigma}$. Consider arbitrary $v, v' \in V$ and $\bar{\sigma}', \bar{\sigma}'' \in \overline{\Sigma}$.

$\Rightarrow$: Assume $vR_{\alpha(M)}v'$. Either $(a)$ $\iota(v) \notin M$ or $(b)$ $\iota(v) \in M$. If the former, by fact 5.7, for some $v'' \in V$, $vR_cv''R_{\alpha(M)}v'$. If $(b)$, also by fact 5.7, there is a $v'' \in V$ such that $vR_{\iota(v)}v''R_{\alpha(M)}v'$. In either case, in virtue of the induction hypothesis, we may assume the existence of a strategy profile $\bar{\sigma}' \in \overline{\Sigma}$ such that $\bar{\sigma}[\bigcup_{i \in M} V_i]\bar{\sigma}'$ and $\bar{\bar{\sigma}}'(v'') = v'$. Now define $\bar{\sigma}'' : V \backslash Z \to V$ as:

$$\bar{\sigma}''(w) := \begin{cases} v'' & \text{if } w = v \\ \bar{\sigma}'(w) & \text{otherwise} \end{cases}$$

Clearly, both $\bar{\sigma}'' \in \overline{\Sigma}$ and $h(v'') < h(v)$. Note further that $\bar{\sigma}'[\{v\}]\bar{\sigma}''$. If $(b)$, it follows immediately that $\bar{\sigma}[\bigcup_{i \in M} V_i]\bar{\sigma}''$. If $(a)$, observe that from $vR_cv''$ and definition 4.3 we obtain $\bar{\sigma}(v) = v'' = \bar{\sigma}''(v)$. So in this case $\bar{\sigma}[\bigcup_{i \in M} V_i]\bar{\sigma}''$. In either case: $\bar{\bar{\sigma}}''(v) =_{h(v)>0} \bar{\bar{\sigma}}''(\bar{\sigma}''(v)) = \bar{\bar{\sigma}}''(v'') = \bar{\bar{\sigma}}'(v'') =_{i.h.} v'$. Note that the last equality holds because of fact 2.6 as $v \notin V_{v''}$ and so $\bar{\sigma}' \sim_{V_{v''}\backslash} \bar{\sigma}''$. $\dashv$

The following theorem establishes that $R_c$ satisfies the property of being (totally) $i$ beneficial in a frame $F$, exactly if the corresponding strategy profile $\bar{\sigma}$ comprises a (subgame perfect) best response strategy for $i$. In a similar fashion, $R_c$ being (totally) Nash induced can be proved to reflect that the strategy profile concerned is a (subgame perfect) Nash-equilibrium.

**Theorem 5.9** For each game on a finite extensive form, $G = \langle V, \prec, Z, N, \iota, u\rangle$, with $v_0$ as the root node, each $\bar{\sigma} \in \overline{\Sigma}_G$, and each frame $F = \langle S, \{\xrightarrow{a}\}_{a \in \Pi_0}, \{\leq_i\}_{i \in N}\rangle$ such that $G \simeq_{\bar{\sigma}} F$, and each $i \in N$:

| | | | |
|---|---|---|---|
| $(i)$ | $R_c$ is $i$ beneficial in $v_0$ | $\Longleftrightarrow$ | $\bar{\sigma}$ comprises a best response for $i$ |
| $(ii)$ | $R_c$ is totally $i$ beneficial | $\Longleftrightarrow$ | $\bar{\sigma}$ comprises a s.p. best response for $i$ |
| $(iii)$ | $R_c$ is Nash induced in $v_0$ | $\Longleftrightarrow$ | $\bar{\sigma}$ is a Nash-equilibrium |
| $(iv)$ | $R_c$ is totally Nash induced | $\Longleftrightarrow$ | $\bar{\sigma}$ is in s.p. Nash-equilibrium |

**Proof:** We restrict ourselves to proving $(iv)$ only as the proofs for $(i)$–$(iii)$ are analogous. Consider an arbitrary game on a finite extensive form $G = \langle V, \prec, Z, N, \iota, u\rangle$, as well as an equally arbitrary $\bar{\sigma} \in \overline{\Sigma}_G$, and frame $F = \langle S, \{\xrightarrow{a}\}_{a \in \Pi_0}, \{\leq_i\}_{i \in N}\rangle$ such that $G \simeq_{\bar{\sigma}} F$.

$\Rightarrow$: (Contraposition) Suppose that $R_c$ is *not* totally Nash induced. Then there is an $i \in N$ as well as there are $v, v', v'' \in V$ such that:

$$(a) \quad vR_{\alpha(\varnothing)}v' \qquad (b) \quad vR_{\alpha(i)}v'' \qquad (c) \quad v'' \not\leq_i v'$$

Since $G \simeq_{\bar{\sigma}} F$, $F \in \mathcal{C}$ (fact 4.4) and so $(c)$ implies: $(d)$ $v' <_i v''$. It follows, from $(a)$ and 5.8.$(iii)$, that: (1) $\bar{\bar{\sigma}}(v) = v'$, and, from $(b)$ and 5.8.$(iv)$:

(2) there is some $\bar{\sigma}' \in \overline{\Sigma}$ such that $\bar{\sigma}[V_i]\bar{\sigma}'$ and $\bar{\bar{\sigma}}'(v) = v''$.

Consider this $\bar{\bar{\sigma}}'$. From $(d)$ we obtain:

(3) $u(i)(v') < u(i)(v'')$, i.e. $u(i)(\bar{\bar{\sigma}}(v)) < u(i)(\bar{\bar{\sigma}}'(v))$. From (1)–(3) together follows that $\bar{\sigma}$ is *not* a subgame perfect Nash-equilibrium.

$\Leftarrow$: (Contraposition) Suppose that $\bar{\sigma}$ does not comprise a subgame perfect Nash-equilibrium, which means that for some $v \in V \backslash Z$, some $i \in N$ and some $\bar{\sigma}' \in \overline{\Sigma}$ both: $(a)$ $\bar{\sigma}[V_i]\bar{\sigma}'$, and: $(b)$ $u(i)(\bar{\bar{\sigma}}(v)) < u(i)(\bar{\bar{\sigma}}'(v))$. Consider these $i$, $v$ and $\bar{\sigma}'$. Since $G \simeq_{\bar{\sigma}} F$, from the latter: $(c)$ $\bar{\bar{\sigma}}(v) <_i \bar{\bar{\sigma}}'(v)$. Moreover, from $(a)$ and 5.8.$(iv)$ we obtain that: $(d)$ $vR_{\alpha(i)}\bar{\bar{\sigma}}'(v)$. With 5.8.$(iii)$: $(e)$ $vR_{\alpha(\varnothing)}\bar{\bar{\sigma}}(v)$. Finally, $(c)$–$(e)$ together entail that $R_c$ is *not* totally Nash induced. $\dashv$

The results of the last two subsection can be combined and we can top things of with the following corollary:

**Corollary 5.10** Let $\vartheta_i \equiv (\boxminus_i\varphi \wedge \langle\alpha(i)\rangle\varphi) \to [\alpha(\varnothing)]\varphi$. Then for all games $G$, with $v_0$ as root node, $\bar{\sigma} \in \overline{\Sigma}$ and each frame $F$, such that $G \simeq_{\bar{\sigma}} F$ and for each $i \in N$:

| | | | |
|---|---|---|---|
| $(i)$ | $\bar{\sigma}$ comprises a best response for $i$ | $\Longleftrightarrow$ | $F, v_0 \models \vartheta_i$ |
| $(ii)$ | $\bar{\sigma}$ comprises a s.p. best response for $i$ | $\Longleftrightarrow$ | $F \models \vartheta_i$ |
| $(iii)$ | $\bar{\sigma}$ is a Nash-equilibrium | $\Longleftrightarrow$ | $F, v_0 \models \bigwedge_{i \in N} \vartheta_i$ |
| $(iv)$ | $\bar{\sigma}$ is a s.p. Nash-equilibrium | $\Longleftrightarrow$ | $F \models \bigwedge_{i \in N} \vartheta_i$ |

**Proof:** Immediately from the theorems 5.9 and 5.6, above. $\dashv$

This result establishes that some model checking settles the question whether, in circumstances that can be described as an extensive game of perfect information, the strategies the agents adopt are (subgame perfect) best responses or constitute a (subgame perfect) Nash equilibrium. One can also view the matter from an opposite angle. The program $c$ could be regarded as a specification of agents are required to decide on strategies that are in (subgame perfect) Nash-equilibria. An interesting question in this respect is whether the program $c$ can be formulated as a complex program that is employed by the players as an algorithm to compute a Nash-optimal choice in each possible circumstance. Still, this issue should be committed to future research.

## 6 Related and Future Research

Under the heading of related research, Bonanno's paper on prediction and backward induction (cf. Bonanno [1998]) should come first and foremost. His work inspired the writing of this paper and his method is comparable to ours in that his papers also deal with the formalization of the concept of a subgame perfect Nash-equilibrium within a logical framework. It differs, however, in three respects. Firstly, Bonanno uses computational tree logic (*CTL*) rather than dynamic logic. Moreover, his emphasis is on the logical foundations of game-theory rather than the incorporation of game-theoretical notions in logic. Thirdly, his analyses are confined to the notion of backward induction, an algorithm designed to generate subgame perfect Nash-equilibria. Backward induction, however, is only guaranteed to provide a solution in *generic* games, i.e. games in which the payoff a player receives is different in each leaf node.

Independent investigations into the logical formalization of Nash-equilibria, which are, nevertheless, quite congenial to our approach, are Alexandru Baltag's as reported in Baltag [1999]. Although his concern is primarily with the epistemic aspects of games, he also proposes a dynamic logical framework in which Nash-equilibria and related concepts can be characterized. The main difference with our work is the way he maps games in extensive form onto Kripke structures.

In our future research we will address other game-theoretical concepts, such as *dominating* and *dominated* strategies, strategy profiles that give rise to *Pareto optimal* outcomes or *coordination equilibria* (cf. Lewis [1969]), to name only a few. We trust that the logical analyses of these notions can be conducted within a logical framework very similar to the one presented in this paper. A similar remark applies to the issue raised in the last paragraph of the previous section. So far, our attention has been concentrated on extensive games of perfect information without either repetition or chance moves. In the light of purported applications to the specification of fully-fledged multi-agent systems, this could be taken to be a considerable concession. One of the areas where the agent metaphor particularly bears fruit is where the players can only be ascribed partial knowledge of their environment. Similar caveats are apposite with respect to topics as synchronous actions, stochastic games, repeated games and chance moves. These matters merit thorough investigation, as do the intricate epistemic issues of game theory and those related to coalition formation.

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