How Much Lookahead is Needed to Win Infinite Games?

(Partially) joint work with Felix Klein (Saarland University)

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Aalborg University, Aalborg, Denmark

Introduction

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The Delay Game $\Gamma_f(L)$

- Delay function: $f \colon \mathbb{N} \to \mathbb{N}_+$.
- ω-language $L \subseteq (\Sigma_I \times \Sigma_O)^\omega$.
- Two players: Input (I) vs. Output (O).

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Questions we are interested in:

- Given L, is there an f such that O wins $\Gamma_f(L)$?
- How *large* does f have to be?
- How hard is the problem to solve?

Another Example

- $\Sigma_I = \{0, 1, \#\}$ and $\Sigma_O = \{0, 1, *\}$.
- Input block: $\#w$ with $w \in \{0, 1\}^+$. Length: $|w|$.
- Output block:

$$\binom{\#}{\alpha(n)}\binom{\alpha(1)}{*}\binom{\alpha(2)}{*}\cdots\binom{\alpha(n-1)}{*}\binom{\alpha(n)}{\alpha(n)}$$

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O wins $\Gamma_f(L_0)$ for every unbounded f:

- If I produces arbitrarily long input blocks, then the lookahead will contain arbitrarily long input blocks.
- Thus, O can produce arbitrarily long output blocks.

Previous Results

Theorem (Hosch & Landweber '72)

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1. *TFAE for L given by deterministic parity automaton $\mathcal{A}$:*
   - *O wins $\Gamma_f(L)$ for some f.*
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Theorem (Fridman, Löding & Z. '11)

The following problem is undecidable: Given (one-counter, weak, visibly, deterministic) context-free L, does O win $\Gamma_f(L)$ for some f?

Uniformization of Relations

- A strategy σ for O in $\Gamma_f(L)$ induces a mapping

  $$f_\sigma \colon \Sigma_I^\omega \to \Sigma_O^\omega$$

- σ is winning $\Leftrightarrow \{\binom{\alpha}{f_\sigma(\alpha)} \mid \alpha \in \Sigma_I^\omega\} \subseteq L$ (f_σ uniformizes L)

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Holtmann, Kaiser, Thomas: for ω-regular L

L uniformizable by continuous function

$\Leftrightarrow$

L uniformizable by Lipschitz-continuous function

Outline

1. **ω-regular Winning conditions**
2. Max-regular Winning Conditions
3. Determinacy
4. Conclusion

Our Results: Regular Winning Conditions

Theorem (Klein & Z. '15)

1. *TFAE for L given by deterministic parity automaton $\mathcal{A}$ with k colors:*
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4. *Solving reachability delay games is* PSPACE*-complete.*

Lower Bounds for Reachability Conditions

Theorem

For every $n > 1$ there is a language L_n such that

- *L_n is recognized by some deterministic reachability automaton $\mathcal{A}_n$ with $|\mathcal{A}_n| \in \mathcal{O}(n)$,*
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- $\Sigma_I = \Sigma_O = \{1, \ldots, n\}$.
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- *$w \in \Sigma_O^*$ has no bad j-pair for any $j \Rightarrow |w| \leq 2^n - 1$.*
- *Exists $w_n \in \Sigma_O^*$ with $|w_n| = 2^n - 1$ and without bad j-pair.*

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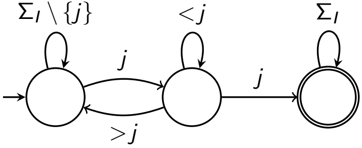

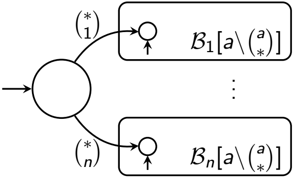

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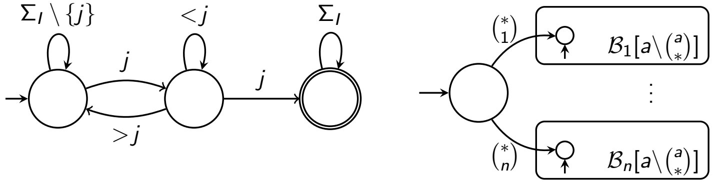

- O wins $\Gamma_f(L_n)$, if $f(0) > 2^n$: In first round, I picks u_0 s.t. u_0 without its first letter has bad j-pair. O picks j in first round.

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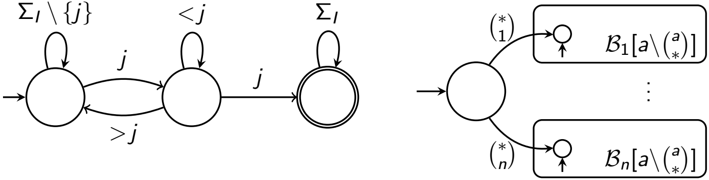

- O wins $\Gamma_f(L_n)$, if $f(0) > 2^n$: In first round, I picks u_0 s.t. u_0 without its first letter has bad j-pair. O picks j in first round.
- I wins $\Gamma_f(L_n)$, if $f(0) \leq 2^n$:
  - I picks prefix of $1w_n$ of length $f(0)$ in first round,
  - O answers by some j.
  - I finishes w_n and then picks some $j' \neq j$ ad infinitum.

Solving Delay Games

Theorem

TFAE for L recognized by a parity automaton with k colors:

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Corollary

Winner can be determined in EXPTIME.

Further Results

Applying both directions of equivalence between $\Gamma_f(L(\mathcal{A}))$ and $\mathcal{G}$ yields upper bound on lookahead.

Corollary

Let $L = L(\mathcal{A})$ where $\mathcal{A}$ is a deterministic parity automaton with k colors. The following are equivalent:

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Note: $f(0) \leq 2^{2|\mathcal{A}|k+2} + 2$ achievable by direct pumping argument.

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The Unbounding Quantifier

Bojańczyk: Let's add a new quantifier to (weak) monadic second order logic (WMSO/MSO)

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Theorem (Bojańczyk '14)

Delay-free games with WMSO+U winning conditions are decidable.

Max-Automata

Equivalent automaton model for WMSO+U on infinite words:

- Deterministic finite automata with counters.
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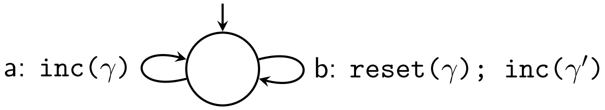

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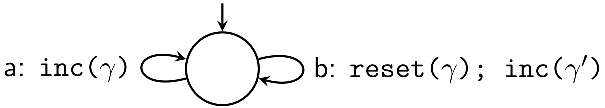

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Theorem (Bojańczyk '09)

The following are (effectively) equivalent:

1. *L WMSO+U-definable.*
2. *L recognized by max-automaton.*

The Case of Bounded Lookahead

Theorem (Z. '15)

The following problem is decidable: given a max-automaton $\mathcal{A}$, does O win $\Gamma_f(L(\mathcal{A}))$ for some constant delay function f.

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$\mathcal{G}$ is delay-free with WMSO+U winning condition.

- Can be solved effectively by reduction to satisfiability problem for WMSO+U with path quantifiers over infinite trees.
- Doubly-exponential upper bound on necessary constant lookahead.

Constant Lookahead is not Sufficient

Recall: O wins $\Gamma_f(L_0)$ for every unbounded f.

- Input block: $\#w$ with $w \in \{0,1\}^+$.
- Output block: $\binom{\#}{\alpha(n)}\binom{\alpha(1)}{*}\binom{\alpha(2)}{*}\cdots\binom{\alpha(n-1)}{*}\binom{\alpha(n)}{\alpha(n)}$
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- Lookahead contains only input blocks of length $f(0)$.
- I can react to O's declaration at beginning of an output block to bound size of output blocks while producing arbitrarily large input blocks.

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TFAE for L recognized by a max automaton with k counters:

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$\mathcal{G}$ is infinite state $\Rightarrow$ cannot solve it to determine winner of delay game w.r.t. unbounded delay functions.

Outline

1. ω-regular Winning conditions
2. Max-regular Winning Conditions
3. **Determinacy**
4. Conclusion

Borel Determinacy for Delay Games

- A game is determined, if one of the players has a winning strategy.
- Borel hierarchy: family of languages constructed from *open* languages $K \cdot \Sigma^\omega$ with $K \subseteq \Sigma^*$ via countable union and complementation.
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Results:

- Tight results for ω-regular conditions
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Open problems:

- Results for other acceptance conditions (Rabin, Streett Muller), non-deterministic or alternating automata.
- Decidability of max-regular delay games w.r.t. unbounded delay functions.
- What are strategies in delay games, e.g., do they have to depend on the delay function under consideration?