“Exactly One” Epistemic Logic

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1 Introduction

Epistemic logic is a modal logic of knowledge (see for example (Fagin et al., 1995)) in which the knowledge of agents, players or processes can be represented. When combined with temporal logics, to represent dynamic aspects, such logics can be used to represent and reason about systems where evolving knowledge plays a key part such as agent based systems, knowledge games, security protocols etc. Many such systems contain subsets of the set of propositions needed to represent the system where exactly one member of each subset holds at any moment. An example of an “exactly one” set in a game playing environment, where all the cards are dealt out to players a, b, c or d and ten_spades_i represents that player i holds the ten of spades card, is the set $\{ten\_spades_a, ten\_spades_b, ten\_spades_c, ten\_spades_d\}$ i.e. the ten of spades card must be held by exactly one player.

Recently we have been investigating, mechanising, and applying, *temporal* logics with additional constraints of the “exactly one” type considered above (Dixon et al., 2007a,b, 2008). In our work, each logic is parametrised by a set of propositions (or, predicates in the first-order case) where exactly one of these propositions is satisfied at any temporal state. We have shown that, if problems can be described in such a logical framework, then not only is the description more succinct, but the decision procedure for the logic is simpler (reducing certain aspects of the decision procedure from *exponential* to *polynomial*).

In this extended abstract we investigate theorem proving for epistemic modal logics allowing *“exactly one”* sets. In particular, we define a tableau procedure for this logic, which incorporates a DPLL (Davis et al., 1962) like mechanism to satisfy the “exactly one” sets and show that this reduces the number of states we must construct. The full details of this work can be found in (Konev et al., 2009).

2 Example

To illustrate the approach we focus on a simple card game from (van Ditmarsch et al., 2005). In this simple game there are three different cards; a heart, a spade and a club. In the most basic scenario, one card is dealt to one player, a further card is placed face down on the table and the final card is returned (face down) to the card holder.

Following (van Ditmarsch et al., 2005) we use simple propositions to represent the position of the cards. So, if $spades_w$ is true, then Wiebe holds a spade, if $clubs_t$ is true, then the clubs card is on the table, if $hearts_h$ is true, then the hearts card is in the holder, etc. Similarly, $K_w spades_w$ means that Wiebe *knows* he holds a spade. And so on.

We can identify six “exactly one” sets, firstly

$$\{spades_w, spades_h, spades_t\}$$

denoting the spades cards can be in exactly one place at any moment, and the same for $clubs_i$ and $hearts_i$ for each of $i = w, h, t$. Further,

$$\{spades_i, hearts_i, clubs_i\}$$

for $i = w, h, t$ denotes for $i = w$ that Wiebe can only hold exactly one card or for $i = h$ exactly one card can be in the holder or $i = t$ exactly one card can be on the table.

3 SX5$_n$— “Exactly One” sets in Epistemic Logic

The logic we consider is called “**SX5**$_n$”. The main novelty in **SX5**$_n$ is that it is parametrised by “exactly-one”-sets $\mathcal{P}_1$, $\mathcal{P}_2$,..., denoted **SX5**$_n(\mathcal{P}_1, \mathcal{P}_2, \ldots)$, which are constructed under the restrictions that *exactly* one proposition from every set $\mathcal{P}_i$ is true in any state. Additionally there may be a set A of normal (unconstrained) propositions.

Assuming a set of agents where $Ag = \{1, \ldots n\}$, formulae are constructed from a set $\textsc{Prop} = \{p, q, r, \ldots\}$ of *atomic propositions*, using the usual Boolean connectives: $\neg$ (not), $\vee$ (or), $\wedge$ (and) and $\Rightarrow$ (implies) plus K_i, for $i \in Ag$ (agent i knows).

A *model structure*, M, for $\mathbf{SX5}_n$ is a structure $M = \langle S, R_1, \ldots, R_n, \pi \rangle$, where: S is a set of states; $R_i \subseteq S \times S$, for all $i \in Ag$, is the agent accessibility relation where R_i is an equivalence relation; and $\pi : S \times \mathcal{P} \rightarrow \{T, F\}$ is a valuation.

As usual, we define the semantics of the language via the satisfaction relation '$\models$'. This relation holds between pairs of the form $\langle M, s \rangle$ (where M is a model structure and $s \in S$), and $\mathbf{S5}_n$-formulae. The rules defining the satisfaction relation are given below where the semantics of the Boolean operators is as usual.

$$\begin{array}{lll} \langle M, s \rangle \models q & \text{iff} & \pi(s, q) = \textbf{true} \text{ (where } q \in \textsc{Prop}) \\ \langle M, s \rangle \models K_i \phi & \text{iff} & \forall s' \in S' \text{ if } (s, s') \in R_i \text{ then } \langle M, s' \rangle \models \phi \end{array}$$

If there is a model structure M and state s such that $\langle M, s \rangle \models \varphi$ then φ is said to be *satisfiable*. If $\langle M, s \rangle \models \varphi$ for all states s and all states s then φ is said to be valid. The set of modal relations (for each agent i) are assumed to be equivalence relations.

4 Results

In (Konev et al., 2009) we present a tableau algorithm for $\mathbf{SX5}_n$. Consider an $\mathbf{SX5}_n$ formula φ to be shown satisfiable. The algorithm constructs sets of *extended assignments* of propositions and modal subformulae i.e. a mapping to true or false, that satisfy both the exactly one sets and φ. However, rather than using the usual alpha and beta rules (see for example the modal tableau in (Halpern and Moses, 1992; Wooldridge et al., 1998)) these are constructed using a DPLL-based expansion (Davis et al., 1962). Next the algorithm attempts to satisfy formulae of the form $\neg K_i \psi$ made true in such an extended assignment by constructing R_i successors which are themselves extended assignments which must satisfy particular subformulae (and the exactly one sets).

We show that the tableau algorithm is sound and complete and that given φ an $\mathbf{SX5}_n$ formula then the tableau algorithm runs in time polynomial in

$$\left(k \times |\mathcal{P}_1| \times \ldots \times |\mathcal{P}_n| \times 2^{|A|+k}\right)$$

where $|\mathcal{P}_i|$ is the size of the set $\mathcal{P}_i$ of exactly one propositions, $|A|$ is the size of the set A of non-constrained propositions, and k is the number of K_i operators in φ.

Future work involves applying this logic to more case studies and extending $\mathbf{SX5}_n$with temporal aspects to target evolving knowledge allowing us to be able to represent and reason about problems from complex domains such as security and planning more efficiently.

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