A Notion of Game Composition and Incorporation

Y. Xie and C.-H. L. Ong

Computing Laboratory, University of Oxford

Abstract. In this paper, we investigate the relationship between the bilateral interactions and the overall effect from the perspective of Nash equilibrium in non-cooperative games. We propose the concept of game composition and decomposition for non-cooperative games, which allows the analysis of the Nash equilibrium of a global game to be broken down into the analysis of the Nash equilibria of local bilateral games and how these local Nash equilibria can lead to a global Nash equilibrium in a relationship which we call *game incorporation*. We seek composition methods that are *sound* (*i.e.* the method ensures a composition of local Nash equilibria to be a global one) and *complete* (*i.e.* any global Nash equilibrium can be constructed as a composition of local Nash equilibria). We consider two natural methods of composition, *additive* and *multiplicative*. We show that they are not sound in general, but by imposing some constraints which we call *strictness* conditions, we obtain much positive results.

1 Introduction

Existing game theory typically analyzes games (either big or small) in isolation, and is inadequate in explaining the relationship between the small game (bilateral interactions) and the big game (overall effect) from the perspective of Nash equilibrium.

In this paper, we propose the concept of game composition and decomposition for non-cooperative games. We seek composition methods that are *sound*, in the sense that the big game, which we call *composite game*, preserves the respective Nash equilibria of the small games, which we call *component games*. In addition, we are interested in composition methods that are (constructively) *complete*, in the sense that there is a procedure that, given an n-player game G and one of its Nash equilibrium x^*, decomposes G into a set of component games, together with a Nash equilibrium for each, such that x^* is the composite of the component Nash equilibria. We consider two natural methods of composition, *additive* and *multiplicative*. We show that they are not sound in general, but by imposing some constraints which we call *strictness* conditions, we obtain much more positive results. We then introduce the concept of *game incorporation* and show how game composition is used in analyzing situations in which players can find Nash equilibrium strategies without knowing certain 2-player component games in full; the latter are called *incorporated game*s.

The rest of the paper is organized as follows. We start by introducing the concept of composition of games in Section 3. We then describe the concept of *game incorporation* in Section 4. We also compared our work with related work in Section 5. Finally, we draw a conclusion in Section 6 and briefly discuss future research directions. Where proofs are omitted, details can be found in the Appendix.

2 Definitions and terminology

In this section, we set up the standard definitions and terminology.

An (n-player) ***game***[1] is a triple $G = \langle P, S, U \rangle$ whereby:

- $P = \{ 1, \cdots, n \}$ is a set of n players. The *size* of G is $|G| = |P| = n$.
- Each player $p \in P$ has a finite nonempty set $S_p = \{ 1, \cdots, m_p \}$ of *pure strategies*, with typical element $s_p \in S_p$. Let $S = S_1 \times \cdots \times S_n$ be the set of possible combinations (or *profiles*) of (pure) strategies, with typical element $s \in S$.
- For each player $p \in P$, a payoff function $u_p : S \longrightarrow \mathbb{R}$ that gives p the payoff $u_p(s_1, \cdots, s_n)$ under each combination of pure strategies. Let $U = \{ u_p : p \in P \}$ be the set of payoff functions of all players.

Convention. For convenience, we introduce the *substitution notation* $(s_{-p}; s_p)$, with $p \in P$, to stand for $(s_1, \cdots, s_p, \cdots, s_n)$ where s_{-p} is the $(n-1)$-tuple that is obtained from $(s_1, \cdots, s_n)$ by removing the component s_p, and we write S_{-p}

[1] We focus on finite strategic-form non-cooperative game, and refer to it as *game* for convenience

for the set of all such $(n-1)$-tuples. In the same vein, we use $(s_{-pp'}; s_p; s_{p'})$, with $1 \leq p \neq p' \leq n$, to stand for $(s_1, \cdots, s_p, \cdots, s_{p'}, \cdots, s_n)$ where $s_{-pp'}$ is the $(n-2)$-tuple obtained from $(s_1, \cdots, s_n)$ by removing the components s_p and $s_{p'}$, and we write $S_{-pp'}$ for the set of all such $(n-2)$-tuples. Unless otherwise indicated, symbols that appear as superscripts denote sets. E.g. we shall write $G^P = \langle P, S^P, U^P \rangle$ to emphasize the player-set P. In case the superscript P refers to a particular set (e.g. $\{1, 2, 3\}$), we shall omit the set braces for simplicity, writing G^{123} instead of $G^{\{1,2,3\}}$.

A ***mixed strategy*** x_p for player p, with $S_p = \{1, \cdots, m_p\}$, is a probability distribution over S_p, *i.e.* $x_p = (x_p(1), \cdots, x_p(m_p))$ such that $x_p(i) \geq 0$ for each $1 \leq i \leq m_p$ and $\sum_{i=1}^{m_p} x_p(i) = 1$. Let X_p be the set of mixed strategies for player p. We write $X = X_1 \times \cdots \times X_n$ for the set of all possible combinations (or profiles) of mixed strategies. We call a mixed strategy $x_p \in X_p$ ***pure*** if $x_p(i) = 1$ for some $i \in S_p$, and $x_p(j) = 0$ for $j \neq i$. We denote such a pure strategy as $\pi_{p,i}$. Let $x(s) = \prod_{p=1}^{n} x_p(s_p)$ be the probability of combination $s = (s_1, \cdots, s_n) \in S$ under mixed profile x. We also use the substitution notation $(x_{-p}; \pi_{p,i})$ to stand for $(x_1, \cdots, x_{p-1}, \pi_{p,i}, x_{p+1}, \cdots, x_n)$. In a game G, the ***expected payoff*** of player p under a mixed strategy profile $x = (x_1, \cdots, x_n) \in X$ is $\sum_{s \in S} x(s) \cdot u_p(s)$, which we will denote by $u_p(x)$ by abuse of notation (it should be clear from the context what is meant).

A ***Nash equilibrium***, $x^* = (x_1^*, \cdots, x_n^*)$, is a profile of mixed strategies in which no player can increase his payoff by unilaterally changing his mixed strategies. A necessary and sufficient condition (e.g. [14]) for $(x_1^*, \cdots, x_n^*)$ to be a Nash equilibrium is: for each $p \in P$, and for each $1 \leq i \neq j \leq m_p$, we have

$$u_p(x_{-p}^*; \pi_{p,i}) > u_p(x_{-p}^*; \pi_{p,j}) \implies x_p^*(j) = 0 \tag{1}$$

Two games $G = \langle P, S, U \rangle$ and $G' = \langle P', S', U' \rangle$ are said to be ***joinable***[2] if for each $p \in P \cap P'$ we have $S_p = S'_p$, *i.e.* every player that is involved in both games has the same set of pure strategies for both. Let x^* and y^* be mixed-strategy profiles for (joinable) G and G' respectively. We say that they are *joinable* just if for each $p \in P \cap P'$ we have $x_p^* = y_p^*$. Generalizing the idea to any finite number of games, we say that a set $\widetilde{G} = \{G^{P_1}, \cdots, G^{P_m}\}$ of games is ***pairwise joinable*** just if $P_i = P_j$ iff $i = j$, and for each $1 \leq i \neq j \leq m$, we have G^{P_i} and G^{P_j} are joinable. Further we say that a set of mixed-strategy profiles, one for each G^{P_i}, is *pairwise joinable* just if for each $1 \leq i \neq j \leq m$, the respective mixed-strategy profiles of G^{P_i} and G^{P_j} are joinable.

3 Game composition and decomposition

Given a pairwise joinable set $\widetilde{G} = \{G^{P_1}, \cdots, G^{P_m}\}$ of games and a *method of composition* F (to be specified later), we define the F***-composite*** of $\widetilde{G}$ to be the game $\bowtie_F \widetilde{G} = \langle P, S^P, U^P \rangle$ with $P = P_1 \cup \cdots \cup P_m$ whereby for each $p \in P$, we have $S_p^P = S_p^{P_i}$ where i satisfies $p \in P_i$ (note that the choice of i is immaterial); and for each $p \in P$, player p's utility function u_p^P (in U^P) is defined in terms of the respective utility functions of the component games in which p is involved (*i.e.* functions $u_p^{P_i} \in U^{P_i}$ as i ranges over those satisfying $p \in P_i$), according to the method F.

Take a set $\widetilde{G} = \{G^{P_1}, \cdots, G^{P_m}\}$ of pairwise joinable games, and given a pairwise joinable set $Eq = \{(x^{P_i})^* : 1 \leq i \leq m\}$ of Nash equilibria, one for each component game G^{P_i} in $\widetilde{G}$. Let $P = P_1 \cup \cdots \cup P_m$; we define the ***join*** of the Nash-equilibrium set to be the $|P|$-tuple $(x^P)^*$ such that for each $p \in P$, we have $(x^P)_p^* = (x^{P_i})_p^*$ for any P_i with $p \in P_i$ (the choice of i is immaterial). For example, (x_1^*, x_2^*, x_3^*) is the join of the the Nash-equilibrium set $\{(x_1^*, x_2^*), (x_1^*, x_3^*), (x_2^*, x_3^*)\}$, where (x_1^*, x_2^*), (x_1^*, x_3^*), and (x_2^*, x_3^*) are Nash equilibria for G^{12}, G^{13}, and G^{23} respectively.

We seek methods F of composition that preserve the given pairwise joinable set of Nash equilibria *i.e.* we require the join of the Nash-equilibrium set, $(x^P)^*$, to be a Nash equilibrium of the composite game $\bowtie_F \widetilde{G}$. We refer to this property as ***soundness*** of the method F. In addition, we are interested in methods F that are ***complete*** in the sense that for every n-player game G, and for every Nash equilibrium x^* of G, there is a pairwise joinable set $\widetilde{G}$ of component games and a pairwise joinable set Eq of Nash equilibria, one for each component game, such that $G = \bowtie_F \widetilde{G}$ and x^* is the join of Eq. Further if a method is complete, it would be desirable to have an algorithm to construct such a *decomposition*. Thus, in this sense, we are interested in the ***constructive completeness*** of composition methods.

Remark 1. In principle the component games in $\widetilde{G}$ need not be of the same size, but for convenience in this paper, we shall restrict component games to be *2-player games*, and $\widetilde{G}$ to be the largest such set (of 2-player games) *i.e.* for each pair p and p' of players in P, there is exactly one component game with player-set $\{p, p'\}$ in $\widetilde{G}$, and so, $\widetilde{G}$ has $\binom{|P|}{2}$ elements.

[2] The concept of *join* here is analogous to the *join* operation in relational database model [2]

Henceforth we shall call such a pairwise joinable set $\widetilde{G}$ of 2-player games a ***microscopic set*** w.r.t. the player-set P. For example, the microscopic set w.r.t. $P = \{1, 2, 3, 4\}$ is $\widetilde{G} = \{G^{12}, G^{13}, G^{14}, G^{23}, G^{24}, G^{34}\}$.

In the following, we shall consider two such methods of composition, namely, additive and multiplicative.

3.1 Additive composition

In additive composition, the utilities of the composite game are just weighted sums of the (relevant) respective utilities of the component games.

Definition 1. Let $\widetilde{G} = \{G^{pq} : 1 \leq p \neq q \leq n\}$ be a microscopic set of 2-player games w.r.t. the player-set $P = \{1, \cdots, n\}$. An ***additive composite*** of $\widetilde{G}$, written $\bowtie_+ \widetilde{G}$, is an n-player game $G^P = \langle P, S^P, U^P \rangle$ whose utility functions u_p^P ($\in U^P$) are defined as follows: for each $p \in P$ and each (profile of pure strategies) $s \in S^P$

$$u_p^P(s) \quad = \quad \sum_{p \neq p'} \gamma_p^{pp'}(s) \cdot u_p^{pp'}(s_p, s_{p'}) \tag{2}$$

where each $\gamma_p^{pp'}(s)$ is a constant. We shall refer to the $\gamma_p^{pp'}(s)$'s as the *weights*[3] of the composition. □

Example 1. The first example of additive composites are those whose weights are uniformly 1. These are precisely the *polymatrix games* of Yanovskaya [18]. *Every* joinable set of Nash equilibria of the component games is preserved by such an additive composite. Consider the simple case of the 3-player setting with $P = \{1, 2, 3\}$. It is straightforward to check that in the composite game, the *expected payoff* of player 1 satisfies

$$u_1^{123}(x_1^*, x_2^*, x_3^*) \;=\; u_1^{12}(x_1^*, x_2^*) + u_1^{13}(x_1^*, x_3^*) \tag{3}$$

where x_i^* is a mixed-strategy profile of player i. Thus if x_1^* is 1's best response to x_2^* in G^{12}, and if it is also 1's best response to x_3^* in G^{13}, then it must be 1's best response to (x_2^*, x_3^*) in the composite game G^{123}. This claim follows from the necessary and sufficient condition (1). Indeed one could say that polymatrix games are *strongly sound* in the sense that the composite preserves *every* joinable set of Nash equilibria of the component games. An example of polymatrix game is shown in Table 1. In general, however, additive composition is not sound.

P1, P2	I	II
I	2, 1	4, 9
II	4, 9	2, 3

P1, P3	I	II
I	3, 6	0, 2
II	1, 1	1, 4

P2, P3	I	II
I	4, 3	3, 4
II	2, 2	4, 1

P1, (P2 & P3)	I & I	I & II	II & I	II & II
I	5, 5, 9	2, 4, 6	7, 11, 8	4, 13, 3
II	5, 13, 4	5, 12, 8	3, 5, 3	3, 7, 5

Table 1. Sample additive composition (polymatrix game). $(3/7, 4/7)(1/2, 1/2)$ is a Nash equilibrium for G^{12}; $(3/7, 4/7)(1/3, 2/3)$ is a Nash equilibrium for G^{13}; $(1/2, 1/2)(1/3, 2/3)$ is a Nash equilibrium for G^{23}; and $(3/7, 4/7)(1/2, 1/2)(1/3, 2/3)$ is a Nash equilibrium of the composite game G^{123}.

Theorem 1. *Additive composition is not sound.*

Given a joinable set of Nash equilibria of the component games, is there a way by which we can so constrain the weights that the component Nash equilibria are preserved by the additive composite? A major result of this paper is that there are indeed such constraints. We shall see that they can be exploited to demonstrate the *constructive completeness* of additive composition. The way to understand these constraints is that they control the weights $\gamma_p^{pp'}(s)$ by another layer of weights $const^{pp'}$, which should be understood as the weight of the component game $G^{pp'}$ in the composite G^P.

[3] The weights can be used as a measure of the impact of a particular outcome in the component game on the composite game w.r.t a particular player. In other words, in an additively-composed game, the payoff to a player p for any combination of *pure* strategies (from all players) is a weighted sum of the corresponding two-player game payoffs with impact.

Using the same notation of microscopic set of 2-player games w.r.t. player-set $P = \{1, \cdots, n\}$, we note that any joinable set of Nash equilibria for the component games can be specified by an n-tuple $(x_1^*, \cdots, x_n^*)$ such that for any $1 \le p \ne p' \le n$, we have x_p^* is a Nash equilibrium of the component game $G^{pp'}$.

Definition 2. Given a joinable set of Nash equilibria for the component games as specified by $(x_1^*, \cdots, x_n^*)$, we say that an additive composite is ***strict*** for $(x_1^*, \cdots, x_n^*)$ just if:

- For each $1 \le p \ne p' \le n$, there is a positive real constant $const^{pp'}$.
- For each $1 \le p \ne p' \le n$, and for each $s_p \in S_p$ and $s_{p'} \in S_{p'}$, there is an equational constraint:

$$\sum_{s_{-pp'} \in S_{-pp'}} \left(\gamma_p^{pp'}(s_{-pp'}; s_p; s_{p'}) \cdot \prod_{q \in P^{-pp'}} x_q^*(s_q) \right) \quad = \quad const^{pp'} \tag{4}$$

where $P^{-pp'} = P \setminus \{p, p'\}$. A sample 3-player strict additive composition is shown in Appendix 8.

The constant $const^{pp'}$ should be viewed as the *weight* of the component game $G^{pp'}$ in the composite G^P. Note that there are $\binom{n}{2}$ constants of type $const^{pp'}$; $\sum_p \sum_{p' \ne p} m_p \cdot m_{p'}$ number of constraint equations where $m_p = |S_p|$; and $2 \cdot \binom{n}{2} \cdot \prod_p m_p$ number of constants of type $\gamma_p^{pp'}(s)$. It is straightforward to verify that if for some constant k, we have $\gamma_p^{pp'}(s) = k = const^{qq'}$ as p, p', q, q' and s vary, then (4) holds trivially. Indeed this is the case for polymatrix games with the constant $k = 1$. Thus polymatrix games are additive composites that are strict for every joinable set of component Nash equilibria.

Lemma 1. *Strict additive composition is sound. I.e. Given any microscopic set* $\widetilde{G} = \{G^{pp'} : 1 \le p \ne p' \le n\}$ *w.r.t. the player set* $P = \{1, \cdots, n\}$*, and given any joinable set of component Nash equilibria (i.e. for each* $1 \le p \ne p' \le n$ *we have* $(x_p^*, x_{p'}^*)$ *is a Nash equilibrium of the component game* $G^{pp'}$*), then* $(x_1^*, \cdots, x_n^*)$ *is a Nash equilibrium for the strict-for-*$(x_1^*, \cdots, x_n^*)$ *additively composed game* G^P*.*

Lemma 2. *Strict additive composition is constructively complete. I.e. there is a procedure that, given any* n*-player game* G^P *with player-set* $P = \{1, \cdots, n\}$ *and given any Nash equilibrium* $(x_1^*, \cdots, x_n^*)$ *of* G^P*, produces a microscopic set* $\widetilde{G}$ *w.r.t.* P *whose (strict) additive composite is* G^P*, and such that for each* $1 \le p \ne p' \le n$*, we have* $(x_p^*, x_{p'}^*)$ *is a Nash equilibrium of* $G^{pp'}$*.*

Theorem 2. *Strict additive composition is both sound and constructively complete.*

Proof. Combining Lemma 1 and 2, the claim follows. ☐

3.2 Multiplicative composition

In an multiplicatively-composed game, the payoff to a player p for any combination of *pure* strategies (from all players) is the weighted product of the corresponding two-player game payoffs.

Definition 3. Let $\widetilde{G} = \{G^{pq} : 1 \le p \ne q \le n\}$ be a microscopic set of 2-player games w.r.t. the player-set $P = \{1, \cdots, n\}$. The ***multiplicative composite*** of $\widetilde{G}$, written $\bowtie_* \widetilde{G}$, is the game $\langle P, S, U \rangle$ whose utility functions u_p^P $(\in U^P)$ are defined as: for each $p \in P$, and each $s \in S$, $u_p^P(s) = \prod_{q \ne p} \gamma_p^{pp'}(s) \cdot u_p^{pq}(s_p, s_q)$. ☐

Theorem 3. *Multiplicative composition is not sound.*

Similar to the additive composition case, we define the strictness condition for multiplicative composition.

Definition 4. An multiplicative composite is ***strict*** just if:

- For each $1 \le p \le n$, and for each $s_p \in S_p$, there is a positive real constant $const_p^{s_p}$.

– For each $1 \le p \ne p' \le n$, and for each $s_{-p} \in S_{-p}$, there is an equational constraint:

$$\prod_{p'} \gamma_p^{pp'}(s_{-p}; s_p) \quad = \quad const_p^{s_p} \tag{5}$$

where all γ's are positive real numbers.

Note that, the constraint (5) is not w.r.t. a particular Nash equilibrium as in the additive case, thus it can be applied to *all* Nash equilibria of a given n-player game, *i.e.* strongly sound. However, the tradeoff is that the payoffs in the n-player game cannot be chosen arbitrarily, *i.e.* not complete.

Theorem 4. *Strict multiplicative composition is strongly sound but not complete.*

4 Game incorporation

By applying game composition, players can view a global game as a composition of smaller games. Consider a game with 3 players 1, 2, and 3, playing bilateral game with each other, *i.e.* G^{12}, G^{13}, and G^{23}, assuming that the global game G^{123} is a composite (*e.g.* polymatrix game) of the three bilateral games, if (x_1^*, x_2^*) and (x_1^*, x_3^*) are Nash equilibria of G^{12} and G^{13} respectively, but (x_2^*, x_3^*) may *not* be a Nash equilibrium of G^{23}, that is, the Nash equilibria for player 2 in G^{12} and G^{23} may not be joinable, then

– How can player 2 ensures his strategy x_2^* is a best response to player 1 and 3's strategies x_1^* and x_3^*.
– If the details of G^{23} are hidden from player 1, or the payoff matrix in G^{23} is perturbed, how can player 1 ensures that (x_1^*, x_2^*, x_3^*) is a global Nash equilibrium?

To answer these questions, we introduce the concept of *game incorporation*. For simplicity, we use 3-player polymatrix game for illustration purpose, and the concept can be easily generalized to n-player case (details can be found in Appendix 13), and to strict additive and strict multiplicative compositions as well.

Definition 5. Given any 3-player polymatrix game G^{123} and a microscopic set $\widetilde{G} = \{ G^{12}, G^{13}, G^{23} \}$ whose composite $\bowtie_+ \widetilde{G}$ is G^{123}, and suppose (x_1^*, x_2^*) is a Nash equilibrium of G^{12} and (x_1^*, x_3^*) is a Nash equilibrium of G^{13}, we say that $G^{12} \bowtie_+ G^{13}$ ***incorporates*** G^{23} ***for player 2***, written $G^{12} \bowtie_+ G^{13} \lhd_2 G^{23}$, just if for each $1 \le j \ne \hat{j} \le m_2$

$$(u_2^{12}(x_1^*; \pi_{2,j}) - u_2^{12}(x_1^*; \pi_{2,\hat{j}})) + (u_2^{23}(x_3^*; \pi_{2,j}) - u_2^{23}(x_3^*; \pi_{2,\hat{j}})) > 0 \implies x_2^*(\hat{j}) = 0 \tag{6}$$

Similarly for player 3. We say $G^{12} \bowtie_+ G^{13}$ ***incorporates*** G^{23}, written $G^{12} \bowtie_+ G^{13} \lhd G^{23}$ (equivalently G^{23} is the *incorporated game* of $G^{12} \bowtie_+ G^{13}$), just if $G^{12} \bowtie_+ G^{13} \lhd_2 G^{23}$ and $G^{12} \bowtie_+ G^{13} \lhd_3 G^{23}$. □

Theorem 5. *Given any 3-player game G^{123} and a microscopic set $\widetilde{G} = \{ G^{12}, G^{13}, G^{23} \}$ whose composite $\bowtie_+ \widetilde{G}$ is G^{123}, and suppose (x_1^*, x_2^*) is a Nash equilibrium of G^{12}, and (x_1^*, x_3^*) is a Nash equilibrium of G^{13}, then (x_1^*, x_2^*, x_3^*) is a Nash equilibrium of the composite game G^{123} if $G^{12} \bowtie_+ G^{13} \lhd G^{23}$.* □

Theorem 5 says that to verify that (x_1^*, x_2^*, x_3^*) is a Nash equilibrium of the composite game G^{123}, the payoff matrix of the component game G^{23} is not required be fully revealed to player 1 or unchangeable. Indeed we just need to check the validity of (6) by regarding the differences, $(u_2^{23}(x_3^*; \pi_{2,\hat{j}}) - u_2^{23}(x_3^*; \pi_{2,j}))$, as suitable *abstractions* (instead of the full details) of the component game G^{23}.

5 Comparisons with related work

The notion of composition and decomposition was first proposed by von Neumann and Morgenstern [15] in the context of cooperative games, as opposed to non-cooperative games in our case. Our notion is also different from the *payoff decomposition* [8] and *belief decomposition* [7] for Bayesian games.

As mentioned earlier, polymatrix game [18] is a special case of our strict additive composition of games: in polymatrix games, all payoffs in the component 2-player games are uniformly weighted 1 in the composite game. Recent work on the

r-Nash Problem [1, 3, 6, 5] is concerned with the complexity of computing Nash equilibrium, while our work focuses on the issue of ensuring global Nash equilibrium with hidden information.

The concept of graphical games was proposed by Kearns, Littman and Singh [11] as a compact graph-theoretic representation for multi-player games. The game incorporation concept complements the graphical game model by allowing the interactions (component games) between each player and the players outside his neighborhood to be treated as incorporated games.

Recent work by Daskalakis and Papadimitriou [4] considers polymatrix graphical games in which all the component games are zero-sum games, and the authors showed that a Nash equilibrium of this class of games can be computed in polynomial time.

6 Conclusion

We introduce two methods of composing games, additive and multiplicative, and analyze the extent to which each preserves and reflects component Nash equilibria. We then introduce *game incorporation*, which provides suitable abstraction of certain component games from the perspective of the composite game. As for further directions, the notion of completeness for strict-additive composition here is with respect to a *particular* Nash equilibrium. An important question is: Can additive composition be so constrained as to be *universally* complete, *i.e.* decomposable with respect to *all* Nash equilibria? We also intend to relate our work with the work on partially-specified large games [10], nested potential games [13], epistemic conditions for equilibrium concepts in games with a communication/interaction structure [17], and apply game composition and incorporation to applications such as multi-agent systems [9] and e-commerce [16]. So far, our work is motivated by the phenomena on the Internet, and an interesting question is: What implication our findings may have for the existing game theory literature.

References

1. Xi Chen and Xiaotie Deng. Settling the complexity of two-player nash equilibrium. In *FOCS*, pages 261–272. IEEE Computer Society, 2006.
2. E. F. Codd. A relational model of data for large shared data banks. *Communications of the ACM*, 13(6):377–387, 1970.
3. C. Daskalakis and C. H. Papadimitriou. Three-player games are hard. *Electronic Colloquium on Computational Complexity (ECCC)*, (139), 2005.
4. C. Daskalakis and C. H. Papadimitriou. On a network generalization of the minmax theorem. In *ICALP '09: Proceedings of the 36th Internatilonal Collogquium on Automata, Languages and Programming*, pages 423–434, Berlin, Heidelberg, 2009. Springer-Verlag.
5. Constantinos Daskalakis, Paul W. Goldberg, and Christos H. Papadimitriou. The complexity of computing a nash equilibrium. [12], pages 71–78.
6. Paul W. Goldberg and Christos H. Papadimitriou. Reducibility among equilibrium problems. In Kleinberg [12], pages 61–70.
7. J. C. Harsanyi. Games with incomplete information played by "bayesian" players, i-iii. part i. the basic model. *Management Science*, 14(3):159–182, 1967.
8. J. C. Harsanyi and R. Selten. *A General Theory of Equilibrium Selection in Games*. MIT Press, Cambridge, MA, USA, 1988.
9. N. R. Jennings. An agent-based approach for building complex software systems. *Commun. ACM*, 44(4):35–41, 2001.
10. E. Kalai. Partially-specified large games. In X. Deng and Y. Ye, editors, *WINE '05: First International Workshop on Internet and Network Economics*, volume 3828 of *Lecture Notes in Computer Science*, pages 3–13. Springer, 2005.
11. M. Kearns, M. L. Littman, and S. Singh. Graphical models for game theory. In *UAI '01: Proceedings of the Uncertainty in Artificial Intelligence Conference*, pages 253–260, 2001.
12. Jon M. Kleinberg, editor. *Proceedings of the 38th Annual ACM Symposium on Theory of Computing, Seattle, WA, USA, May 21-23, 2006*. ACM, 2006.
13. H. Uno. Nested potential games. *Economics Bulletin*, 3(19):1–8, 2007.
14. E. van Damme. *Stability and perfection of Nash equilibria*. Springer-Verlag, London, UK, 1991.
15. J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.
16. N. Vulkan. *The Economics of E-Commerce: A Strategic Guide to Understanding and Designing the Online Marketplace*. Princeton University Press, 2003.
17. A. Witzel, K. R. Apt, and J. A. Zvesper. Strategy elimination in games with interaction structures. In X. He, J. F. Horty, and E. Pacuit, editors, *LORI*, volume 5834 of *Lecture Notes in Computer Science*, pages 302–315. Springer, 2009.
18. E. B. Yanovskaya. Equilibrium points in polymatrix games. *Litovskii Matematicheskii Sbornik (in Russian)*, (8):381–384, 1968.

7 Proof of Theorem 1: unsoundness of general additive composition

We construct a counter-example shown in Table 2. Since the weights in the additive composition are arbitrary, we can choose $\gamma_1^{12}(I,I,I) = \gamma_2^{12}(I,I,I) = \gamma_1^{13}(I,I,I) = \gamma_3^{13}(I,I,I) = \gamma_2^{23}(I,I,I) = \gamma_3^{23}(I,I,I) = 0$, and the rest γ's are all 1. Thus, $(1,0)(1,0)$ is a Nash equilibrium for G^{12}; $(1,0)(1,0)$ is a Nash equilibrium for G^{13}; $(1,0)(1,0)$ is a Nash equilibrium for G^{23}; but $(1,0)(1,0)(1,0)$ is not a Nash equilibrium of the additive composite game G^{123}. □

P1, P2	I	II
I	100, 100	1, 1
II	1, 1	1, 1

P1, P3	I	II
I	100, 100	1, 1
II	1, 1	1, 1

P2, P3	I	II
I	100, 100	1, 1
II	1, 1	1, 1

P1, (P2 & P3)	I & I	I & II	II & I	II & II
I	0, 0, 0	101, 101, 2	101, 2, 101	101, 2, 2
II	2, 101, 101	2, 101, 2	2, 2, 101	2, 2, 2

Table 2. Sample counter-example to show additive composition is not sound.

8 Proof of Lemma 1: soundness of strict additive composition

We start with the 3-player case for illustration, and then generalize the proof to n-player games.

3-player strict additive composition – soundness

We begin with the simplest case of a 3-player setting. Let $u_p^{123}(i,j,k)$ be the payoff of player p in the strict-additively composed game G^{123} with players 1, 2 and 3 playing pure strategies i, j and k respectively. We have: for each $1 \le i \le m_1, 1 \le j \le m_2, 1 \le k \le m_3$,

$$u_1^{123}(i,j,k) \quad = \quad \gamma_1^{12}(i,j,k) \cdot u_1^{12}(i,j) + \gamma_1^{13}(i,j,k) \cdot u_1^{13}(i,k) \tag{7}$$

subject to the following constraints:

$$\text{For each } 1 \le i \le m_1, 1 \le j \le m_2, \qquad \sum_{k=1}^{m_3} \gamma_1^{12}(i,j,k) \cdot x_3^*(k) \quad = \quad \mathit{const}^{12} \tag{8}$$

$$\text{For each } 1 \le i \le m_1, 1 \le k \le m_3, \qquad \sum_{j=1}^{m_2} \gamma_1^{13}(i,j,k) \cdot x_2^*(j) \quad = \quad \mathit{const}^{13} \tag{9}$$

where both const^{12} and const^{13} are positive. Similarly for u_2^{123} and u_3^{123}.

Lemma 3. *Strict additive composition is sound in the 3-player setting. I.e. for every joinable set of Nash equilibria for the component games, say, (x_1^*, x_2^*) for G^{12}, (x_2^*, x_3^*) for G^{23} and (x_1^*, x_3^*) for G^{13}, the mixed-strategy profile (x_1^*, x_2^*, x_3^*) is a Nash equilibrium of any additive composite G^{123} that is strict-for-(x_1^*, x_2^*, x_3^*).*

Proof. Suppose (x_1^*, x_2^*), (x_1^*, x_3^*) and (x_2^*, x_3^*) are Nash equilibrium (mixed) strategies for G^{12}, G^{13}, and G^{23} respectively, *i.e.* for each $1 \le i \ne \hat{i} \le m_1$

$$\underbrace{\sum_{j=1}^{m_2} u_1^{12}(i,j) \cdot x_2^*(j)}_{10.1} > \underbrace{\sum_{j=1}^{m_2} u_1^{12}(\hat{i},j) \cdot x_2^*(j)}_{10.2} \quad \Longrightarrow \quad x_1^*(\hat{i}) = 0 \tag{10}$$

$$\underbrace{\sum_{k=1}^{m_3} u_1^{13}(i,k) \cdot x_3^*(k)}_{11.1} > \underbrace{\sum_{k=1}^{m_3} u_1^{13}(\hat{i},k) \cdot x_3^*(k)}_{11.2} \quad \Longrightarrow \quad x_1^*(\hat{i}) = 0 \tag{11}$$

Similarly for players 2 and 3. We want to show (x_1^*, x_2^*, x_3^*) is a Nash equilibrium for the strict-additively composed game G^{123}, *i.e.* for each $1 \le i \ne \hat{i} \le m_1$

$$\underbrace{\sum_{j=1}^{m_2}\sum_{k=1}^{m_3} u_1^{123}(i,j,k)\cdot x_2^*(j)\cdot x_3^*(k)}_{12.1} > \underbrace{\sum_{j=1}^{m_2}\sum_{k=1}^{m_3} u_1^{123}(\hat{i},j,k)\cdot x_2^*(j)\cdot x_3^*(k)}_{12.2} \quad \Longrightarrow \quad x_1^*(\hat{i}) = 0 \tag{12}$$

according to the constraints (8) and (9), we have

$$\begin{aligned}(12.1) \quad &= \quad \left[\sum_{j=1}^{m_2} u_1^{12}(\hat{i},j)x_2^*(j)\cdot\sum_{k=1}^{m_3}\gamma_1^{12}(i,j,k)x_3^*(k)\right] + \left[\sum_{k=1}^{m_3} u_1^{13}(\hat{i},k)x_3^*(k)\cdot\sum_{j=1}^{m_2}\gamma_1^{13}(i,j,k)x_2^*(j)\right]\\ &= \quad \mathit{const}^{12}\cdot(10.1) + \mathit{const}^{13}\cdot(11.1)\end{aligned}$$

Similarly, we have $(12.2) = \mathit{const}^{12}\cdot(10.2) + \mathit{const}^{13}\cdot(11.2)$. Thus, if (10) and (11) hold, then (12) must hold. Similarly for players 2 and 3. The claim follows. □

n-player strict additive composition – soundness

Without loss of generality, we look at player 1, and all other players are similar. We denote s_p as the index of player p's pure strategies. Suppose (x_1^*, x_p^*) ($2 \le p \le n$) is a Nash equilibrium (mixed) strategies for G^{1p}, *i.e.* for each $1 \le s_1 \ne \hat{s}_1 \le m_1$

$$\underbrace{\sum_{s_p=1}^{m_p} u_1^{1p}(s_1,s_p)\cdot x_p^*(s_p)}_{13.1} > \underbrace{\sum_{s_p=1}^{m_p} u_1^{1p}(\hat{s}_1,s_p)\cdot x_p^*(s_p)}_{13.2} \quad \Longrightarrow \quad x_1^*(\hat{s}_1) = 0 \tag{13}$$

We want to show $(x_1^*, \cdots, x_n^*)$ is a Nash equilibrium for the strict-additively composed game $G^{1\cdots n}$, *i.e.* for each $1 \le s_1 \ne \hat{s}_1 \le m_1$

$$\underbrace{\sum_{s_{-1}\in S_{-1}}\left[u_1^{1\cdots n}(s_{-1};s_1)\cdot\prod_{p=2}^{n} x_p^*(s_p)\right]}_{14.1} > \underbrace{\sum_{s_{-1}\in S_{-1}}\left[u_1^{1\cdots n}(s_{-1};\hat{s}_1)\cdot\prod_{p=2}^{n} x_p^*(s_p)\right]}_{14.2} \quad \Longrightarrow \quad x_1^*(\hat{s}_1) = 0 \tag{14}$$

We observe that

$$\begin{aligned}(14.1) \quad &= \quad \sum_{s_{-1}\in S_{-1}}\left(\left[\sum_{p=2}^{n}\gamma_1^{1p}(s_{-1};s_1)\cdot u_1^{1p}(s_1,s_p)\right]\cdot\left[\prod_{p=2}^{n} x_p^*(s_p)\right]\right)\\ &= \quad \sum_{p=2}^{n}\mathit{const}^{1p}\cdot\sum_{s_p=1}^{m_p} u_1^{1p}(s_1,s_p)\cdot x_p^*(s_p)\\ &= \quad \sum_{p=2}^{n}\mathit{const}^{1p}\cdot(13.1)\end{aligned}$$

Similarly, we have

$$(14.2) \quad = \quad \sum_{p=2}^{n}\mathit{const}^{1p}\cdot(13.2)$$

So if (13) hold, then (14) must hold. Similarly for other players. Thus, the strict additive composition is sound in preserving Nash equilibrium for 3-player games. □

9 Proof of Lemma 2: completeness of strict additive composition

We start with the 3-player case for illustration, and then generalize the proof to n-player games.

3-player strict additive composition – completeness

Lemma 4. *Strict additive composition is constructively complete in the 3-player setting. I.e. there is a procedure that, for every 3-player game G^{123} and for every Nash equilibrium (x_1^*, x_2^*, x_3^*) of G^{123}, produces a microscopic set $\widetilde{G} = \{ G^{12}, G^{13}, G^{23} \}$ whose (strict) additive composite is G^{123}, and such that (x_1^*, x_2^*) is a Nash equilibrium of G^{12}, (x_1^*, x_3^*) is a Nash equilibrium of G^{13}, and (x_2^*, x_3^*) is a Nash equilibrium of G^{23}.*

Proof. We establish the completeness by proving the existence of real roots to a system of linear equations. From the strict additive composition rules, we have a system of linear equations (7) in matrix form

$$\gamma \quad \cdot \quad \mathbf{X} \quad = \quad \mathbf{B}$$

where $\gamma : \gamma_1^{12}(i,j,k), \gamma_1^{13}(i,j,k)$; $\mathbf{X} : u_1^{12}(i,j), u_1^{13}(i,k)$; $\mathbf{B} : u_1^{123}(i,j,k)$. There are $(2m_1 \cdot m_2 \cdot m_3)$ number of equations with $3(m_1 \cdot m_2 + m_1 \cdot m_3 + m_2 \cdot m_3)$ variables in $\mathbf{X}$. Our task is to show that given $\mathbf{B}$, we can configure γ, such that there are real roots for $\mathbf{X}$, subject to the constraints in (8) and (9). We will only look at player 1, because other players are similar. Also, since the equations in (7) can be grouped into mutually independent sets according to different i's, *i.e.* pure strategies for player 1, we can focus on a subset of equations, $\gamma \cdot \mathbf{X}_i = \mathbf{B}_i$. There are $(m_2 \cdot m_3)$ equations with $(m_2 + m_3)$ variables in $\mathbf{X}_i$. They can be solved in m_2 steps, and in the 1st step $(m_3 + 1)$ number of variables are solved, and in the rest of the steps, one variable is solved in each step, as shown in Table 3.

Step #	$\mathbf{X}_i$								$\mathbf{B}_i$
j	$u_1^{12}(i,1)$	$u_1^{12}(i,2)$	$\cdots$	$u_1^{12}(i,m_2)$	$u_1^{13}(i,1)$	$u_1^{13}(i,2)$	$\cdots$	$u_1^{13}(i,m_3)$	$u_1^{123}(i,j,k)$
1	$\gamma_1^{12}(i,1,1)$ $\gamma_1^{12}(i,1,2)$ $\vdots$ $\gamma_1^{12}(i,1,m_3)$				$\gamma_1^{13}(i,1,1)$	$\gamma_1^{13}(i,1,2)$	$\ddots$	$\gamma_1^{13}(i,1,m_3)$	$u_1^{123}(i,1,1)$ $u_1^{123}(i,1,2)$ $\vdots$ $u_1^{123}(i,1,m_3)$
2		$\gamma_1^{12}(i,2,1)$ $\gamma_1^{12}(i,2,2)$ $\vdots$ $\gamma_1^{12}(i,2,m_3)$			$\gamma_1^{13}(i,2,1)$	$\gamma_1^{13}(i,2,2)$	$\ddots$	$\gamma_1^{13}(i,2,m_3)$	$u_1^{123}(i,2,1)$ $u_1^{123}(i,2,2)$ $\vdots$ $u_1^{123}(i,2,m_3)$
$\vdots$			$\ddots$				$\ddots$		$\vdots$
m_2				$\gamma_1^{12}(i,m_2,1)$ $\gamma_1^{12}(i,m_2,2)$ $\vdots$ $\gamma_1^{12}(i,m_2,m_3)$	$\gamma_1^{13}(i,m_2,1)$	$\gamma_1^{13}(i,m_2,2)$	$\ddots$	$\gamma_1^{13}(i,m_2,m_3)$	$u_1^{123}(i,m_2,1)$ $u_1^{123}(i,m_2,2)$ $\vdots$ $u_1^{123}(i,m_2,m_3)$

Table 3. System of equations for the 3-player game decomposition.

More specifically, the detailed algorithm is as follows:

- **Step 1**: for each $1 \leq k \leq m_3$, $u_1^{123}(i,1,k)$ are given in G^{123}.
  1. Arbitrarily choose $u_1^{12}(i,1)$ to be a positive real number such that

$$u_1^{12}(i,1) \cdot x_2^*(1) \quad \leq \quad \frac{1}{2m_2} \sum_j \sum_k u_1^{123}(i,j,k) \cdot x_2^*(j) \cdot x_3^*(k)$$

  2. Arbitrarily choose $u_1^{13}(i,1), u_1^{13}(i,2), \cdots u_1^{13}(i,m_3)$ to be positive, such that $(11.1) = 1/2 \cdot (12.1)$.
  3. Arbitrarily choose $\gamma_1^{12}(i,1,1), \gamma_1^{12}(i,1,2), \cdots \gamma_1^{12}(i,1,m_3)$ which satisfy the constraint in (8).
  4. Solve for $\gamma_1^{13}(i,1,1), \gamma_1^{13}(i,1,2), \cdots, \gamma_1^{13}(i,1,m_3)$.
- **Step 2**: for each $1 \leq k \leq m_3$, $u_1^{123}(i,2,k)$ are given in G^{123} and for each $1 \leq k \leq m_3$, $u_1^{13}(i,k)$ are given from step 1.

1. Arbitrarily choose $u_1^{12}(i,2)$ to be a positive real number such that

$$u_1^{12}(i,2)\cdot x_2^*(2) \quad \leq \quad \frac{1}{2m_2}\sum_j\sum_k u_1^{123}(i,j,k)\cdot x_2^*(j)\cdot x_3^*(k)$$

2. Arbitrarily choose $\gamma_1^{12}(i,2,1),\gamma_1^{12}(i,2,2),\cdots,\gamma_1^{12}(i,2,m_3)$ which satisfy the constraint (8).
3. Solve for $\gamma_1^{13}(i,2,1),\gamma_1^{13}(i,2,2),\cdots,\gamma_1^{13}(i,2,m_3)$.

– **Step** j ($3\leq j\leq m_2-1$): for each $1\leq k\leq m_3$, $u_1^{123}(i,j,k)$ are given in G^{123} and for each $1\leq k\leq m_3$, $u_1^{13}(i,k)$ are given from step 1. Similar to step 2, we can compute $\gamma_1^{13}(i,j,1),\gamma_1^{13}(i,j,2),\cdots,\gamma_1^{13}(i,j,m_3)$.

– **Step** m_2: for each $1\leq k\leq m_3$, $u_1^{123}(i,m_2,k)$ are given in G^{123} and for each $1\leq k\leq m_3$, $u_1^{13}(i,k)$ are given from step 1. Also, for each $1\leq j\leq m_2-1$, $1\leq k\leq m_3$, $\gamma_1^{13}(i,j,k)$ are given from previous (m_2-1) steps.

1. Solve for $\gamma_1^{13}(i,m_2,1),\gamma_1^{13}(i,m_2,2),\cdots,\gamma_1^{13}(i,m_2,m_3)$ using the constraint (9). Note that, without loss of generality we assume $x_2^*(m_2)>0$, because there is at least a j, $1\leq j\leq m_2$ such that $x_2^*(j)>0$, in which case we will solve the $\gamma_1^{13}(i,j,1),\gamma_1^{13}(i,j,2),\cdots,\gamma_1^{13}(i,j,m_3)$ in the final step. Now, we are left with the following system of equations: for each $1\leq k\leq m_3$

$$\gamma_1^{12}(i,m_2,k)\cdot u_1^{12}(i,m_2) \quad = \quad u_1^{123}(i,m_2,k)-\gamma_1^{13}(i,m_2,k)\cdot u_1^{13}(i,k) \tag{15}$$

2. According to constraint (8), we have

$$\mathit{const}^{12}\cdot u_1^{12}(i,m_2) \quad = \quad \sum_{k=1}^{m_3}[u_1^{123}(i,m_2,k)-\gamma_1^{13}(i,m_2,k)\cdot u_1^{13}(i,k)]\cdot x_3^*(k)$$

To satisfy it, we only need to choose

$$u_1^{12}(i,m_2) \quad = \quad \sum_{k=1}^{m_3}[u_1^{123}(i,m_2,k)-\gamma_1^{13}(i,m_2,k)\cdot u_1^{13}(i,k)]\cdot x_3^*(k)/\mathit{const}^{12}$$

Then, for each $1\leq k\leq m_3, \gamma_1^{12}(i,m_2,k)$ can be solved by substituting $u_1^{12}(i,m_2)$ back to (15).

Note that this algorithm computes all $u_1^{12}(i,j)$ and $u_1^{13}(i,k)$ by assigning $\gamma_1^{12}(i,j,k)$ and $\gamma_1^{13}(i,j,k)$ appropriate values, without any restriction on $u_1^{123}(i,j,k)$. Thus, real roots are guaranteed for **X** no matter what values **B** are. Also, by the construction step 1, we have $\mathit{const}^{13}\cdot(11.1)=1/2\cdot(12.1)$. Also, according to the composition rule and constraints, we have $(12.1)=\mathit{const}^{12}\cdot(10.1)+\mathit{const}^{13}\cdot(11.1)$. Then, $\mathit{const}^{12}\cdot(10.1)=1/2\cdot(12.1)$. Similarly for (10.2), (11.2), and (12.2). So, if (12) is satisfied, then (10) and (11) must also hold as well.

As a side note, we have chosen $u_1^{12}(i,j)$ $(1\leq j\leq m_2-1)$ such that

$$\sum_{j=1}^{m_2-1}u_1^{12}(i,j)\cdot x_2^*(j) \quad \leq \quad \frac{m_2-1}{2m_2}\sum_j\sum_k u_1^{123}(i,j,k)\cdot x_2^*(j)\cdot x_3^*(k)$$

so we have

$$u_1^{12}(i,m_2)\cdot x_2^*(m_2) \quad \geq \quad \frac{1}{2m_2}\sum_j\sum_k u_1^{123}(i,j,k)\cdot x_2^*(j)\cdot x_3^*(k) \quad > \quad 0$$

which implies $u_1^{12}(i,m_2)>0$.

So far, we have shown by construction that for each Nash equilibrium point (x_1^*,x_2^*,x_3^*) of G^{123}, there is a strict additive composition such that (x_1^*,x_2^*), (x_1^*,x_3^*), and (x_2^*,x_3^*) are Nash equilibrium of G^{12}, G^{13}, and G^{23} respectively.

□

n-player strict additive composition – completeness

Similar to the proof in the 3-player case, we show the completeness by proving the existence of real roots to a system of linear equations. From the strict additive composition rules, we have a system of linear equations (2) in matrix form

$$\gamma \quad \cdot \quad \mathbf{X} \quad = \quad \mathbf{B}$$

where for each $1 \leq p \neq p' \leq n$, $s \in S$

$$\begin{aligned} \gamma: &\quad \gamma_p^{pp'}(s) \\ \mathbf{X}: &\quad u_p^{pp'}(s_p, s_{p'}) \\ \mathbf{B}: &\quad u_p^{1\cdots n}(s) \end{aligned}$$

There are $(n \cdot \prod_{p=1}^{n} m_p)$ number of equations with $(2 \cdot \sum_{p \neq p'} m_p \cdot m_{p'})$ variables in $\mathbf{X}$. Our task is to show that given $\mathbf{B}$, we can configure γ, such that there are real roots for $\mathbf{X}$, subject to the constraints in (4). We only need to look at player 1, because other players are similar. Also, since the equations in (2) can be grouped into mutually independent sets according to different s_1's, *i.e.* pure strategies for player 1, we can focus on a subset of equations, that is

$$\gamma \quad \cdot \quad \mathbf{X}_{s_1} \quad = \quad \mathbf{B}_{s_1} \tag{16}$$

and there are $(\prod_{p=2}^{n} m_p)$ equations with $(\sum_{p=2}^{n} m_1 \cdot m_p)$ variables in $\mathbf{X}_{s_1}$. To solve them, we follow the same strategy in the 3-player game decomposition case, in which we divide the whole computation into m_2 steps. The intuition is that we first choose $u_1^{12}(s_1, 1), \cdots u_1^{12}(s_1, m_2 - 1)$ and $u_1^{1p}(s_1, s_p)$, $3 \leq p \leq n$. Then we tune the γ's to make the equations in the first $(m_2 - 1)$ steps hold. In the last step, we choose a specific value of $u_1^{12}(s_1, m_2)$ to make the rest of the equations hold.

More specifically, the detailed algorithm is as follows:

- **Step 1**: $\forall s_{-12} \in S_{-12}$, $u_1^{1\cdots n}(s_{-12}; s_1; 1)$ are given in $G^{1\cdots n}$.
  1. Arbitrarily choose $u_1^{12}(s_1, 1)$ to be a positive real number such that

$$u_1^{12}(s_1, 1) \cdot x_2^*(1) \quad \leq \quad \frac{1}{(n-1) \cdot m_2} \sum_{s_{-12} \in S_{-12}} \left(u_1^{1\cdots n}(s_{-12}; s_1; 1) \cdot \prod_{p=2}^{n} x_p^*(s_p) \right)$$

  2. Arbitrarily choose $\forall s_p \in S_p$, $u_1^{1p}(s_1, s_p)$, $3 \leq p \leq n$, to be positive, such that

$$const^{1p} \cdot (13.1) \quad = \quad \frac{1}{n-1} \cdot (14.1)$$

  3. Arbitrarily choose $\forall s_{-12} \in S_{-12}$, $\gamma_1^{1p}(s_{-12}; s_1; 1)$, $2 \leq p \leq n-1$ which satisfy the constraint (4).
  4. Solve for $\forall s_{-12} \in S_{-12}$, $\gamma_1^{1n}(s_{-12}; s_1; 1)$.
- **Step 2**: $\forall s_{-12} \in S_{-12}$, $u_1^{1\cdots n}(s_{-12}; s_1; 2)$ are given in $G^{1\cdots n}$ and $\forall s_p \in S_p$, $u_1^{1p}(s_1, s_p)$, $3 \leq p \leq n$, are given from step 1.
  1. Arbitrarily choose $u_1^{12}(s_1, 2)$ to be a positive real number such that

$$u_1^{12}(s_1, 2) \cdot x_2^*(2) \quad \leq \quad \frac{1}{(n-1) \cdot m_2} \sum_{s_{-12} \in S_{-12}} \left(u_1^{1\cdots n}(s_{-12}; s_1; 2) \cdot \prod_{p=2}^{n} x_p^*(s_p) \right)$$

  2. Arbitrarily choose $\forall s_{-12} \in S_{-12}$, $\gamma_1^{1p}(s_{-12}; s_1; 2)$, $2 \leq p \leq n-1$ which satisfy the constraint (4).
  3. Solve for $\forall s_{-12} \in S_{-12}$, $\gamma_1^{1n}(s_{-12}; s_1; 2)$.
- **Step** j **($3 \leq j \leq m_2 - 1$)**: Similar to step 2, we can solve $\forall s_{-12} \in S_{-12}$, $\gamma_1^{1p}(s_{-12}; s_1; j)$, $3 \leq p \leq n$.
- **Step** m_2: $\forall s_{-12} \in S_{-12}$, $\gamma_1^{1p}(s_{-12}; s_1; j)$, $3 \leq p \leq n$, $1 \leq j \leq m_2 - 1$, are given from previous $(m_2 - 1)$ steps.
  1. Solve $\forall s_{-12} \in S_{-12}$, $\gamma_1^{1p}(s_{-12}; s_1; m_2)$, $3 \leq p \leq n$, using the constraint (4). Note that, without loss of generality we assume $x_2^*(m_2) > 0$, because there is at least some j, $1 \leq j \leq m_2$ such that $x_2^*(j) > 0$, in which case we will solve the $\gamma_1^{1p}(s_{-12}; s_1; j)$, $3 \leq p \leq n$, in the final step. Now, we are left with the following system of equations: $\forall s_{-12} \in S_{-12}$, $3 \leq p \leq n$

$$\gamma_1^{12}(s_{-12}; s_1; m_2) \cdot u_1^{12}(s_1, m_2) = u_1^{1\cdots n}(s_{-12}; s_1; m_2) - \sum_{p=3}^{n} \gamma_1^{1p}(s_{-12}; s_1; m_2) \cdot u_1^{1p}(s_1, s_p) \tag{17}$$

  2. According to constraint (4), we have

$$\sum_{s_{-12} \in S_{-12}} \left(\left[u_1^{1\cdots n}(s_{-12}; s_1; m_2) - \sum_{p=3}^{n} \gamma_1^{1p}(s_{-12}; s_1; m_2) \cdot u_1^{1p}(s_1, s_p) \right] \cdot \prod_{p=3}^{n} x_p^*(s_p) \right) \\ = \quad const^{12} \cdot u_1^{12}(s_1, m_2)$$

To satisfy it, we only need to choose $u_1^{12}(s_1, m_2)$ to be

$$\sum_{s_{-12} \in S_{-12}} \left(\frac{u_1^{1 \cdots n}(s_{-12}; s_1; m_2) - \sum_{p=3}^{n} \gamma_1^{1p}(s_{-12}; s_1; m_2) \cdot u_1^{1p}(s_1, s_p)}{const^{12}} \cdot \prod_{p=3}^{n} x_p^*(s_p) \right)$$

Then, $\forall s_{-12} \in S_{-12}$, $\gamma_1^{12}(s_{-12}; s_1; m_2)$ can be solved by substituting $u_1^{12}(s_1, m_2)$ back to (17).

Thus real roots are guaranteed for **X** no matter what values **B** are. Also, by the construction step 1, we have for each $3 \le p \le n$

$$const^{1p} \cdot (13.1) \quad = \quad \frac{1}{n-1} \cdot (14.1)$$

Also, according to the decomposition rule and constraints, we have

$$\begin{aligned} const^{1n} \cdot (13.1) &= (14.1) - \sum_{p=3}^{n} const^{1p} \cdot (13.1) \\ &= \frac{1}{n-1} \cdot (14.1) \end{aligned}$$

Similarly for (13.2) and (14.2). So, if (14) is satisfied, then for each $2 \le p \le n$, (13) must also hold as well.

As a side note, it can be verified that $u_1^{12}(s_1, m_2) > 0$, because we have chosen $u_1^{12}(s_1, s_2)$ $(1 \le s_2 \le m_2 - 1)$ such that

$$\begin{aligned} u_1^{12}(s_1, m_2) \cdot x_2^*(m_2) &= \frac{1}{n-1} \cdot u_1^{1 \cdots n}(s) \cdot \prod_{p=2}^{n} x_p^*(s_p) - \sum_{s_2=1}^{m_2-1} u_1^{12}(s_1, s_2) \cdot x_2^*(s_2) \\ &\ge \frac{1}{m_2 \cdot (n-1)} \cdot u_1^{1 \cdots n}(s) \cdot \prod_{p=2}^{n} x_p^*(s_p) \quad > \quad 0 \end{aligned}$$

So far, we have shown by construction that for each Nash equilibrium point $(x_1^*, \cdots, x_n^*)$ of $G^{1 \cdots n}$, there is a strict additive composition such that for each $1 \le p \ne p' \le n$, $(x_p^*, x_{p'}^*)$ is Nash equilibrium of $G^{pp'}$. □

10 Proof of Theorem 3: unsoundness of general multiplicative composition

We construct a counter-example shown in Table 4. Since the weights in the multiplicative composition are arbitrary, we can choose $\gamma_1^{12}(I, I, I) = \gamma_2^{12}(I, I, I) = \gamma_1^{13}(I, I, I) = \gamma_3^{13}(I, I, I) = \gamma_2^{23}(I, I, I) = \gamma_3^{23}(I, I, I) = 0$, and the rest γ's are all 1. Thus, $(1, 0)(1, 0)$ is a Nash equilibrium for G^{12}; $(1, 0)(1, 0)$ is a Nash equilibrium for G^{13}; $(1, 0)(1, 0)$ is a Nash equilibrium for G^{23}; but $(1, 0)(1, 0)(1, 0)$ is not a Nash equilibrium of the multiplicative composite game G^{123}. □

P1, P2	I	II
I	100, 100	1, 1
II	1, 1	1, 1

P1, P3	I	II
I	100, 100	1, 1
II	1, 1	1, 1

P2, P3	I	II
I	100, 100	1, 1
II	1, 1	1, 1

P1, (P2 & P3)	I & I	I & II	II & I	II & II
I	0, 0, 0	100, 100, 1	100, 1, 100	100, 1, 1
II	1, 100, 100	1, 100, 1	1, 1, 100	1, 1, 1

Table 4. Sample counter-example to show multiplicative composition is not sound.

11 Proof of Theorem 4: soundness and non-completeness of strict multiplicative composition

For the soundness part, we start with the 3-player case for illustration, and then generalize the proof to n-player games; we show the non-completeness by giving a counter-example.

3-player strict multiplicative composition – soundness

We begin with the simplest case of strict multiplicative-decomposing a 3-player game into three 2-player games. The rule for 3-player strict multiplicative game composition is defined as follows: for each $1 \le i \le m_1, 1 \le j \le m_2, 1 \le k \le m_3$,

$$u_1^{123}(i,j,k) \quad = \quad \gamma_1^{12}(i,j,k) \cdot u_1^{12}(i,j) \cdot \gamma_1^{13}(i,j,k) \cdot u_1^{13}(i,k)$$

subject to the following constraints:

$$\text{For each } 1 \le i \le m_1, 1 \le j \le m_2, 1 \le k \le m_3, \qquad \gamma_1^{12}(i,j,k) \cdot \gamma_1^{13}(i,j,k) \quad = \quad const_1^i \tag{18}$$

where $const_1^i$ is a positive real number. Similarly for player 2 and 3.

Lemma 5. *Strict multiplicative composition is strongly sound. For any G^{123} and its microscopic set $\widetilde{G} = \{ G^{12}, G^{13}, G^{23} \}$, if (x_1^*, x_2^*) is a Nash equilibrium of G^{12}, (x_1^*, x_3^*) is a Nash equilibrium of G^{13}, and (x_2^*, x_3^*) is a Nash equilibrium of G^{23}, then (x_1^*, x_2^*, x_3^*) is Nash equilibrium of G^{123}.*

Proof. Suppose (x_1^*, x_2^*), (x_1^*, x_3^*) and (x_2^*, x_3^*) are Nash equilibria for G^{12}, G^{13}, and G^{23} respectively, *i.e.* for each $1 \le i \ne \hat{i} \le m_1$

$$\underbrace{\sum_{j=1}^{m_2} u_1^{12}(i,j) \cdot x_2^*(j)}_{19.1} > \underbrace{\sum_{j=1}^{m_2} u_1^{12}(\hat{i},j) \cdot x_2^*(j)}_{19.2} \quad \Longrightarrow \quad x_1^*(\hat{i}) = 0 \tag{19}$$

$$\underbrace{\sum_{k=1}^{m_3} u_1^{13}(i,k) \cdot x_3^*(k)}_{20.1} > \underbrace{\sum_{k=1}^{m_3} u_1^{13}(\hat{i},k) \cdot x_3^*(k)}_{20.2} \quad \Longrightarrow \quad x_1^*(\hat{i}) = 0 \tag{20}$$

(Similarly for players 2 and 3.) We want to show (x_1^*, x_2^*, x_3^*) is a Nash equilibrium for the composite game G^{123}, *i.e.* for each $1 \le i \ne \hat{i} \le m_1$

$$\underbrace{\sum_{j=1}^{m_2}\sum_{k=1}^{m_3} u_1^{123}(i,j,k) \cdot x_2^*(j) \cdot x_3^*(k)}_{21.1} > \underbrace{\sum_{j=1}^{m_2}\sum_{k=1}^{m_3} u_1^{123}(\hat{i},j,k) \cdot x_2^*(j) \cdot x_3^*(k)}_{21.2} \quad \Longrightarrow \quad x_1^*(\hat{i}) = 0 \tag{21}$$

We observe that

$$\begin{aligned}
(21.1) \quad &= \quad \sum_{j=1}^{m_2}\sum_{k=1}^{m_3} \gamma_1^{12}(i,j,k) \cdot u_1^{12}(i,j) \cdot \gamma_1^{13}(i,j,k) \cdot u_1^{13}(i,k) \cdot x_2^*(j) \cdot x_3^*(k) \\
&= \quad const_1^i \cdot \sum_{j=1}^{m_2}\sum_{k=1}^{m_3} u_1^{12}(i,j) \cdot u_1^{13}(i,k) \cdot x_2^*(j) \cdot x_3^*(k) \\
&= \quad const_1^i \cdot (19.1) \cdot (20.1)
\end{aligned}$$

Similarly, we have $(21.2) = const_1^i \cdot (19.2) \cdot (20.2)$. Note that, we can safely assume that the payoffs are positive, so if (19) and (20) hold, then (21) must hold, and it holds for *all* Nash equilibria of the composite game G^{123}. Similarly for players 2 and 3. Thus, the claim follows. ◻

n-player strict multiplicative composition – soundness

Lemma 6. *Strict multiplicative composition is strongly sound. I.e. Given any microscopic set $\widetilde{G} = \{ G^{pp'} : 1 \le p \ne p' \le n \}$ w.r.t. the player set $P = \{ 1, \cdots, n \}$, and given any joinable set of component Nash equilibria (i.e. for each $1 \le p \ne p' \le n$ we have $(x_p^*, x_{p'}^*)$ is a Nash equilibrium of the component game $G^{pp'}$), then $(x_1^*, \cdots, x_n^*)$ is a Nash equilibrium for the strict multiplicatively composed game G^P.*

Proof. Without loss of generality, we look at player 1, and all other players are similar. We denote s_p as the index of player p's pure strategies. Suppose (x_1^*, x_p^*) $(2 \leq p \leq n)$ is a Nash equilibrium (mixed) strategies for G^{1p}, *i.e.* for each $1 \leq s_1 \neq \hat{s}_1 \leq m_1$

$$\underbrace{\sum_{s_p=1}^{m_p} u_1^{1p}(s_1, s_p) \cdot x_p^*(s_p)}_{22.1} > \underbrace{\sum_{s_p=1}^{m_p} u_1^{1p}(\hat{s}_1, s_p) \cdot x_p^*(s_p)}_{22.2} \quad \Longrightarrow \quad x_1^*(\hat{s}_1) = 0 \tag{22}$$

We want to show $(x_1^*, \cdots, x_n^*)$ is a Nash equilibrium for the strict-multiplicatively composed game $G^{1\cdots n}$, *i.e.* for each $1 \leq s_1 \neq \hat{s}_1 \leq m_1$

$$\underbrace{\sum_{s_{-1} \in S_{-1}} \left[u_1^{1\cdots n}(s_{-1}; s_1) \cdot \prod_{p=2}^{n} x_p^*(s_p) \right]}_{23.1} > \underbrace{\sum_{s_{-1} \in S_{-1}} \left[u_1^{1\cdots n}(s_{-1}; \hat{s}_1) \cdot \prod_{p=2}^{n} x_p^*(s_p) \right]}_{23.2} \quad \Longrightarrow \quad x_1^*(\hat{s}_1) = 0 \tag{23}$$

We observe that

$$\begin{aligned}
(23.1) \quad &= \quad \sum_{s_{-1} \in S_{-1}} \left(\left[\prod_{p=2}^{n} \gamma_1^{1p}(s_{-1}; s_1) \cdot u_1^{1p}(s_1, s_p) \right] \cdot \left[\prod_{p=2}^{n} x_p^*(s_p) \right] \right) \\
&= \quad const_1^{s_1} \cdot \prod_{p=2}^{n} \sum_{s_p=1}^{m_p} u_1^{1p}(s_1, s_p) \cdot x_p^*(s_p) \\
&= \quad const_1^{s_1} \cdot \prod_{p=2}^{n} \cdot (22.1)
\end{aligned}$$

Similarly, we have

$$(23.2) \quad = \quad const_1^{s_1} \cdot \prod_{p=2}^{n} \cdot (22.2)$$

Note that, we can safely assume that the payoffs are positive, so if (22) hold, then (23) must hold, and it holds for *all* Nash equilibria of the composite game G^P. Similarly for other players. Thus, the strict multiplicative composition is strongly sound in preserving Nash equilibrium. ☐

Strict multiplicative composition – non-completeness

Lemma 7. *Strict multiplicative composition is not complete.*

Proof. We construct a strict multiplicative decomposition of a 3-player game as a counter-example: each player has two strategies $\{1, 2\}$,

$$\begin{aligned}
T_1^{123}(1,1,1) \quad &= \quad \gamma_1^{12}(1,1,1) \cdot T_1^{12}(1,1) \cdot \gamma_1^{13}(1,1,1) \cdot T_1^{13}(1,1) \\
T_1^{123}(1,1,2) \quad &= \quad \gamma_1^{12}(1,1,2) \cdot T_1^{12}(1,1) \cdot \gamma_1^{13}(1,1,2) \cdot T_1^{13}(1,2) \\
T_1^{123}(1,2,1) \quad &= \quad \gamma_1^{12}(1,2,1) \cdot T_1^{12}(1,2) \cdot \gamma_1^{13}(1,2,1) \cdot T_1^{13}(1,1) \\
T_1^{123}(1,2,2) \quad &= \quad \gamma_1^{12}(1,2,2) \cdot T_1^{12}(1,2) \cdot \gamma_1^{13}(1,2,2) \cdot T_1^{13}(1,2)
\end{aligned}$$

and given that

$$\begin{aligned}
\gamma_1^{12}(1,1,1) \cdot \gamma_1^{13}(1,1,1) \quad &= \quad \gamma_1^{12}(1,1,2) \cdot \gamma_1^{13}(1,1,2) \\
&= \quad \gamma_1^{12}(1,2,1) \cdot \gamma_1^{13}(1,2,1) \\
&= \quad \gamma_1^{12}(1,2,2) \cdot \gamma_1^{13}(1,2,2) \\
&= \quad const_1^1
\end{aligned}$$

let all of the Ts be positive, then we have

$$T_1^{123}(1,2,2) \quad = \quad \frac{T_1^{123}(1,1,2) \cdot T_1^{123}(1,2,1)}{T_1^{123}(1,1,1)}$$

Thus, $T_1^{123}(1,2,2)$ cannot be an arbitrary real number. In other words, strict multiplicative composition is not complete. □

12 Proof of Theorem 5: 3-player game incorporation

Proof. By rearranging the terms in (6), we have the necessary and sufficient condition for Nash equilibrium:

$$u_2^{12}(x_1^*; \pi_{2,j}) + u_2^{23}(x_3^*; \pi_{2,j}) > u_2^{12}(x_1^*; \pi_{2,\hat{j}}) + u_2^{23}(x_3^*; \pi_{2,\hat{j}}) \quad \Longrightarrow \quad x_2^*(\hat{j}) = 0$$

Thus the claim follows. □

13 n-player game incorporation

Definition 6. Given any G^P ($P = \{1, \cdots, n\}$) and a microscopic set $\widetilde{G} = \{G^{pp'} : 1 \leq p \neq p' \leq n\}$ whose composite $\bowtie_+ \widetilde{G}$ is G^P, and suppose for each $2 \leq q \leq n-1$, (x_1^*, x_q^*) is a Nash equilibrium of G^{1q}, and (x_q^*, x_n^*) is a Nash equilibrium of G^{qn}, let $\widetilde{G}' = \{G^{1q} : 2 \leq q \leq n-1\} \cup \{G^{nq} : 2 \leq q \leq n-1\}$, we say that $\bowtie_+ \widetilde{G}'$ ***incorporates*** G^{1n} ***for player*** 1, written $\bowtie_+ \widetilde{G}' \lhd_1 G^{1n}$, just if for each $1 \leq s_1 \neq \hat{s}_1 \leq m_1$

$$\sum_{p=2}^{n} (u_1^{1p}(x_p^*; \pi_{1,s_1}) - u_1^{1p}(x_p^*; \pi_{1,\hat{s}_1})) > 0 \quad \Longrightarrow \quad x_1^*(\hat{s}_1) = 0 \tag{24}$$

Similarly for player n. We say $\bowtie_+ \widetilde{G}'$ ***incorporates*** G^{1n}, written $\bowtie_+ \widetilde{G}' \lhd G^{1n}$, if $\bowtie_+ \widetilde{G}' \lhd_1 G^{1n}$ and $\bowtie_+ \widetilde{G}' \lhd_n G^{1n}$. □

Theorem 6. *Given any G^P ($P = \{1, \cdots, n\}$) and a microscopic set $\widetilde{G} = \{G^{pp'} : 1 \leq p \neq p' \leq n\}$ whose composite $\bowtie_+ \widetilde{G}$ is G^P, and suppose for each $2 \leq q \leq n-1$, (x_1^*, x_q^*) is a Nash equilibrium of G^{1q}, and (x_q^*, x_n^*) is a Nash equilibrium of G^{qn}, then $(x_1^*, \cdots, x_n^*)$ is a Nash equilibrium of $G^{1\cdots n}$ if $\bowtie_+ \widetilde{G}' \lhd G^{1n}$, where $\widetilde{G}' = \{G^{1q} : 2 \leq q \leq n-1\} \cup \{G^{nq} : 2 \leq q \leq n-1\}$.* □

Proof. By rearranging the terms in (24), we have the necessary and sufficient condition for Nash equilibrium:

$$\sum_{p=2}^{n} u_1^{1p}(x_p^*; \pi_{1,s_1}) > \sum_{p=2}^{n} u_1^{1p}(x_p^*; \pi_{1,\hat{s}_1}) \quad \Longrightarrow \quad x_1^*(\hat{s}_1) = 0$$

Thus the claim follows. □

The result shows that the payoff matrix of a component game can be hidden or even perturbed; so long as the game incorporation ratios remain unchanged, Nash equilibrium is preserved by the composite game.