

Deceit and Nondefeasible Knowledge: The Case of *Dubitatio*

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Abstract. The current trend in knowledge revision in the Dynamic Epistemic Logic tradition focuses on the *addition* of new knowledge, rather than the possibility of *losing* knowledge. Yet there are natural situations where there is a need to be able to model the retraction of a proposition from a knowledge base. One situation where this is systematically required is the variant of the medieval theory of *obligationes* known as *dubitatio*, where one of the agents in the dialogue is obliged to hold the initial proposition as doubtful. In this paper, we use *dubitatio* as a motivation for studying deceitful agents, and we discuss various ways that an agent can move from a model where φ is known to one where φ is not known.

Keywords: Dynamic Epistemic Logic, *dubitatio*, hard knowledge, higher-order knowledge revision, *obligationes*, soft knowledge, uncertainty

1 Games of ‘Let’s Pretend’

Consider the following scenarios:

Scenario 1

Ophelia’s birthday is tomorrow, and Rebecca has planned a surprise party for the evening. Ophelia comes into the room and asks Rebecca, “What should we do tomorrow night?” Rebecca doesn’t want to spoil the surprise by saying anything that might imply to Ophelia that she knows there are festivities planned, but she also doesn’t want to commit to anything that would interfere with the party.

In this scenario, Rebecca is playing a game of “let’s pretend”. Even though she in fact does know that there will be a party tomorrow evening, she is going to act *as if she does not know this*, and, furthermore, act in such a way that she will *never ‘learn’ this*, no matter what other propositions Ophelia questions her about. Such a scenario seems to be *prima facie* reasonable, and the question naturally arises how Rebecca should reason about the propositions that Ophelia

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questions her about in such circumstances. When she pretends that she does not know whether there is a party tomorrow night, she must revise her knowledge and beliefs accordingly so that she doesn't end up in a situation where Ophelia thinks either that she knows that there is a party or that there is not. That is, Rebecca she must *give up* her knowledge of p and continue to reason and interact with Ophelia without it.

Scenario 2

Ophelia and Rebecca are arguing about a proposition, call it p . The question is whether Rebecca knows whether p is true or not (she in fact does). Knowing that she won't make any further headway with Ophelia unless she does so, Rebecca tells Ophelia, "Just to make you happy, even though I know p and I know that I know p , I'm going to pretend that I don't—that I am uncertain about whether or not I know p . I hope you're satisfied now."

Such a scenario could arise when Ophelia and Rebecca disagree about what counts as 'knowledge' (and Rebecca is in fact right). Then it would be perfectly natural for them to "table" knowledge propositions, to set them aside and not make any inference which would decide the matter one way or another, until they have agreed on a definition.

Both of these scenarios, the propositional one and the higher-order one, have in common an element of guile: one (or more) agent(s) acting in a way which is not in accord with their actual knowledge. Both of the scenarios, furthermore, are relatively natural: We can imagine them arising in fairly ordinary circumstances. Given the plausibility of both, what is surprising is that currently favored logical models for belief and knowledge revision, both static and dynamic, cannot account for either of them: Rebecca's actions, though in principle reasonable, violate a deeply-entrenched principle about knowledge, namely that once you gain it, you cannot lose it. Whereas beliefs can be revised in such a way that a certain belief may go in and out of an agent's belief set, knowledge, as it is standardly defined in the paradigm of Dynamic Epistemic Logic, is robust and monotone: Once it is secured, it cannot be lost.¹ The wide-spread feeling is that if it can be lost, then it is not knowledge. This prejudice is rampant [4, 3, 16, 17], and it is shared with a focus on changes in beliefs and knowledge that are veridical.² And yet, it still seems natural to ask questions about how an agent should reason if she wants to pretend that she does not have some knowledge that she in fact has.

In this paper, we consider a situation where scenarios like 1 and 2 arise: the species of *obligatio* (a medieval game-like type of disputation between two agents) called *dubitatio*. In [14] we introduced a logical framework for modeling different

¹ Except in special cases, e.g., where the agent becomes cognitively disabled, but even these cases are not formally considered.

² E.g., the focus on truthful public announcements in Public Announcement Logic (PAL). Cf. [4, p. 133], [3, p. 47].

types of *obligationes* which is based on Multi-Agent Dynamic Epistemic Logic. The framework provides an advance on previous formalizations of *obligationes* because it allows us to model the agents' knowledge and the moves of the game explicitly and dynamically. We showed that the framework could be used to model various different types of *positio*, the most common version of *obligatio*, in a natural fashion. In the present paper, we extend this framework to the less-commonly discussed—but in our opinion much more interesting—variant *dubitatio*. Our goal in this paper is to give a formal model to make precise why exactly *dubitatio* is interesting and how it is not just a trivial variant of the more common types of *obligationes*, thus giving foundation to the argument in [15] that it is a misrepresentation of the intricacies surrounding doubt, as opposed to concession and rejection, to dismiss *dubitatio* as uninteresting in its own right. In order to give a DEL-style formalization of *dubitatio*, we need to be able to deal with the actions illustrated in the two scenarios: We must give a principled way to allow an agent to shelve parts of her knowledge-base and show how she should continue reasoning. In doing so, we hope to not only shed light on an interesting medieval disputational theory but also make a more general contribution to the literature on knowledge revision, both first- and higher-order.

2 Uncertainty and Nondefeasible Knowledge

We consider a fragment of Dynamic Epistemic Logic (DEL [16]) which combines epistemic operators with programmes (specifically, test programmes for formulas). We call the fragment Deceitful Announcement Logic (DAL) in analogy to Public Announcement Logic (PAL). For a set Φ_0 of propositional letters and A of agents, the set Φ_{DAL} of well-formed formulas of DAL is defined as follows:³

$\varphi := p \in \Phi_0 \mid \neg\varphi \mid \varphi \vee \psi \mid K_a\varphi : a \in A \mid [\varphi?]\psi : \varphi$ does not contain $[\alpha?]$ for any α

$\varphi \wedge \psi$ and $\varphi \rightarrow \psi$ are defined in the usual fashion. $K_a\varphi$ is read ‘agent a knows that φ ’. The dual operator k_a , defined as $\neg K_a\neg$, is read ‘it is consistent with a 's knowledge that...’, representing epistemic possibility. We define the uncertainty operator $U_a\varphi$ as $\neg K_a\varphi \wedge \neg K_a\neg\varphi$, that is, neither φ nor its negation is known by a . $[\varphi?]$ is a programme, interpreted as a test.

Elements of Φ_{DAL} are interpreted in the standard fashion, on Kripke structures for epistemic logic $\mathfrak{M} = \langle W, w^*, \{\sim_a : a \in A\}, V \rangle$, where W is a set, with $w^* \in W$ a designated point (representing the actual world), $\{\sim_a : a \in A\}$ is a collection of equivalence relations on W , one for each member of A , the set of agents, and $V : \Phi_0 \rightarrow 2^W$ is a valuation function associating atomic propositions with subsets of W . For $p \in \Phi_0$, if $w \in V(p)$, we say that ‘ p is true at w ’, and write $\mathfrak{M}, w \models p$. The relation $w \sim_a w'$ is interpreted as ‘ w and w' are epistemically equivalent for agent a ’ or ‘agent a cannot distinguish between w and w' ’.

³ We do not include the operator C_G for common knowledge among a group $G \subseteq A$ of agents as it doesn't play any role in our formalization, but it would be straightforward to add.

The semantics for the propositional connectives are as expected. The truth conditions for tests are given in terms of model reduction:

Definition 1. *The reduction of a model \mathfrak{M} by a formula φ is defined as follows:*

$$\mathfrak{M} \upharpoonright \varphi = \langle W^{\mathfrak{M},\varphi}, \{\sim_a^{\mathfrak{M},\varphi} : a \in A\}, V^{\mathfrak{M},\varphi} \rangle,$$

with $W^{\mathfrak{M},\varphi} := \{w \in W : \mathfrak{M}, w \models \varphi\}$, and the relations and valuation functions are just restrictions of the originals. For a set of ordered propositions Γ_n , let $\mathfrak{M} \upharpoonright \Gamma_n = \mathfrak{M} \upharpoonright \gamma_0 \upharpoonright \dots \upharpoonright \gamma_n$, that is, $\mathfrak{M} \upharpoonright \Gamma_n$ is the result of the sequential restriction of \mathfrak{M} by the elements of Γ_n .

Then we define the truth conditions for the modal operators as follows:

$$\begin{aligned} \mathfrak{M}, w \models K_a \varphi &\text{ iff } \forall w' (\langle w, w' \rangle \in \sim_a \text{ implies } \mathfrak{M}, w' \models \varphi) \\ \mathfrak{M}, w \models [\varphi] \psi &\text{ iff } \forall v \in \mathfrak{M} \upharpoonright \varphi, v \models \psi \end{aligned}$$

Validity is defined as expected. As these models are ordinary epistemic models, they satisfy logical omniscience for all agents. Note that in contrast with PAL, we do not require that the ‘announcements’ (tests) made be truthful; the fact that they are possibly false is what makes this logic (or at least the agents therein) deceitful. This is the first point where our fragment differs from standard DEL systems.

Two types of knowledge are identified and discussed in the epistemic logic literature, *hard* knowledge and *soft* knowledge. Hard knowledge, or *Aumann* knowledge (cf. [1]), is the type of knowledge formalized above with the K operator. We also call it S5-knowledge, because the underlying Kripke structures are S5 structures, that is, the accessibility relation is an equivalence class, so that \sim_a partitions W . Hard knowledge is generally considered to be irrevocable; once it has been gained it cannot be lost [4, p. 126], [17, pp. 230, 235]. We disagree that the assumption of irrevocability is reasonable. Revoking knowledge, or tabling it, is essentially what Rebecca is doing in the scenarios we began with. When we no longer consider truthful, helpful, nondeceitful agents, there arise many scenarios where the revocation of knowledge may be required. Therefore it is interesting to ask how this revocation can (or should) be modeled.

Given an arbitrary model \mathfrak{M} there are in principle many possible ways of changing it to ensure that the truth value of a specific proposition is not known.⁴ These many possible ways can be classified into the following four types of ways that we can modify \mathfrak{M} to \mathfrak{M}' such that $\mathfrak{M}, w \models K_a \varphi$ but $\mathfrak{M}', w' \models \neg K_a \varphi$: (1) Change V ; (2) change W (by making it grow); (3) change \sim_a ; and (4) change K_a . Model change procedures which change V are classified in the literature on belief and knowledge change as *update* processes, and those which change the model by changing \sim_a (possibly in addition to changing V) are called *revision* processes [16, p. 43]. Update, which changes a model’s valuation, is relevant in dynamic situations, where what is being modeled is the change of an agent’s

⁴ It is precisely this multiplicity of ways that makes giving a computer implementation of *dubitatio* so difficult (cf. [7]).

knowledge which results from a changing world. Since the scenarios we outlined above (and in the example of *dubitatio* that we discuss later in the paper; see Rule 3) take place in the context of a static, unchanging world, we are not interested in model-change operations which result in the *update* of knowledge, so we will discount the first option. The quotes above, reflecting the dominant paradigm in belief and knowledge revision, show that the second and third options are generally not considered acceptable.

This brings us to the last possibility, namely, changing our definition of knowledge to something weaker than S5-knowledge. This is the second type of knowledge mentioned above, soft knowledge, also called “safe belief”. Soft knowledge is non-introspective, and is persistent under revision with *true* information. This type of knowledge was introduced by Stalnaker in [12, 13] as a formalization of Lehrer’s defeasibility analysis of knowledge [2, p. 13]. But just because there has been no proposal to date which takes seriously the (apparent) loss of S5 knowledge, it does not follow that any situation where knowledge is revoked or shelved must involve a weaker form of knowledge. We argue that it is not only possible to, but in some cases also sensible and reasonable to, allow agents who have S5-knowledge to act as if they do not, and we propose below a model-change procedure which allows this in the propositional case.⁵

3 A study in uncertainty

In the opening section we gave two examples of agents acting in a deceitful or manipulative fashion, in that they are announcing statements that are not in accord with their knowledge. This can be either done with the knowledge and agreement of the other agent(s) involved in the interaction, or only privately known by the agent making the announcements. In this section we give an example of the former type, namely a particular type of medieval disputation *de obligationibus*.

The first treatises on *obligationes* appeared at the beginning of the 13th century; medieval logicians continued to write treatises on *obligationes* or include sections on *obligationes* in their textbooks until the 16th century.⁶ Briefly, an *obligatio* is a dialogue between two agents, the Opponent O and the Respondent R. Before the dialogue begins, the Opponent and the Respondent may optionally agree on a set of ‘common knowledge’ (the *casus*); then, the Opponent puts forward a proposition. If the Respondent agrees to the proposition, the dialogue begins. In each round the Opponent puts forward a proposition, and the Respondent responds with one of three (in some texts, four) actions: accept/concede, deny/reject, doubt(, and draw distinctions). Which response is correct depends on the rules that the Respondent is obligated (hence the name) to follow. The different types of *obligatio* are determined by the rules for the Respondent. In the most common type of *obligatio*, *positio*, the Respondent’s primary obligation is to hold the initial proposition (called the *positum*) as true, even if in fact it

⁵ We defer consideration of the (more interesting) higher-order case to future work.

⁶ For more information on the history of *obligationes*, see [11].

may be false. In *depositio*, the Respondent’s primary obligation is to hold the initial proposition (called the *depositum*) as false; in *dubitatio*, his obligation is to hold the *dubitatum* as doubtful, that is, he should not accept any proposition whose truth would entail the truth of the *dubitatum*, nor should he reject any proposition whose falsity would entail the falsity of the *dubitatum*.

In what follows, we are only interested in the language where $A = \{R, O\}$ (Respondent and Opponent, respectively). All obligational disputations share four features in common: The set of propositions put forward by O, the set of responses to these propositions by R, and a rule governing the correctness of those responses which generates an “optimal” sequence of actions of R. The set Act of actions of R is given in the language of DAL (see Def. 9 for the formal definition). Formally,

Definition 2. (*Obligatio*). An obligatio is a quadruple $O = \langle \Theta, R, \Gamma, \Gamma^R \rangle$ where

- Θ is a sequence of propositions, such that $\theta_0 \in \Theta$ is the obligatum and $\theta_n \in \Theta$ is the proposition put forward by O at round n .
- $R : \Theta \times \mathbb{N} \rightarrow \text{Act}$ is a function determining R’s response to each element of Θ . We write $R(\theta_n)$ for $R(\theta, n)$ to simplify notation.
- Γ is a sequence of actions, formed by R’s responses to each element of Θ .
- Γ^R is a sequence of actions, formed by the correct response of R to each element of Θ , as given by R .

We will often abuse notation and identify Γ and Γ^R , which are sequences of tests, with the sequences of tested formulas (that is, if $\Gamma_1 = \langle [\theta_0?] \top, [-\theta_1?] \top \rangle$, we will sometimes identify Γ_1 with $\langle \theta_0, -\theta_1 \rangle$).

Dubitatio has hitherto received very little attention from modern scholars [15, §1], and even when it is mentioned, it is generally dismissed; Spade explains the lack of attention to *dubitatio* in modern times by saying that it is a “trivial variation on positing [*positio*]” [11]. Later medieval authors, such as William of Ockham [9], Richard Brinkley [6], and Paul of Venice [10] in the 14th and early 15th centuries, do not treat *dubitatio* at any great length. Paul of Venice, in particular, agrees with Spade’s assessment of *dubitatio*; Paul says that “every *dubitatio* or *petitio* is a *positio*, as was shown at the beginning” [10, p. 327]. In early treatises, however, *dubitatio* is considered in much more detail. For example, in *Obligaciones Parisienses* [8], an anonymous text which can be tentatively dated to the early part of the 13th century, *dubitatio* is treated in equal length to *depositio* (both types meriting about 6–7 pages of discussion). Furthermore, it is discussed after *positio*, and before *depositio*, another indication that it was considered an important type of *obligaciones* in its own right. More interestingly, this text has an argument for the positive answer to the question “whether *dubitatio* is a species of *obligatio*”; were *dubitatio* seen as a just a variant of *positio*, such a question would not only be straightforward to answer, its asking would seem almost insulting.

One of the most clear and complete set of rules for *dubitatio* is given by Nicholas of Paris, whose *Obligaciones* can be dated to around 1240, making it one of the earlier witnesses of the genre. Nicholas gives seven rules for *dubitatio*. Four concern how R is to respond to propositions put forward during the dialogue:

Rule (N-2). *Just as in positio a positum put forward in the form of the positum, and everything convertible to it in the time of positing is to be conceded and its opposite and things convertible with it is to be denied and just as in depositio a depositum put forward in the form of the depositum, with its convertibles, must be denied and its opposite with things convertible with it must be conceded; so in dubitatio for a dubitatum put forward in the form of dubitatum and for its convertibles and moreover for the opposite of the dubitatum with its convertibles must be answered “prove!” [5, p. 223].⁷*

Rule (N-3). *For everything antecedent to the dubitatum the response must be “false” or “prove!” and never “true” [5, p. 224].*

Rule (N-4). *For everything consequent to the dubitatum it is possible to reply “it is true” or “prove!” and never “it is false” [5, p. 224]*

Rule (N-5). *For everything irrelevant to the dubitatum the response must be according to its quality [5, p. 225].*

Impertinent or irrelevant propositions are defined as follows:

Definition 3. *What is neither antecedent to nor consequent from nor convertible with the dubitatum nor is the opposite of or convertible with the opposite of it is called impertinent, which is easily understood [5, p. 225].*

Another is a meta-rule about how the relevance of propositions and the knowledge of the participants is to be evaluated:

Rule (N-7). *All the responses must be directed to the same instant [5, p. 227].*

This rule recognizes that while contingent facts about the world may change during the disputation (for example, R may at some point be sitting and at another point be perambulating), world-knowledge is taken to be static, and the truth value of propositions is to be evaluated with respect to how the world was when the disputation started.

One rule tells R when he should reject a *dubitatum* altogether, and not agree to start a disputation:

Rule (N-1). *Just as in false position it is impossible to put down “a falsehood is posited” nor in deposition “a falsehood is deposited”, by the same reason it is impossible to doubt “a falsehood is doubted” [5, p. 223].*

The final rule is the most peculiar:

Rule (N-6). *The questioning exercise cannot be terminated [5, p. 226].*

Unfortunately, Nicholas does not say anything more about this rule, and for the purposes of this paper we must set discussion of it aside.

⁷ Throughout, all translations of Nicholas’s Latin into English are my own; I am omitting quotation of the Latin for reasons of space.

4 Modeling uncertainty

In the formal framework we introduced above, and which is more fully developed in [14], an *obligatio* is evaluated against epistemic models. We begin with an epistemic model, representing the world and the agents' knowledge of the world at the beginning of the disputation, and with each step of the disputation we revise the model to a new one. The question is how the series of model revisions should be done in the case of *dubitatio*. Since model updates result in the growth of knowledge, even if the dubitatum θ_0 is uncertain in \mathfrak{M} , for an arbitrary Γ there is no guarantee that θ_0 will still be uncertain in $\mathfrak{M} \uparrow \Gamma$. Furthermore, we need to worry about the case where θ_0 is not in fact uncertain in \mathfrak{M} to begin with, that is, $\mathfrak{M}, w^* \models \neg U_R \theta_0$. Supposing that $\mathfrak{M}, w^* \models \neg U_R \theta_0$, in order to model *dubitatio* we must define an operation \uparrow on models such that $\mathfrak{M} \uparrow \theta_0, w^* \models U_R \theta_0$.

Our basic principle is one of minimality: when a set of knowledge needs to be revised, only the smallest changes necessary to obtain the desired result should be made. In the context of introducing uncertainty into a model where there potentially is currently knowledge, this means that we should make as few changes to W or \sim_a as possible.

Definition 4. *The truth set of $\varphi \in \Phi_{\text{DAL}}$ in a model \mathfrak{M} is $\llbracket \varphi \rrbracket^{\mathfrak{M}} = \{w : \mathfrak{M}, w \models \varphi\}$.*

Definition 5. *Given a model $\mathfrak{M} = \langle W, w^*, \{\sim_a : a \in A\}, V \rangle$, define $\mathfrak{M}^{U_a \varphi}$ as follows:*

- *If $\mathfrak{M}, w^* \models U_a \varphi$, then $\mathfrak{M}^{U_a \varphi} = \mathfrak{M}$. (If φ is already in doubt for the agent, then the present model is already sufficient.)*
- *Otherwise, $\mathfrak{M}^{U_a \varphi} = \langle W', w'^*, \{\sim'_a : a \in A\}, V' \rangle$ where:*
 - *$W' = W \cup \{v\}$, where $v \notin W$.*
 - *$w'^* = w^*$.*
 - *\sim'_a is the reflexive, transitive, and symmetric closure of $\sim_a \cup \langle w'^*, v \rangle$.*
 - *For $A \ni a' \neq a$, $\sim'_{a'} = \sim_{a'}$.*
 - *V' is any valuation function minimally extending V in such a way that*
 - * *If $\mathfrak{M}, W^* \models K_a \varphi$, then $v \in \llbracket \neg \varphi \rrbracket$.*
 - * *If $\mathfrak{M}, W^* \models K_a \neg \varphi$, then $v \in \llbracket \varphi \rrbracket$.*

That is, if φ is not already uncertain for a , then either it or its negation is known. Note that this definition is only successful when φ is propositional and not equivalent to either \top or \perp . In this case, its truth value depends only on the world in which it is being evaluated, and since both it and its negation are satisfiable, it is always possible to construct a valuation where its negation is false:

Fact 6. $\mathfrak{M}^{U_a \varphi} \models U_a \varphi$ iff φ is neither a contradiction or a tautology.

Definition 7. *The revision of a model \mathfrak{M} by a formula φ , written $\mathfrak{M} \uparrow \varphi$, is defined as follows:*

$$\begin{array}{ll} \text{If } \varphi \text{ is of the form } U_a \psi: & \mathfrak{M} \uparrow \psi = \mathfrak{M}^{U_a \psi} \\ \text{Otherwise:} & \mathfrak{M} \uparrow \varphi = \mathfrak{M} \uparrow \varphi \end{array}$$

with $W^{\mathfrak{M},\varphi} := \{w \in W : \mathfrak{M}, w \models \varphi\}$, and the relations and valuation functions are just restrictions of the originals. $\mathfrak{M} \upharpoonright \Gamma_n$ is defined in analogy to $\mathfrak{M} \upharpoonright \Gamma_n$.

This approach is much simpler than the possible-worlds analysis of belief contraction given in [16, §3.3]. This analysis introduces a *system of spheres* and a corresponding *entrenchment relation*. For an arbitrary epistemic model \mathfrak{M} , let $B(w)_a = \{\varphi : \mathfrak{M}, w \models B_a\varphi\}$ and $R(w)_a = \{v \in W : v \models B(w)_a\}$ (that is, given an agent's beliefs at w , $R(w)_a$ is the set of worlds he considers possible in w).

Definition 8. A system of spheres for $R(w)$ in a model \mathfrak{M} with domain W is the set $\text{Sphe}(R(w))$ of spheres $Sp \supseteq R(w)$ that are linearly ordered by \supseteq , and contains the maximum W and the minimum $R(w)$, and, for all φ , if some $S \in \text{Sphe}(R(w))$ intersects with $\llbracket \varphi \rrbracket$, then there is a \subseteq -smallest $S' \in \text{Sphe}(R(w))$ that intersects with $\llbracket \varphi \rrbracket$ [16, p. 56].

The system of spheres is intended to model an entrenchment relation of an agent; the smaller spheres are those which the agent is more committed to, that he believes are *more possible* than the ones which are further out. Then, contraction of a belief set by φ can be defined as moving to the first $Sp \supseteq R(w)$ such that $Sp \cap \llbracket \neg\varphi \rrbracket \neq \emptyset$, and revision of φ by doing the same plus then throwing away all of the worlds where φ is true (for the formal definitions, see [16, Defs. 3.30, 3.31]). As with our construction of $\mathfrak{M}^{U_a\varphi}$, this will only work where there is a sphere Sp such that $Sp \cap \llbracket \neg\varphi \rrbracket \neq \emptyset$; sometimes, revision or contraction of the belief set is not possible.

The reason why such heavy machinery is required in order to model belief revision and contraction is that an agent may have sophisticated or complex preferences concerning which sets of possible worlds are more or less likely. It could be that if an agent believes φ , the worlds in the smallest sphere where φ is false are radically different from the current world—that is, the most preferred worlds (as indicated by the closest sphere) may not necessarily be the ones with the smallest amount of changes. In contrast, when an agent knows that φ is true, all worlds where φ is false are equally likely, because they have 0 probability, since none of them can be the actual world. Thus, the parsimonious option is to pick a world which is most similar to the current world and add it to the model, and the most similar world will be one which agrees on all the propositional valuations except where needed to make φ false.

We can now formalize the rules of *dubitatio* as given by Nicholas of Paris (NoP). First, redefine the truth conditions for $[\varphi]\psi$ by replacing \upharpoonright with \upharpoonright . Then,

Definition 9 (Actions of R). Let φ_n be a proposition put forward by O. The possible actions of R (designated Act) are:

$$\begin{aligned} \text{concede:} & \quad [\varphi_n?] \top \\ \text{deny:} & \quad [\neg\varphi_n?] \top \\ \text{doubt:} & \quad [U_R\varphi?] \top \end{aligned}$$

Definition 10. A formula θ_n is relevant (cf. Def. 3) in a *dubitatio* with *dubitatum* θ_0 evaluated on model \mathfrak{M} if any of the following holds:

- $\mathfrak{M} \models \theta_0 \rightarrow \theta_n$.
- $\mathfrak{M} \models \theta_n \rightarrow \theta_0$.
- $\mathfrak{M} \models \theta_0 \rightarrow \neg\theta_n$.
- $\mathfrak{M} \models \neg\theta_n \rightarrow \theta_0$.

It is irrelevant otherwise.

Let $O^{\text{NoP}} = \langle \Theta, R^{\text{NoP}}, \Gamma, \Gamma^{R^{\text{NoP}}} \rangle$, where Θ is any sequence of propositions, Γ is a sequence of actions, and R^{NoP} and $\Gamma^{R^{\text{NoP}}}$ are defined as follows:

Definition 11. For a model \mathfrak{M} and formula $\theta_n \in \Theta$, R^{NoP} is defined as follows:

$$\begin{array}{ll}
\text{If } \mathfrak{M} \uparrow \Gamma_{n-1} \models \theta_n \rightarrow \theta_0: & R^{\text{NoP}}(\theta_n) = \mathbf{doubt} \text{ or } \mathbf{deny} \\
\text{If } \mathfrak{M} \uparrow \Gamma_{n-1} \models \theta_0 \rightarrow \theta_n: & R^{\text{NoP}}(\theta_n) = \mathbf{doubt} \text{ or } \mathbf{concede} \\
\text{If } \mathfrak{M} \uparrow \Gamma_{n-1} \models \neg\theta_n \rightarrow \theta_0: & R^{\text{NoP}}(\theta_n) = \mathbf{doubt} \text{ or } \mathbf{concede} \\
\text{If } \mathfrak{M} \uparrow \Gamma_{n-1} \models \theta_0 \rightarrow \neg\theta_n: & R^{\text{NoP}}(\theta_n) = \mathbf{doubt} \text{ or } \mathbf{deny} \\
\text{Otherwise:} & \\
\text{If } \mathfrak{M}, w^* \models K_R \theta_n: & R^{\text{NoP}}(\theta_n) = \mathbf{concede} \\
\text{If } \mathfrak{M}, w^* \models K_R \neg\theta_n: & R^{\text{NoP}}(\theta_n) = \mathbf{deny} \\
\text{If } \mathfrak{M}, w^* \models \neg(K_R \theta_n \vee K_R \neg\theta_n): & R^{\text{NoP}}(\theta_n) = \mathbf{doubt}
\end{array}$$

Rule 3 can be recovered from the first four cases, by noting that if $\mathfrak{M} \uparrow \Gamma_{n-1} \models \theta_n \leftrightarrow \theta_0$ or $\mathfrak{M} \uparrow \Gamma_{n-1} \models \neg\theta_n \leftrightarrow \theta_0$, then the only option for R is to doubt θ_n . Notice that unlike rules for *positio* (cf. [14]), rules for *dubitatio* are in general not deterministic. Thus $\Gamma^{R^{\text{NoP}}}$ is not unique; it is any sequence that satisfying by the following recursive constraints:

$$\begin{array}{l}
\Gamma_0^{R^{\text{NoP}}} = \langle R^{\text{NoP}}(\theta_0) \rangle \\
\Gamma_n^{R^{\text{NoP}}} = \langle \gamma_0, \dots, \gamma_{n-1}, R^{\text{NoP}}(\theta_n) \rangle
\end{array}$$

We show that, playing by these rules, Respondent can *win* the disputation, that is, there is never any case where she will be forced either to concede or to deny the *dubitatum*.

First, two lemmas:

Lemma 1. $\theta_0 \neq \top \neq \perp$.

Proof. This is a consequence of Rule 3. Here, it is important to note that when *falsum* (which we translated as “falsehood” above) is used unmodified, it generally means “necessary falsehood”; when “contingent falsehood” is intended, then the phrase is *falsum contingens* is used. So we know that $\theta_0 \neq \perp$. Now, suppose $\theta_0 = \top$. Then, by Rule 3, if there is some $\theta_n = \neg\theta_0$, the correct response is **doubt**; but one cannot doubt a contradiction, as Rule 3 says.

Thus, if the *dubitatum* θ_0 is a contradiction or a tautology, Respondent should reject the possibility of debating on this proposition altogether.

Lemma 2. For propositional φ and ψ (that is, φ and ψ not containing any modal operators or tests), the following holds:

- (1) If $\mathfrak{M} \models \varphi \rightarrow \psi$, $\mathfrak{M} \not\models \psi \rightarrow \varphi$, and $\mathfrak{M} \models \neg K_a \varphi$, then $\mathfrak{M} \uparrow \psi \models \neg K_a \varphi$.
- (2) If $\mathfrak{M} \models \varphi \rightarrow \psi$, $\mathfrak{M} \not\models \psi \rightarrow \varphi$, and $\mathfrak{M} \models \neg K_a \neg\varphi$, then $\mathfrak{M} \uparrow \psi \models \neg K_a \neg\varphi$.

Proof. (1) If φ implies ψ but is not equivalent to it, then there is some world w in \mathfrak{M} such that $\mathfrak{M}, w \models \psi \wedge \neg\varphi$. Since $\mathfrak{M} \models \neg K_a\varphi$, $\mathfrak{M}, w \models \neg K_a\varphi$ as well. Now, suppose that $\mathfrak{M} \upharpoonright \psi, w \models K_a\varphi$. Then for all $v \sim_a w$, $\mathfrak{M} \upharpoonright \psi, v \models \varphi$. But then since $w \sim_a w$, it follows that $\mathfrak{M} \upharpoonright \psi, w \models \varphi$. But this contradicts $\mathfrak{M}, w \models \psi \wedge \neg\varphi$, since $\mathfrak{M}, w \models \psi \wedge \neg\varphi$ implies $\mathfrak{M} \upharpoonright \psi, w \models \neg\varphi$, since φ is propositional.

(2) Since $\mathfrak{M} \models \neg K_a\neg\varphi$, there is v in \mathfrak{M} such that $\mathfrak{M}, v \models \varphi$. Since $\varphi \rightarrow \psi$, $\mathfrak{M}, v \models \psi$, and hence v is in $\mathfrak{M} \upharpoonright \psi$. Thus, it cannot be the case that $\mathfrak{M} \upharpoonright \psi \models K_a\neg\varphi$, since there is at least one world where φ is true.

Theorem 1. *For arbitrary O^{NoP} and epistemic model \mathfrak{M} , $\mathfrak{M} \upharpoonright \Gamma^{R^{\text{NoP}}} \models U_R\theta_0$.*

Proof. The proof is a straightforward one by induction on the length of $\Gamma^{R^{\text{NoP}}}$.

Basis case: $\Theta_0 = \langle \theta_0 \rangle$. By Lemma 1, $\theta_0 \neq \top \neq \perp$. Since $\mathfrak{M} \models \theta_0 \leftrightarrow \theta_0$, $R^{\text{NoP}}(\theta_0) = \mathbf{doubt}$. Thus, $\Gamma_0^{R^{\text{NoP}}} = \langle [U_R\varphi?] \top \rangle$. $\mathfrak{M} \upharpoonright \Gamma_0 = \mathfrak{M}^{U_R\theta_0}$, and by Fact 6, $\mathfrak{M}^{U_R\theta_0} \models U_R\theta_0$, as required.

Inductive step: Assume that $\mathfrak{M} \upharpoonright \Gamma_{n-1} \models U_R\theta_0$. There are three possibilities for $R^{\text{NoP}}(\theta_n)$:

1. $R^{\text{NoP}}(\theta_n) = \mathbf{concede}$: Then $\mathfrak{M} \upharpoonright \Gamma_n = \mathfrak{M} \upharpoonright \Gamma_{n-1} \upharpoonright \theta_n$, and either (a) $\mathfrak{M} \upharpoonright \Gamma_{n-1} \models \theta_0 \rightarrow \theta_n$, (b) $\mathfrak{M} \upharpoonright \Gamma_{n-1} \models \neg\theta_n \rightarrow \theta_0$, or (c) neither and $\mathfrak{M}, w^* \models K_R\theta_n$. If (a), since $R^{\text{NoP}}(\theta_n) = \mathbf{concede}$, we know that $\theta_n \leftrightarrow \theta_0$. Thus, we can apply both cases of Lemma 2 to conclude that $\mathfrak{M} \upharpoonright \Gamma_n \models U_R\theta_0$. The case for (b) follows an argument analogous to the argument for Lemma 2. For (c), if θ_n is irrelevant, it follows that there are worlds w, v such that $\mathfrak{M} \upharpoonright \Gamma_{n-1}, w \models \theta_n \wedge \theta_0$ and $\mathfrak{M} \upharpoonright \Gamma_{n-1}, v \models \theta_n \wedge \neg\theta_0$. Both w, v will be in $\mathfrak{M} \upharpoonright \Gamma_n$, and hence the global uncertainty of θ_0 will be preserved.
2. $R^{\text{NoP}}(\theta_n) = \mathbf{deny}$: This case is analogous to the previous.
3. $R^{\text{NoP}}(\theta_n) = \mathbf{doubt}$: Then $\mathfrak{M} \upharpoonright \Gamma_n = \mathfrak{M} \upharpoonright \Gamma_{n-1}^{U_a\theta_n}$, and either (a) $\mathfrak{M} \upharpoonright \Gamma_{n-1} \models \theta_n \leftrightarrow \theta_0$ or (b) θ_n and θ_0 are independent in $\mathfrak{M} \upharpoonright \Gamma_{n-1}$ and $\mathfrak{M}, w^* \models \neg(K_R\theta_n \vee K_R\neg\theta_n)$. In both cases, $\mathfrak{M} \upharpoonright \Gamma_n = \mathfrak{M} \upharpoonright \Gamma_{n-1}$, and by the inductive hypothesis, $\mathfrak{M} \upharpoonright \Gamma_{n-1} \models U_R\theta_0$.

5 Conclusion

In this paper we have presented some natural scenarios where a notion of retraction or revocation of knowledge seems plausible, as well as a formal situation where such a notion is in fact required. We have given a definition of model revision which moves from a model where φ is known to one where it is uncertain for an agent, that is, neither it nor its negation is known. This model revision process works any time that φ is neither a tautology nor a contradiction. We applied this new revision process to the medieval theory of *dubitatio*, and proved that in the propositional case, if the Respondent follows the rules of Nicholas of Paris, the Opponent can never force him into admitting certainty about the dubitatum θ_0 . Thus, Respondent can always win a *dubitatio* disputation if he plays according to Nicholas of Paris's rules.

We have restricted our attention to the propositional case (that is, where the propositions θ_n put forward by \mathbf{O} do not contain any modal operators). It is quite interesting to ask how this could be extended to the higher-order case, where we want to be able to introduce uncertainty about knowledge claims. As with higher-order knowledge statements in general, such an extension will be non-trivial. We hope to explore this case in further research.

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