A linear time logic for reasoning about political consensus

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Abstract. The aim of the paper is to present a sound and strongly complete temporal logic that can model reasoning about reaching political consensus.

1 Introduction

The present paper introduces a formalism that can reason about reaching consensus within a coalition and forming a government. We use a model of political consensus introduced by Eklund et. al. in [2]. Since that model is dynamic (parties can compromise in order to reach consensus), we developed propositional logic with temporal features. For temporal logics we refer the reader to [3].

We proved that our logic is sound and strongly complete (every consistent set of formulas has a model). From the technical point of view, we modified some of our previously developed completion techniques presented in [1, 4, 5].

The rest of the paper is organized as follows. In Section 2 we present some basic concepts concerning consensus reaching within a coalition. In Section 3 we introduce the propositional linear-time logic and define the class of models. Section 4 contains an axiomatization, that is proved to be sound and complete in Section 5.

2 Preliminaries

In this section we recall some basic notions and definitions from [2], that are necessary for introducing our logic.

The coalition $S = \{1, \dots, n\}$ is the set of parties with sufficient number of seats in order to form a government (we assume that a government consists of a coalition and a policy that is acceptable to all the parties in the coalition). We denote by G_S^* the set of governments acceptable to the coalition S (since our aim is to formalize consensus reaching within a coalition, we identify G_S^* with the set of policies acceptable to S).

Each party evaluates governments with respect to the set of criteria $Crit$. Formally, for $i \in S$ we define the function $\alpha_i : Crit \to [0,1]$, such that

$$\sum_{c \in Crit} \alpha_i(c) = 1.$$

Furthermore, each party evaluates each government with respect to the set of criteria:

- $f_i : Crit \times G_S^* \to [0,1]$,
- $\sum_{g \in G_S^*} f_i(c,g) = 1$, $c \in Crit$.

Next, for $i \in S$ we define the functions $\beta_i : G_S^* \to [0,1]$, the evaluation of the governments by parties:

$$\beta_j(g_k) = \alpha_j(c_1) f_j(c_1, g_k) + \ldots + \alpha_j(c_p) f_j(c_p, g_k).$$

It is easy to see that $\sum_{g \in G_S^*} \beta_i(g) = 1$.

A distance function $d : \{\beta_i | i \in S\}^2 \to [0,1]$ is defined by

$$d(\beta_i, \beta_j) = \sqrt{\frac{1}{|G_S^*|} \sum_{g \in G_S^*} (\beta_i(g) - \beta_j(g))^2}.$$

A generalized consensus degree δ_S is defined by $\delta_S = 1 - d_S$, where

$$d_S = max\{d(\beta_i, \beta_j) | i, j \in S\}.$$

We say that coalition S reaches consensus if $\delta_S \geq \tilde{\delta}$, where $\tilde{\delta} \in (0,1)$ is required consensus degree.

If the consensus is not reached, the chairman (a person which is indifferent between all the parties) decides which party will be advised to change the preferences. If that party does not agree to adjust it's preferences, the chairman proposes another change to the same party, or chooses another party.

We suppose that parties are sufficiently willing to compromise, and the procedure finishes in reaching consensus. After S reaches consensus, the chairman chooses the government g such that

$$\sum_{j \in S} \frac{k(j)}{k(1) + \ldots + k(n)} \sum_{c \in Crit} \alpha_j(w_i)(c) F_j(w_i)(c, g)$$

is maximal, where $k(i) > 0$ is the relative power of the party i(if there are two such governments, the chairman chooses one of them).

Although the consensus is always reached in a finite time, we define a time model as infinite sequence, since the time required for reaching consensus is not bounded by a fixed positive integer. Nevertheless, we do not allow changes in a model after the consensus is reached.

3 Syntax and semantics

Let $S = \{1, \ldots, n\}$, $G_S^* = \{g_1, \ldots, g_m\}$, $Crit = \{c_1, \ldots, c_p\}$ and $g_0 \notin G_S^*$.

Definition 1. The set $Term$ of terms is defined recursively as follows:

– $Term(0) = \{\alpha_i(c) | c \in Crit, i \in S\} \cup \{k(i) | i \in S\} \cup \{\beta_i(g) | g \in G_S^*, i \in S\} \cup \{f_i(c,g) | c \in Crit, g \in G_S^*, i \in S\} \cup \{\underline{s} \mid s \in \mathbb{Q}\}$,
– $Term(n+1) = Term(n) \cup \{(\mathtt{f} + \mathtt{g}), (\mathtt{f} \cdot \mathtt{g}), (-\mathtt{f}) | \mathtt{f}, \mathtt{g} \in Term(n)\}$,
– $Term = \bigcup_{n=0}^{\infty} Term(n)$. □

We will denote terms by $\mathtt{f}, \mathtt{g}$ and $\mathtt{h}$, possibly with indices. Furthermore, we introduce the usual abbreviations : $\mathtt{f} + \mathtt{g}$ is $(\mathtt{f} + \mathtt{g})$, $\mathtt{f} + \mathtt{g} + \mathtt{h}$ is $((\mathtt{f} + \mathtt{g}) + \mathtt{h})$, $\mathtt{f} \cdot \mathtt{g}$ is $(\mathtt{f} \cdot \mathtt{g})$ and $(\mathtt{f} \cdot \mathtt{g}) \cdot \mathtt{h} = \mathtt{f} \cdot (\mathtt{g} \cdot \mathtt{h})$, $-\mathtt{f}$ is $(-\mathtt{f})$, $\mathtt{f} - \mathtt{g}$ is $(\mathtt{f} + (-\mathtt{g}))$ and so on.

Definition 2. The set For of formulas is defined recursively as the smallest set that satisfies the following conditions:

– expressions of the form $\mathtt{f} \geqslant \underline{s}$, $\mathtt{f} \in Term$ and $Gov(g)$, $g \in G_S^* \cup \{g_0\}$ are formulas,
– if ϕ and ψ are formulas, then $\neg\phi$, $\phi \wedge \psi$, $\bigcirc\phi$ and $\phi U \psi$ are formulas. □

The intended meaning of $Gov(g_i)$ is that a government g_i is chosen (if $i = 0$ then no government is formed). As in the propositional case, $\neg$ and $\wedge$ are the primitive connectives, while all of the other connectives are introduced in the usual way. We introduce temporal operators F (sometimes) and G (always):

– $F\phi$ is $\top U \phi$.
– $G\phi$ is $\neg F \neg \phi$.

To simplify notation, we define the following abbreviations:

– $\mathtt{f} \leqslant \underline{s}$ is $-\mathtt{f} \geqslant \underline{s}$, $\mathtt{f} > \underline{s}$ is $\neg(\mathtt{f} \leqslant \underline{s})$. Similarly are defined $\mathtt{f} < \underline{s}$, $\mathtt{f} = \underline{s}$ and $\mathtt{f} \neq \underline{s}$.
– $\mathtt{f} \geqslant \mathtt{g}$ is $\mathtt{f} - \mathtt{g} \geqslant 0$. Similarly are defined $\mathtt{f} \leqslant \mathtt{g}$, $\mathtt{f} > \mathtt{g}$, $\mathtt{f} < \mathtt{g}$, $\mathtt{f} = \mathtt{g}$ and $\mathtt{f} \neq \mathtt{g}$.

Suppose that $\tilde{\delta} \in (0,1) \cap \mathbb{Q}$ is the *required consensus degree.*

A *model* $\mathcal{M}$ is any tuple $\langle S, K, G_S^*, Crit, \{\hat{\alpha}_i | i \in S\}, W, \{F_i | i \in S\}, \tilde{g}, g_{fin} \rangle$ such that:

1. $S = \{1, \ldots, n\}$ is a nonempty set of *parties*,
2. K is a function $K : S \to \mathbb{R}^+$,
3. $G_S^* = \{g_1, \ldots, g_m\}$ is a nonempty set of *governments*,
4. $Crit = \{c_1, \ldots, c_p\}$ is a nonempty set of *criteria*,
5. for every $i \in S$, $\hat{\alpha}_i$ is a function defined by:
   – $\hat{\alpha}_i : Crit \to [0,1]$,
   – $\hat{\alpha}_i(c_1) + \ldots + \hat{\alpha}_i(c_p) = 1$,

6. $W = w_0, w_1, \ldots$ is an ω-sequence of *time instants*,
7. for every $i \in S$, F_i associates to every $w_j \in W$, a function which evaluates the governments with respect to all the criteria, i.e.,
   – $F_i(w_j) : Crit \times G^*_S \rightarrow [0, 1]$,
   – $F_i(w_j)(c_k, g_1) + \ldots + F_i(w_j)(c_k, g_m) = 1$, for all $c_k \in Crit$
8. $\tilde{g}$ associates a government $\tilde{g}(w_i) \in G^*_S \cup \{g_0\}$ to a time instant w_i, such that
   – $\tilde{g}(w_i) = g_0$ iff

$$max_{j,k} \sum_{g \in G^*_S} \Big(\sum_{c \in Crit} \alpha_j(w_i)(c)F_j(w_i)(c, g) - \sum_{c \in Crit} \alpha_k(w_i)(c)F_k(w_i)(c, g)\Big)^2 > n(1-\tilde{\delta}),$$

   – otherwise, $\tilde{g}(w_i) = g \in G^*_S$ such that the number

$$\sum_{j \in S} \frac{K(j)}{K(1) + \ldots + K(n)} \sum_{c \in Crit} \alpha_j(w_i)(c)F_j(w_i)(c, g)$$

   is maximal,
9. there exists i such that $\tilde{g}(w_i) \neq g_0$,
10. the sequence

$$a_i = max_{j,k} \sum_{g \in G^*_S} \Big(\sum_{c \in Crit} \alpha_j(w_i)(c)F_j(w_i)(c, g) - \sum_{c \in Crit} \alpha_k(w_i)(c)F_k(w_i)(c, g)\Big)^2$$

    is decreasing (so the consensus degree never decrease),
11. if $\tilde{g}(w_i) \neq g_0$, then $F_j(w_k) \equiv F_j(w_i)$ and $\tilde{g}(w_k) = \tilde{g}(w_i)$, for all j and all $k > i$,
12. for each i there is at most one j such that $F_j(w_i) \not\equiv F_j(w_{i+1})$,
13. g_{fin} is the unique element from G^*_S such that $g_{fin} = \tilde{g}(w_i)$, for some i (it represents the chosen government).

Let $\mathcal{M}$ be any model and $i \in \omega$. For a term f define the *interpretation* $f^{\mathcal{M},w_i}$ as follows:

– $\underline{s}^{\mathcal{M},w_i} = s$, $s \in \mathbb{Q}$,
– $k(i)^{\mathcal{M},w_i} = K(i)$, $i \in S$,
– $\alpha_j(c)^{\mathcal{M},w_i} = \hat{\alpha}_j(c)$, $c \in Crit$,
– $f_j(c, g)^{\mathcal{M},w_i} = F_j(w_i)(c, g)$, $c \in Crit$, $g \in G^*_S$,
– $\beta_j(g)^{\mathcal{M},w_i} = \alpha_j(w_i)(c_1)F_j(w_i)(c_1, g) + \ldots + \alpha_j(w_i)(c_p)F_j(w_i)(c_p, g)$, $g \in G^*_S$,
– $(\mathtt{f} + \mathtt{g})^{\mathcal{M},w_i} = \mathtt{f}^{\mathcal{M},w_i} + \mathtt{g}^{\mathcal{M},w_i}$,
– $(\mathtt{f} \cdot \mathtt{g})^{\mathcal{M},w_i} = \mathtt{f}^{\mathcal{M},w_i} \cdot \mathtt{g}^{\mathcal{M},w_i}$,
– $(-\mathtt{f})^{\mathcal{M},w_i} = -(\mathtt{f}^{\mathcal{M},w_i})$.

Definition 3. Let $\mathcal{M} = \langle S, G^*_S, Crit, W, \{\hat{\alpha}_i | i \in S\}, \{F_i | i \in S\}, \tilde{g}\rangle$ be a model and $i \in \omega$. The satisfiability relation $\models$ is inductively defined as follows:

– $\mathcal{M}, w_i \models \mathtt{f} \geqslant \underline{s}$ if $\mathtt{f}^{\mathcal{M},w_i} \geqslant s$,
– $\mathcal{M}, w_i \models Gov(g)$ if $\tilde{g}(w_i) = g$,

- $\mathcal{M}, w_i \models \neg\phi$ if $\mathcal{M}, w_i \not\models \phi$,
- $\mathcal{M}, w_i \models \phi \wedge \psi$ if $\mathcal{M}, w_i \models \phi$ and $\mathcal{M}, w_i \models \psi$,
- $\mathcal{M}, w_i \models \bigcirc\phi$ if $\mathcal{M}, w_{i+1} \models \phi$,
- $\mathcal{M}, w_i \models \phi U \psi$ if there is $j \in \omega$ such that $\mathcal{M}, w_{i+j} \models \psi$,and for every $k \in \omega$ such that $k < j$, $\mathcal{M}, w_{i+k} \models \phi$. $\square$

A set of formulas T is *satisfiable* if there is if there is a model $\mathcal{M}$ and a time instant w_i in $\mathcal{M}$ such that for every formula $\phi \in T$, $\mathcal{M}, w_i \models \phi$. A formula ϕ is *satisfiable* if the set $\{\phi\}$ is satisfiable. A formula is *valid* if it is satisfied in each time instant in each model.

4 Axiomatization

Propositional axioms

A0. $\tau(\phi_1, \ldots, \phi_n)$, where $\tau(p_1, \ldots, p_n) \in For_C$ is any propositional tautology.
A1. $\mathtt{f} = \mathtt{g} \rightarrow (\phi(\ldots, \mathtt{f}, \ldots) \rightarrow \phi(\ldots, \mathtt{g}, \ldots))$

Axioms about consensus

A3. $k(i) > \underline{0}$.
A4. $\alpha_i(g_j) \geqslant \underline{0}$.
A5. $\alpha_i(c_1) + \ldots + \alpha_i(c_p) = \underline{1}$.
A6. $f_i(g_j) \geqslant \underline{0}$.
A7. $f_i(c_k, g_1) + \ldots + f_i(c_k, g_m) = \underline{1}$.
A8. $\beta_j(g_k) = \alpha_j(c_1) f_j(c_1, g_k) + \ldots + \alpha_j(c_p) f_j(c_p, g_k)$.
A9. $Gov(g_i) \rightarrow \neg Gov(g_j)$, $i \neq j$.
A10. $\bigvee_{i,j \in S} (\sum_{g \in G_S^*} (\beta_i(g) - \beta_j(g))^2 > \underline{n(1 - \tilde{\delta})}) \leftrightarrow Gov(g_0)$.
A11. $(\neg Gov(g_0) \wedge \bigwedge_{g' \in G_S^*} \sum_{i \in S} k(i)\beta_i(g) \geq \sum_{i \in S} k(i)\beta_i(g')) \rightarrow Gov(g)$, $g \in G_S^*$.

Axioms about commutative ordered rings

A11. $\underline{0} < \underline{1}$.
A12. $\mathtt{f} + \mathtt{g} = \mathtt{g} + \mathtt{f}$.
A13. $(\mathtt{f} + \mathtt{g}) + \mathtt{h} = \mathtt{f} + (\mathtt{g} + \mathtt{h})$.
A14. $\mathtt{f} + \underline{0} = \mathtt{f}$.
A15. $\mathtt{f} - \mathtt{f} = \underline{0}$.
A16. $\mathtt{f} \cdot \mathtt{g} = \mathtt{g} \cdot \mathtt{f}$.
A17. $\mathtt{f} \cdot (\mathtt{g} \cdot \mathtt{h}) = (\mathtt{f} \cdot \mathtt{g}) \cdot \mathtt{h}$.
A18. $\mathtt{f} \cdot \underline{1} = \mathtt{f}$.
A19. $\mathtt{f} \cdot (\mathtt{g} + \mathtt{h}) = (\mathtt{f} \cdot \mathtt{g}) + (\mathtt{f} \cdot \mathtt{h})$.
A20. $\mathtt{f} \geqslant \mathtt{f}$.
A21. $\mathtt{f} \geqslant \mathtt{g} \ \vee \ \mathtt{g} \geqslant \mathtt{f}$.
A22. $(\mathtt{f} \geqslant \mathtt{g} \ \wedge \ \mathtt{g} \geqslant \mathtt{h}) \rightarrow \mathtt{f} \geqslant \mathtt{h}$.
A23. $\mathtt{f} \geqslant \mathtt{g} \ \rightarrow \ \mathtt{f} + \mathtt{h} \geqslant \mathtt{g} + \mathtt{h}$.
A24. $(\mathtt{f} \geqslant \mathtt{g} \ \wedge \ \mathtt{h} > \underline{0}) \ \rightarrow \mathtt{f} \cdot \mathtt{h} \geqslant \mathtt{g} \cdot \mathtt{h}$.
A25. $(\mathtt{f} \geqslant \mathtt{g} \ \wedge \ \mathtt{h} < \underline{0}) \ \rightarrow \mathtt{f} \cdot \mathtt{h} \leqslant \mathtt{g} \cdot \mathtt{h}$.

A26. $\mathtt{f} = \mathtt{g} \rightarrow (\phi(\ldots, \mathtt{f}, \ldots) \rightarrow \phi(\ldots, \mathtt{g}, \ldots))$.

Temporal axioms

A27. $\bigcirc(\phi \rightarrow \psi) \rightarrow (\bigcirc\phi \rightarrow \bigcirc\psi)$.
A28. $\neg \bigcirc \phi \leftrightarrow \bigcirc\neg\phi$.
A29. $\phi U\psi \leftrightarrow \psi \vee (\phi \wedge \bigcirc(\phi U\psi))$.
A30. $\phi U\psi \rightarrow F\psi$.
A31. $F\neg Gov(g_0)$.
A32. $\mathtt{f} \geqslant \underline{r} \leftrightarrow \bigcirc(\mathtt{f} \geqslant \underline{r})$, if f does not contain an occurrence of f_i and β_j.
A33. $(f_i(c,g) \geq \underline{r} \leftrightarrow \bigcirc f_i(c,g) \geq \underline{r}) \vee (f_j(c,g) \geq \underline{r} \leftrightarrow \bigcirc f_j(c,g) \geq \underline{r})$, $i \neq j$.
A34. $\neg Gov(g_0) \rightarrow (\phi \leftrightarrow \bigcirc\phi)$.
A35. $\bigcirc \bigvee_{i,j\in S}(\sum_{g\in G_S^*}(\beta_i(g) - \beta_j(g))^2 \geq \underline{r} \rightarrow \bigvee_{i,j\in S}(\sum_{g\in G_S^*}(\beta_i(g) - \beta_j(g))^2 \geq \underline{r}$.

Inference rules

R1. From ϕ and $\phi \rightarrow \psi$ infer ψ.
R2. From the set of premises $\{\phi \rightarrow \bigcirc^m \mathtt{f} \geqslant \underline{-n^{-1}} \mid n = 1,2,3,\ldots\}$ infer $\phi \rightarrow \bigcirc^m \mathtt{f} \geqslant \underline{0}$ (for any $m \in \omega$).
R3. From ϕ infer $\bigcirc\phi$, if ϕ is a theorem.
R4. From the set of premises $\{\phi \rightarrow \bigcirc^n\psi \mid n = 1,2,3,\ldots\}$ infer $\phi \rightarrow G\psi$.

Let us briefly discuss the axioms and rules listed above. The axioms. Propositional axioms provide syntactical verification of tautology instances and substitution of provably equal terms in formulas. Axioms about consensus describe properties of the evaluation functions and the choice of the government depending on those functions. Axioms about commutative ordered rings formally provide the usual manipulations with terms (commutativity, associativity etc). Temporal axioms can be divided into two parts. The axioms A27–A30 are usual axioms for temporal logics, while the axioms A31–A35 describe changing of the system over time. In particular, A31 states that a government will be chosen in some time instant, A32 states that formulas that only the evaluations of governments by the parties and choosing a government are not time-independent, A33 provides that at most one party may change preferences in a time instance, while A34 provides that after reaching required consensus degree any change is impossible. Finally, A35 states that the consensus degree can not decrease over time. The rules R1 and R3 are Modus Ponens and Necessitation, respectively. The rule R2 intuitively says that if value of a term is arbitrary closed to 0, then it is at least 0, while R4 characterizes the always operator.

Definition 4. A formula ϕ is deducible from a set T of sentences ($T \vdash \phi$) if there is an at most countable sequence of formulas $\phi_0, \phi_1, \ldots, \phi$, such that every ϕ_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule. A formula ϕ is a theorem ($\vdash \phi$) if it is deducible from the empty set. A set T of formulas is consistent if there is at least one formula from For that is not deducible from T, otherwise T is inconsistent.

A consistent set T of sentences is said to be maximally consistent if for every $\phi \in For$, either $\phi \in T$ or $\neg\phi \in T$. A set T is deductively closed if for every $\phi \in For$, if $T \vdash \phi$, then $\phi \in T$. □

Note that the length of inference may be any successor ordinal lesser than the first uncountable ordinal ω_1.

5 Completeness

Using a straightforward induction on the length of the inference, one can easily prove that the above axiomatization is sound with respect to the class of models.

In this section, we will give the proof of Completeness theorem.

Lemma 5 (Deduction theorem). *Suppose that T is an arbitrary set of formulas and that $\phi, \psi \in For$. Then, $T \cup \{\phi\} \vdash \psi$ implies $T \vdash \phi \to \psi$.*

Proof. We can prove the statement using the transfinite induction on the length of the inference. Suppose that $T \cup \{\phi\} \vdash \psi$. If ψ is a theorem,the claim is obviously true.

Note that the form of the inference rules R2 is modified in order to allow proving this theorem. Namely, if ψ is obtained by R2, then $\psi = \theta \to \bigcirc^m \mathtt{f} \geqslant \underline{0}$. Moreover, $T \cup \{\phi\} \vdash \theta \to \bigcirc^m \mathtt{f} \geqslant -n^{-1}$, for all $n \in \omega$, hence $T \vdash \phi \to (\theta \to \bigcirc^m \mathtt{f} \geqslant \underline{-n^{-1}})$ (the induction hypothesis). Consequently, $T \vdash (\phi \wedge \theta) \to \bigcirc^m \mathtt{f} \geqslant \underline{-n^{-1}}$, so, by R2, $T \vdash (\phi \wedge \theta) \to \bigcirc^m \mathtt{f} \geqslant \underline{0}$, or, equivalently, $T \vdash \phi \to (\theta \to \bigcirc^m \mathtt{f} \geqslant \underline{0})$.

The case when ψ is obtained by application of R2, while the cases when we apply inference rule R1 and R3 are standard. □

Theorem 6 (Strong completeness theorem). *Every consistent set T of formulas is satisfiable.*

Proof. Let T be a consistent set of formulas and let $For = \{\phi_i \mid i = 0, 1, 2, 3, \ldots\}$. We define a theory T_0^* as follows:

- $T_0 = T$.
- If $T_i \cup \{\phi_i\}$ is consistent , then $T_{i+1} = T_i \cup \{\phi_i\}$.
- If $T_i \cup \{\phi_i\}$ is inconsistent , then:
  1. If $\phi_i = \psi \to G\theta$, then
  $$T_{i+1} = T_i \cup \{\psi \to \neg \bigcirc^{n_1} \theta\},$$
  where n_1 is a positive integer such that T_{i+1} is consistent.
  2. Otherwise, if $\phi_i = \psi \to \bigcirc^m \mathtt{f} \geqslant 0$, then
  $$T_{i+1} = T_i \cup \{\psi \to \bigcirc^m \mathtt{f} < \underline{-n_2^{-1}}\},$$
  where n_2 is a positive integer such that T_{i+1} is consistent.
  3. Otherwise, $T_{i+1} = T_i$.
- $T_0^* = \bigcup_{n \in \omega} T_n$.

Note that existence of the positive integers n_1 and n_2 (in 1. and 2.) is provided by Lemma 5. Using Lemma 5 and the fact that each T_i is consistent, it is easy to prove that for each ϕ, either $\phi \in T_0^*$ or $\neg\phi \in T_0^*$. Moreover, T_0^* is consistent, since it is deductively closed (if T_0^* is inconsistent, then, by deductive closeness, $\perp \in T_0^*$, so T_i would be inconsistent, for some i). We will prove only that T_0^* is closed under the inference rule R2, i.e., that $A = \{\phi \ \rightarrow \bigcirc^m \mathtt{f} \geqslant \underline{-n^{-1}} | n \in \omega\} \subseteq T_0^*$ implies $\phi \ \rightarrow \bigcirc^m \mathtt{f} \geqslant \underline{0} \in T_0^*$. Suppose that $A \subseteq T_0^*$ and $\phi \ \rightarrow \bigcirc^m \mathtt{f} \geqslant \underline{0} \notin T_0^*$. Since T_0^* is maximal, $\neg(\phi \ \rightarrow \bigcirc^m \mathtt{f} \geqslant \underline{0}) \in T_0^*$, so there is $i \in \omega$ such that $\neg(\phi \ \rightarrow \bigcirc^m \mathtt{f} \geqslant \underline{0}) \in T_i$; it follows that $T_i \vdash \phi$. Since $\phi \ \rightarrow \bigcirc^m \mathtt{f} < \underline{-m^{-1}} \in T_{j+1}$, for some $m \in \omega$, for sufficiently large k we have $T_k \vdash \phi$, $T_k \vdash \bigcirc^m \mathtt{f} < \underline{-m^{-1}}$ and $T_k \vdash \bigcirc^m \mathtt{f} \geqslant \underline{-m^{-1}}$; a contradiction.

Thus, we proved that T_0^* is maximally consistent. Now we define a sequence T_0^*, T_1^*, $T_2^* \ldots$ such that $T_{i+1}^* = \{\phi | \bigcirc \phi \in T_i^*\}$. It follows from the axioms A27–A30 (and the fact that T_0^* is maximally consistent) that T_i^* is maximally consistent for every i.

We define a model $\mathcal{M} = \langle S, K, G_S^*, Crit, \{\hat{\alpha}_i | i \in S\}, W, \{F_i | i \in S\}, \tilde{g}, g_{fin}\rangle$ as follows:

- $K(i) = sup\{r \in \mathbb{Q} | T_0^* \vdash k(i) \geq \underline{r}\}$,
- $\hat{\alpha}_i(c) = sup\{r \in \mathbb{Q} | T_0^* \vdash \alpha_i(c) \geq \underline{r}\}$,
- $F_i(w_j)(c, g) = sup\{r \in \mathbb{Q} | T_j^* \vdash f_i(c, g) \geq \underline{r}\}$,
- $\tilde{g}(w_i) = g$ if $T_i^* \vdash Gov(g)$
- g_{fin} is the unique element from G_S^* such that $T_0^* \vdash FGov(g)$

The proof that $\mathcal{M}$ is a model is straightforward.

For example, $K(i) > 0$ follows from A0.

Let us prove that $\hat{\alpha}_i(c_1) + \ldots + \hat{\alpha}_i(c_p) = 1$ (for a fixed i).

Using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, for every $k \leq p$ we may chose increasing sequence $\underline{a}_0^k < \underline{a}_1^k < \underline{a}_2^k < \cdots$ and decreasing sequence $\overline{a}_0^k > \overline{a}_1^k > \overline{a}_2^k > \cdots$ in $\mathbb{Q}$ such that $\lim \underline{a}_n^k = \lim \overline{a}_n^k = \hat{\alpha}_i(c_k)$. By the definition of $\hat{\alpha}_i$ and completeness of T_0^*, we obtain

$$T_0^* \vdash \alpha_i(c_k) \geqslant \underline{a_n^k} \wedge \alpha_i(c_k) < \underline{\overline{a}_n^k}$$

for all n.

Using axioms about commutative ordered rings, we have

$$T_0^* \vdash \sum_{k \leq p} \alpha_i(c_k) \geqslant \sum_{k \leq p} \underline{a_n^k} \wedge \sum_{k \leq p} \alpha_i(c_k) < \sum_{k \leq p} \underline{\overline{a}_n^k}$$

for all n. Using A4 we obtain that

$$T_0^* \vdash \underline{1} \geqslant \sum_{k \leq p} \underline{a_n^k} \wedge \underline{1} < \sum_{k \leq p} \underline{\overline{a}_n^k}$$

for all n.

Now $\hat{\alpha}_i(c_1) + \ldots + \hat{\alpha}_i(c_p) = 1$ follows from

$$\lim \sum_{k \leq p} \underline{a}_n^k = \lim \sum_{k \leq p} \overline{a}_n^k = \sum_{k \leq p} \hat{\alpha}_i(c_k).$$

We can use same technique to prove that

$$F_i(w_j)(c_k, g_1) + \ldots + F_i(w_j)(c_k, g_m) = 1$$

holds for all j.

The other properties of the model (items 8.–12. from the definition of a model) follows from the axioms A8–A10, A31–A33 and A35.

It remains to prove that $\mathcal{M}, w_i \models \phi$ iff $\phi \in T_i^*$ holds for every formula ϕ. That can be proved using the induction on the complexity of formulas. We will prove the most interesting case, when $\phi = \mathtt{f} \geq \underline{r}$. If $\mathtt{f} \geqslant \underline{r} \in T_i^*$. We can use the axioms for ordered commutative rings to prove that that $\vdash \mathtt{f} = \underline{r_1}\mathtt{g}_1 + \ldots + \underline{r_{n_\mathtt{f}}}\mathtt{g}_{n_\mathtt{f}}$,, for some $n_\mathtt{f} \in \omega$, where each $\mathtt{g}_i$ is of the form $\mathtt{g}_i = \mathtt{h}_1 \cdots \mathtt{h}_{n_i}$, for some $\mathtt{h}_j \in Term(0) \setminus \mathbb{Q}$. Since $\vdash \mathtt{g}_i \geqslant \underline{0}$ and $\mathtt{h}_i^{\mathcal{M}, w_i} = \sup\{r \in [0,1] \cap \mathbb{Q} \mid T_i^* \vdash \mathtt{h}_i \geqslant r\}$, using the technique with increasing and decreasing sequences of rational numbers as above, we can prove that $\mathtt{g}_j^{\mathcal{M}, w_i} = \sup\{r \in \mathbb{Q} \mid T_i^* \vdash \mathtt{g}_j \geqslant \underline{r}\}$.

Once again, using the monotone sequences of rational numbers we can show that $\mathtt{f}^{\mathcal{M}, w_i} = \sup\{r \in \mathbb{Q} \mid T_i^* \vdash \mathtt{f} \geqslant \underline{r}\}$, so $\mathtt{f}^{\mathcal{M}, w_i} \geqslant r$.

For the other direction, we can use the contraposition, the fact that T_i^* is maximally consistent and the proven direction.

For the cases when the formula is of the form $\bigcirc\phi$ or of the form $\phi U \psi$, we refer the reader to [4].

References

1. D. Doder, Z. Marković, Z. Ognjanović, A. Perović, M. Rašković, A probabilistic temporal logic that can model reasoning about evidence, Lecture Notes in Computer Science 5956, pp 9–24, 2010.
2. P. Eklund, A. Rusinowska, H. de Swart, A consensus model of political decision-making, Annals of Operations Research 158(1), pp 5–20, 2008.
3. D. Gabbay, I. Hodkinson, M. Reynolds. Temporal logic. Mathematical Foundations and Computational Aspects (volume 1). Clarendon Press, 1994.
4. Z. Ognjanović. Discrete linear-time probabilistic logics: completeness, decidability and complexity. J. Log. Comput. 16(2), pp 257–285, 2006.
5. A. Perović, Z. Ognjanović, M. Rašković, Z. Marković. A probabilistic logic with polynomial weight formulas. FoIKS 2008, pp 239–252.