Impossibility results for infinite-electorate abstract aggregation rules*

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Abstract. It is well known that the literature on judgment aggregation inherits the impossibility results from the aggregation of preferences that it generalises. This is due to the fact that the typical judgment aggregation problem induces an ultrafilter on the the set of individuals, as was shown in a model theoretic framework by Herzberg and Eckert (2009), generalising the Kirman-Sondermann correspondence and extending the methodology of Lauwers and Van Liedekerke (1995). In the finite case, dictatorship then immediately follows from the principality of an ultrafilter on a finite set. This is not the case for an infinite set of individuals, where there exist free ultrafilters, as Fishburn already stressed in 1970. The main problem associated with free ultrafilters in the literature on aggregation problems is however, the arbitrariness of their selection combined with the limited anonymity they guarantee (which already led Kirman and Sondermann (1972) to speak about invisible dictators). Following another line of Lauwers and Van Liedekerke's (1995) seminal paper, this note explores another source of impossibility results for free ultrafilters: The domain of an ultraproduct over a free ultrafilter extends the individual factor domains, such that the preservation of the truth value of some sentences by the aggregate model — if this is as usual to be restricted to the original domain — may again require the exclusion of free ultrafilters, leading to dictatorship once again.

Key words: Arrow-type preference aggregation; judgment aggregation; model theory; first-order predicate logic; filter; ultrafilter; reduced product; ultraproduct; existential quantifier

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1 Introduction

In the last decades, the literature on social choice theory has seen important generalisations of the classical Arrovian problem of preference aggregation, starting with isolated contributions on abstract and algebraic aggregation theory by Wilson [14] resp. by Rubinstein and Fishburn [13] and culminating in the new field of judgment aggregation (for a survey see List and Puppe [12]). An essential feature of these generalisations is the extension of the problem of aggregation from the aggregation of preferences to the aggregation of arbitrary information. It thus seems natural to exploit the potential of model theory which, broadly speaking, studies the relation between abstract structures and statements about them (for an introduction to model theory see Bell and Slomson [2]) and to analyse the problem of judgment aggregation as the problem of aggregating the models that satisfy these judgments. This approach is justified by the fact that one of the major tools of model theory, namely the ultraproduct construction can be shown to be equivalent to the construction of an aggregation rule satisfying properties in the spirit of the conditions of Arrow's impossibility theorem, an equivalence which is based on the role of ultrafilters in both cases. Thus a generalisation of the Kirman-Sondermann [10] correspondence between Arrovian aggregation rules and ultrafilters on the set of individuals was obtained (Herzberg and Eckert [7]). For the case of a finite set of individuals, this equivalence immediately allows to derive a dictatorship result, as ultrafilters on finite sets are necessary principal, whence the ultrafilter on a finite set of individuals always is the set of all supersets of a singleton — the dictator.

Whilst this dictatorship result does not carry over to the case of an infinite set of individuals (where free ultrafilters exist), it is well known since Kirman and Sondermann's [10] identification of "invisible dictators" that free ultrafilters only guarantee a limited amount of anonymity (as was also shown by Lauwers and Van Liedekerke [11] in their model theoretic framework and by Dietrich and Mongin [5] in the framework of judgment aggregation). On the other hand, the selection of one of the numerous free ultrafilters entails some striking inherent arbitrariness as was also pointed out by Lauwers and Van Liedekerke [11]. Perhaps even more interestingly, the latter also have suggested another source of impossibility results, viz. the preservation of non-universal formulae (e.g. formulae which describe the existence of a best alternative or continuity of preferences), leading to dictatorship results once again.

In this short note, we explore this suggestion by Lauwers and Van Liedekerke [11] further: In a framework of abstract aggregation theory (which also allows for the analysis of propositional and modal propositional judgment aggregation), we prove a theorem about the general impossibility of non-dictatorial Arrovian aggregators which preserve certain non-universal formulae.

2 A model-theoretic framework for abstract aggregation theory

In this short note, we shall work within the framework of a previous paper (Herzberg and Eckert [7]), in which Lauwers' and Van Liedekerke's [11] model-theoretic approach to preference aggregation (with a recent correction by Herzberg *et al.* [8]) is carried over to more abstract aggregation problems.

Let A be an arbitrary set. Let $\mathfrak{L}$ be a language consisting of at most countably many predicate symbols $\dot{P}_n$, $n \in \mathbf{N}$, and constant symbols $\dot{a}$ for all elements a of A. The arity of $\dot{P}_n$ will be denoted $\delta(n)$, for all $n \in \mathbf{N}$. (Following common practice in mathematical logic, we use dots to distinguish symbols of the formal object language from the symbols of the meta language.)

Let T be a consistent set of universal (i.e. Π_1) sentences in $\mathfrak{L}$.[4] (In the case of preference aggregation, for example, A would be the set of alternatives, there would be just one binary predicate symbol, and T would consist of the weak order axioms.)

The relational structure $\mathfrak{B} = \langle B, \{P_n \ : \ n \in \mathbf{N}\}\rangle$ with $A \subseteq B$ is called a *realisation of* $\mathfrak{L}$ *with domain* B or an $\mathfrak{L}$*-structure* if and only if the arities of the relations P_n correspond to the arities of the predicate symbols $\dot{P}_n$, that is if $P_n \subseteq B^{\delta(n)}$ for each n. The interpretation of the constant symbols does not need to be specified, but will be fixed uniformly for all $\mathfrak{L}$-structures: For each $\mathfrak{L}$-structure $\mathfrak{B}$, the interpretation of the constant symbol $\dot{a}$ is, for every $a \in A$, just $\dot{a}^{\mathfrak{B}} = a$. In other words, in this article, all $\mathfrak{L}$-structures are understood to have a domain $\supseteq A$ and to interpret the constant symbols canonically (i.e. $\dot{a}$ is always interpreted by a, for all $a \in A$).

An $\mathfrak{L}$-structure $\mathfrak{B}$ is a *model* of the theory T if $\mathfrak{B} \models \varphi$ for all $\varphi \in T$, i.e. if all sentences of the theory hold true in $\mathfrak{B}$ (with the usual Tarski definition of truth[5]).

Let $\mathfrak{B} = \langle B, \{P_n \ : \ n \in \mathbf{N}\}\rangle$ be an $\mathfrak{L}$-structure with domain B. (Note that this entails $A \subseteq B$ by our convention.) According to standard model-theoretic termininology (cf. e.g. Bell and Slomson [2, p. 73]), the *restriction* of $\mathfrak{B}$ to A is the $\mathfrak{L}$-structure $\left\langle A, \{P_n \cap A^{\delta(n)} \ : \ n \in \mathbf{N}\}\right\rangle$ and will be denoted by $\mathrm{res}_A \mathfrak{B}$. (In other words, the restriction of $\mathfrak{B}$ to A is the $\mathfrak{L}$-structure that is obtained by restricting the interpretations of the relation symbol to the domain $B \subseteq A$.)

Suppose now that $\mathfrak{B} = \langle B, \{P_n \ : \ n \in \mathbf{N}\}\rangle$ is a relational structure with $P_n \subseteq B^{\delta(n)}$ for each n and such that there exists an injective map $i : A \to B$. Then, the *restriction of* $\mathfrak{B}$ *to* A *under* i is the $\mathfrak{L}$-structure $\left\langle A, \{i^{-1}\left[P_n \cap i[A]^{\delta(n)}\right] \ : \ n \in \mathbf{N}\}\right\rangle$ and will be denoted by $\mathrm{res}_{i,A}\mathfrak{B}$. If B is the reduced product of A with respect to some filter D and $i : A \to B, \quad a \mapsto [(a)]_D$,

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[4] A sentence is *universal* if it (in its prenex normal form) has the form $(\dot{\forall}\dot{v}_{k_1})\cdots(\dot{\forall}\dot{v}_{k_m})\psi$ for some formula ψ that does not contain any quantifiers.

[5] For instance, if $\mathfrak{B} = \langle B, \{P_n \ : \ n \in \mathbf{N}\}\rangle$ is an $\mathfrak{L}$-structure, then for all $a_1, \ldots, a_{\delta(n)} \in A$, one has

$$\mathfrak{B} \models \dot{P}_n(\dot{a}_1, \ldots, \dot{a}_{\delta(n)}) \Leftrightarrow \langle a_1, \ldots, a_{\delta(n)}\rangle \in P_n.$$

is the canonical embedding, then we will drop the subscript i and simply write $\mathrm{res}_A\mathfrak{B}$ instead of $\mathrm{res}_{i,A}\mathfrak{B}$.[6]

Let Ω be the collection of models of T with domain A.

Let I be a (finite or infinite) set. Elements of I will be called *individuals*, elements of Ω^I will be called *profiles*.

An *aggregator* is a map $f : \mathrm{dom}(f) \to \mathrm{ran}(f)$ whose domain $\mathrm{dom}(f)$ is a subset of Ω^I and whose range $\mathrm{ran}(f)$ is a subset of Ω.

As Herzberg and Eckert [7] have pointed out, this framework is sufficiently general to cover the cases of preference aggregation, propositional judgment aggregation, and modal aggregation.

Generalising the Kirman-Sondermann [10] correspondence between Arrovian social welfare functions and ultrafilters of *decisive coalitions*[7] on the set of individuals, Herzberg and Eckert [7] — following a seminal paper by Lauwers and Van Liedekerke [11] as well as recent work by Dietrich and Mongin [5] — have shown that given certain rationality axioms, inspired by Arrow [1], on f and some assumptions on the expressivity of $\mathfrak{L}$, every aggregator is in fact given by a restricted reduced product construction with respect to the *filter of decisive coalitions*. Under additional assumptions, this filter will be an ultrafilter.

Hence, in this note we assume that there is some filter $\mathcal{D}_f$ on I such that for all $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$,

$$f(\underline{\mathfrak{A}}) = \mathrm{res}_A \prod_{i\in I} \mathfrak{A}_i/\mathcal{D}_f .$$

[6] One could also define the restriction of $\mathfrak{B}$ to A as follows: Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathfrak{L}$-structures where the domain A of $\mathfrak{A}$ is a subset of the domain B. If the inclusion mapping i is an elementary embedding, then $\mathfrak{A}$ is the restriction of $\mathfrak{B}$ to A and will be denoted $\mathrm{res}_A\mathfrak{B}$. This alternative definition is more general since it can also be used where $\mathfrak{L}$-structures are allowed to have different, non-canonical interpretations for the constant symbols $\dot{a}$, $a \in A$ (which in our framework is excluded by definition).

[7] In our framework, a subset $S \subseteq I$ of individuals is a *decisive* coalition if there exists some $\mathfrak{L}$-sentence ψ such that both $f(\underline{\mathfrak{A}}) \models \psi$ and

$$S = \{i \in I \ : \ \mathfrak{A}_i \models \psi\} .$$

If f satisfies some rationality assumptions inspired by Arrow [1], one can show that the set of decisive coalitions forms a *filter*, i.e. a collection of non-empty subsets of I which is closed under finite intersections and supersets, and under additional conditions even an *ultrafilter*, i.e. a maximal filter (cf. Herzberg and Eckert [7], generalising similar findings by Kirman and Sondermann [10], Lauwers and Van Liedekerke [11], Dietrich and Mongin [5]). Note that under these conditions on f, it even makes no difference if one replaces the "$S =$" in the above definition of a decisive coalition by "$S \subseteq$" and "there exists some ψ" by "for all ψ".

From the ultrafilter property of the set of decisive coalitions, one can immediately deduce Arrow's theorem by noting that every ultrafilter on a finite set is *principal*, i.e. its intersection equals a singleton (the element of this singleton being the dictator if the ultrafilter is a set of decisive coalitions). Non-principal ultrafilters are called *free*.

Observe that the restriction to A is important since it is a necessary condition (for f to be an aggregator) that the aggregate model $f(\underline{\mathfrak{A}})$ belongs to Ω and thus must have A as its domain. Moreover, if $\mathfrak{D}_f$ is an ultrafilter, then, by application of Łos's theorem, for every $\mathfrak{L}$-sentence ψ,

$$\prod_{i \in I} \mathfrak{A}_i / \mathfrak{D}_f \models \psi \Leftrightarrow \{i \in I : \mathfrak{A}_i \models \psi\} \in \mathfrak{D}_f,$$

which guarantees that $\prod_{i \in I} \mathfrak{A}_i / \mathfrak{D}_f \models T$ and hence $f(\underline{\mathfrak{A}}) = \mathrm{res}_A \prod_{i \in I} \mathfrak{A}_i / \mathfrak{D}_f \models T$ since T consists only of universal sentences. Therefore, if f is given as the restriction of an ultraproduct to A, then $f(\underline{\mathfrak{A}}) \in \Omega$ for all profiles $\underline{\mathfrak{A}} \in \Omega^I$.

3 Impossibility theorems for infinite populations

In the case of a finite number of individuals dictatorship results immediately follow from the principality of any ultrafilter on a finite set. For the case of an infinite set of individuals there exist free ultrafilters and therefore Arrow's impossibility theorem does not apply (as was already shown by Fishburn [6]).

However, the very construction of an ultraproduct bears another source of impossibility results as remarked by Lauwers and Van Liedekerke [11]: Ultraproducts with respect to free ultrafilters have a strictly larger domain than the factor structures, and thus, witnesses to certain existential statements in the ultraproduct do not need to belong to the domain of the factor structures (cf. Hodges [9] for a more comprehensive discussion of the role of ultraproducts for the construction of extensions of given structures). Therefore, if an aggregator is the restriction (to the factor-domain) of an ultraproduct[8] and is required to preserve some non-universal statement (for example: existence of a best alternative; continuity; etc.), it must be the restriction of an ultraproduct with respect to a principal ultrafilter and will thus be dictatorial.

Indeed, Lauwers and Van Liedekerke [11] have remarked that in the setting of preference aggregation, the preservation of non-Π_1 formulae generically leads to dictatorial impossiblity results (e.g. Campbell's theorem on the translation of the Arrovian dictatorship result to infinite populations when preferences are assumed to be continuous [3]). The same phenomenon can be observed in the more general setting of first-order predicate aggregation theory.

In order to illustrate this, let us consider the simplest case, viz. preservation of a Σ_1-formula with one existential quantifier in a restricted ultraproduct construction. Suppose hence $\psi = (\dot{\exists}\dot{v})\phi(\dot{v})$ for some $\mathfrak{L}$-formula $\phi(\dot{v})$ with one free variable, assume I is infinite, let $\mathfrak{D}$ be an ultrafilter on I, and consider a family $\underline{\mathfrak{A}} = \langle \mathfrak{A}_i \rangle_{i \in I}$ of models of T, all with the same domain A. Suppose that whilst $(\dot{\exists}\dot{v})\phi(\dot{v})$ is true in all models $\mathfrak{A}_i$, there does not exist an almost uniform witness, i.e. there exists no $a \in A$ such that $\phi[\dot{a}]$ would be true in $\mathfrak{D}$-almost

[8] For instance, Arrovian preference aggregators always map every profile to the restriction — to the set of alternatives — of its ultraproduct with respect to the ultrafilter of decisive coalitions, cf. Lauwers and Van Liedekerke [11].

all models $\mathfrak{A}_i$. Then, Łoś's theorem teaches that $\phi[\dot{a}]$ fails in $\prod_{i\in I}\mathfrak{A}_i/\mathcal{D}$ for all $a \in A$, and therefore ψ cannot be true in the restriction of $\prod_{i\in I}\mathfrak{A}_i/\mathcal{D}$ to A.

This phenomenon can be used as a source of more general impossibility theorems in abstract aggregation theory: In this note, we will prove an impossibility theorem for aggregators which preserves some Π_2-formula outside Π_1 (e.g. some Σ_1-formula which is not Δ_1).

Consider an arbitrary $\mathfrak{L}$-sentence which is not Π_1. In its prenex normal form it can be written as $\psi \equiv (\dot{\forall}\dot{x}_1)\ldots(\dot{\forall}\dot{x}_1)(\dot{\exists}\dot{y})\phi(\dot{x}_1,\ldots,\dot{x}_m;\dot{y})$, wherein m is a nonnegative integer and $\phi(\dot{x}_1,\ldots,\dot{x}_m;\dot{y})$ is an $\mathfrak{L}$-formula with $m+1$ free variables. For the rest of this paper, ψ and ϕ are fixed in this manner.

We say that a profile $\underline{\mathfrak{A}} \in \Omega^I$ has *finite witness multiplicity* with respect to ψ if and only if $\mathfrak{A}_i \models \psi$ for all $i \in I$, but for all $a_1,\ldots,a_m,a' \in A$, the coalition $\{i \in I \ : \ \mathfrak{A}_i \models \phi(\dot{a}_1,\ldots,\dot{a}_m;\dot{a}')\}$ is finite.

An aggregator f is said to *preserve* an $\mathfrak{L}$-sentence ψ if and only if for all $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$, one has $f(\underline{\mathfrak{A}}) \models \psi$ whenever $\mathfrak{A}_i \models \psi$ for all $i \in I$.

We say that ϕ is *free of negation, disjunction and universal quantification* if and only if its non-abbreviated form does not contain the symbols $\dot{\neg}$, $\dot{\vee}$ nor $\dot{\forall}$, in other words, if the only logical symbols appearing in it are $\dot{\wedge}$ and $\dot{\exists}$.

With this terminology, we have the following impossibility theorem:

Theorem 1. *Let f be an aggregator that preserves ψ, and assume that there exists some $\underline{\mathfrak{A}} \in \mathrm{dom}(f)$ with finite witness multiplicity with respect to ψ.*

1. *If $\mathcal{D}_f$ is an ultrafilter, then even principal (whence f is a dictatorship).*
2. *If $\mathcal{D}_f$ is merely a filter, but ϕ is free of negation, disjunction and universal quantification, then $\mathcal{D}_f$ contains a finite coalition (whence f is an oligarchy).*

Proof (Proof of Theorem 1).

1. Since $f(\underline{\mathfrak{A}})$ is just the A-restriction of the ultraproduct of $\underline{\mathfrak{A}}$ with respect to $\mathcal{D}_f$, Łoś's theorem readily yields the equivalence

$$\begin{aligned} & f(\underline{\mathfrak{A}}) \models \phi(\dot{a}_1,\ldots,\dot{a}_m;\dot{a}') \\ \Leftrightarrow & \{i \in I \ : \ \mathfrak{A}_i \models \phi(\dot{a}_1,\ldots,\dot{a}_m;\dot{a}')\} \in \mathcal{D}_f \end{aligned} \tag{1}$$

   for all $a_1,\ldots,a_m,a' \in A$. Since $\underline{\mathfrak{A}}$ is assumed to have finite witness multiplicity with respect to ψ, we know that $\{i \in I \ : \ \mathfrak{A}_i \models \phi(\dot{a}_1,\ldots,\dot{a}_m;\dot{a}')\}$ is finite for all $a_1,\ldots,a_m,a' \in A$, and that $\mathfrak{A}_i \models \psi$ for all $i \in I$, whence $f(\underline{\mathfrak{A}}) \models \psi$ as f preserves ψ. Therefore, for all $a_1,\ldots,a_m \in A$ there is some $a' \in A$ such that $f(\underline{\mathfrak{A}}) \models \phi[\dot{a}_1,\ldots,\dot{a}_m;\dot{a}']$, hence

$$\{i \in I \ : \ \mathfrak{A}_i \models \phi[\dot{a}_1,\ldots,\dot{a}_m;\dot{a}']\} \in \mathcal{D}_f$$

   by equivalence (1), although

$$C_{\boldsymbol{a},a'} := \{i \in I \ : \ \mathfrak{A}_i \models [\dot{a}_1,\ldots,\dot{a}_m;\dot{a}']\}$$

   is finite. Thus, the ultrafilter $\mathcal{D}_f$ contains a finite subset of I, viz. $C_{\boldsymbol{a},a'}$. But then, $\mathcal{D}_f$ must already be principal, namely $\mathcal{D}_f = \{C \subseteq I \ : \ i \in C\}$ for some individual $i \in C_{\boldsymbol{a},a'}$. The individual i is the dictator.

2. By assumption, $f(\underline{\mathfrak{A}})$ is just the A-restriction of the reduced product of $\underline{\mathfrak{A}}$ with respect to $\mathcal{D}_f$. If ϕ is free of negation, disjunction and universal quantification, an analysis of the proof of Łoś's theorem reveals that we must have

$$\begin{aligned} & f(\underline{\mathfrak{A}}) \models \phi(\dot{a}_1, \ldots, \dot{a}_m; \dot{a}') \\ \Leftrightarrow & \{i \in I \ : \ \mathfrak{A}_i \models \phi(\dot{a}_1, \ldots, \dot{a}_m; \dot{a}')\} \in \mathcal{D}_f \end{aligned} \tag{2}$$

for all $a_1, \ldots, a_m, a' \in A$. Hence, as before one can show that the filter $\mathcal{D}_f$ contains a finite subset of I, viz. $C_{\boldsymbol{a},a'}$. But then, $\mathcal{D}_f = \{C \subseteq I \ : \ C' \subset C\}$ for some $C' \subseteq C_{\boldsymbol{a},a'}$. This C', necessarily a finite set, is the set of oligarchs.

Already Lauwers and Van Liedekerke [11, p. 230, Property 4 (of aggregation functions)] had obtained a dictatorial impossibility theorem for preference aggregators that preserve certain non-Π_1-formulae. However, their theorem is based on a syntactic condition which is quite restrictive as it entails that A is countable and that I is the set of nonnegative integers $\mathbb{N}$. (Lauwers and Van Liedekerke's [11] proof strategy consisted essentially in constructing an aggregator based on a free ultrafilter which does not preserve the truth value of the non-Π_1 formula in question, because the element which satisfies it does, by construction, not belong to A.) Our condition allows uncountable sets of alternatives and uncountable populations.

Moreover, even in the special setting of countably many alternatives and individuals, our condition is at least as general as the one of Lauwers and Van Liedekerke [11]:

Theorem 2. *Let $I = \mathbb{N}$ and $A = \{\alpha_i\}_{i\in\mathbb{N}}$. For all $n \in \mathbb{N}$, let ψ_n be the formula*

$$(\dot{\forall}\dot{x}_1)\ldots(\dot{\forall}\dot{x}_m)(\dot{\forall}\dot{y})(\phi(\dot{x}_1, \ldots, \dot{x}_m; \dot{\alpha}_{n+1}) \dot{\wedge} (\phi(\dot{x}_1, \ldots, \dot{x}_m; \dot{y}) \rightarrow \dot{\bigwedge}_{j=0}^{n} \dot{y} \dot{\neq} \dot{\alpha}_j))).$$

If $T \cup \{\psi_n\}$ is consistent for all $n \in \mathbb{N}$, then there exists some $\underline{\mathfrak{A}} \in \Omega^I$ with finite witness multiplicity with respect to ψ.

Proof (Proof of Theorem 2). Suppose that $T \cup \{\psi_n\}$ is consistent for all $n \in \mathbb{N}$. Then there exists for every $n \in \mathbb{N}$ some model $\mathfrak{A}_n$ of $T \cup \{\psi_n\}$ with domain A.[9] Then, for every $k \in \mathbb{N}$ and arbitrary $a_1, \ldots, a_m \in A$, the set

$$\{n \in \mathbb{N} \ : \ \mathfrak{A}_n \models \phi\,[\dot{a}_1, \ldots, \dot{a}_m; \dot{\alpha}_k]\}$$

must contain $k-1$, but none of the integers $\geq k$. It is therefore finite. Since $A = \{\alpha_k\}_{k\in\mathbb{N}}$, we conclude that for all $a \in A$ and all $a_1, \ldots, a_m \in A$, the set

$$\{n \in \mathbb{N} \ : \ \mathfrak{A}_n \models \phi\,[\dot{a}_1, \ldots, \dot{a}_m; \dot{a}]\}$$

[9] For, by completeness, there exists for every $n \in \mathbb{N}$ some model $\mathfrak{A}_n$ of $T \cup \{\psi_n\}$ with domain A_n, relational interpretations $R^m \subseteq A_n^{\delta(m)}$ ($m \in \mathbb{N}$) and pairwise distinct constant interpretations $c_a^n \in A_n$ ($a \in A$). Since $T \cup \{\psi_n\}$ is universal, the restriction of this relational structure to $\{c_n^a \ : \ a \in A\}$ will still be a model of $T \cup \{\psi_n\}$. Without loss of generality, one may assume that $c_n^a = a$ for all $a \in A$.

is finite. On the other hand, ψ_n implies ψ, so each of the $\mathfrak{A}_n$ is a model of ψ. This proves that $\langle \mathfrak{A}_n \rangle_{n \in I}$ has finite witness multiplicity with respect to ψ.

Let us finally consider some applications of our impossibility theorem (Theorem 1):

- In preference aggregation, as already remarked by Lauwers and Van Liedekerke [11, p. 231], any Arrovian aggregator which preserves either continuity or the existence of upper bounds or lower bounds must be dictatorial. The reason is that one can devise profiles with finite witness multiplicity with respect to the formula expressing continuity of preferences, and there exist also profiles with finite witness multiplicity with respect to the formula describing the existence of an upper/lower bound. In particular, this yields an alternative proof of Campbell's theorem [3] (which asserts the impossibility of non-dictatorial, Arrovian and continuity-preserving aggregators — regardless of the electorate's cardinality).
- In propositional judgment aggregation à la Dietrich and List [4], this result means that a judgment aggregator which satisfies certain rationality axioms and preserves some existential conjunctive statement about the elements of the agenda must be oligarchic, provided usual agenda conditions are met and there exists a profile with finite witness multiplicity. Under stronger agenda conditions, we even have a dictatorial impossibility result for aggregators which preserve some non-Π_1 statement.
- In modal propositional judgment aggregation, any rational aggregator preserves some existential conjunctive statement about possible worlds (in the Kripke semantics) must be oligarchic, provided there exists a profile with finite witness multiplicity. Under stronger agenda conditions, we even have a dictatorial impossibility result for aggregators which preserve some non-Π_1 statement about possible worlds.

4 Conclusion

As shown in a companion paper [7], in a model-theoretic framework for the analysis of aggregation problems the ultraproduct construction allows to derive the correspondence between abstract aggregation rules in an Arrovian spirit and ultrafilters of winning coalitions on the set of individuals. Whilst this construction immediately reveals why dictatorship results do not carry over to the infinite case — where free ultrafilters exist —, it opens up another source of impossibility results, which we have analyzed in this paper: Non-universal statements are generically not preserved under aggregation. This problem is, of course, hardly surprising from the vantage point of model theory (given that an important use of ultraproducts is the enlargement of a given structure). However, it challenges one of the usual conditions on aggregation rules — viz. that the aggregate model has exactly the same domain as the individual models (the factor domains of the ultraproduct) —, as this requirement can only be met for sufficiently rich theories if the ultrafilter of decisive coalitions is principal, i.e. the aggregation rule is dictatorial.

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