Of what one cannot speak, must one pass over in silence?

Logical Perspective on universal knowledge structures.

Preliminary report.

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Abstract. Putting together a number of known results from epistemic game theory and modal logic, we argue for a language-based perspective on the question of the existence of a *universal knowledge structure*. We first survey known results on universal structures, and explain why the question is important for the foundations of game theory and interactive epistemology. We then observe that the notion universality that is used in the current literature is based on a very fine-grained, model-based description of the agents' information. In a more coarse-grained, language-based perspective, universal structures do exist, and most of the structures used to show non-existence turns out to be informationally equivalent. We finish with some general remarks on the source of language-dependency, and observe that similar phenomena also occur for Harsanyi type spaces.

1 Introduction

It is a recurrent issue in epistemic game theory whether the formal models used to formalize the players strategic uncertainty and higher-order information make *substantive* assumptions about what the players know about each other, and in particular about how information is impaired to them—see [1, 2] and references therein. For Harsanyi *type spaces*, the issue has been solved by showing that there exists a so-called *universal* type space, see for instance [3, 4]. For qualitative structures, a.k.a. Aumman, Kripke or partition structures, it has been shown that in general no such universal structure exists [5–7].

In this preliminary report we focus on qualitative structures, and argue that the existence of universal structures of this kind depends on the language one uses to describe them. Our argument is based on a number of technical observations, well-known in the interactive epistemology and epistemic logic literature. In Section 2 we define qualitative structures, explain what a universal structure is, and state what is known about them, namely that they do not exist. In Section 4 we explain why the question of the existence of universal structures is important. Our main argument is in Section 5. There we provide an alternative notion on universality, based on finitary modal epistemic languages, and observe that according to this definition universal qualitative structures do exist. We conclude with some general remarks, pointing out important open problems and noting that the issue of language-dependency bears on Harsanyi type spaces as well.

The contribution of this report is conceptual rather than formal: conveying that the question of the existence of a universal knowledge structure is important for the foundations of game theory and interactive epistemology, and arguing for a language-based perspective on this question. The mathematics we use is well-known, and at this level our contribution is to connect the dots. In the conclusion, we do point to interesting and open mathematical questions, which will be discussed in the full version of this paper.

2 Basic Definitions

knowledge structures are qualitative models of information and higher-order information. They are part of the standard toolkit in the foundations of game theory [8, 9] and in logic [10, 11], where they have been extensively used in epistemic characterization of solution concepts.

Definition 1 (Knowledge Structure). *A* knowledge structure $\mathcal{M}$ *is a tuple* $\langle W, N, \mathcal{R}, V\rangle$ *where* W *is a nonempty set of* states*,* N *is a finite set of* agents*,* $\mathcal{R}$ *is a collection of* binary relations *on* W *and* $V : W \to 2^{\textsc{prop}}$ *is a* valuation *function from* W *to subsets of a given countable set of atomic proposition* PROP*. We write* $R[w]$ *for* $\{w' : wRw'\}$*. A* pointed knowledge structure *is a pair* $\mathcal{M}, w$*.*

Each state of a knowledge structure describes the basic facts of the situation being modeled as well as the information each agent has about these facts, and about the information of the other agents. The valuation function V assigns to each state a set of basic facts (i.e., atomic propositions) that are true at that state. The relations encode the information that the agents have at each state. Unless otherwise specified, the collection $\mathcal{R}$ is a set of indexed relations R_i, one for each agent i in N. These relations are sometimes assumed to be equivalence relations, and the partitions they induce are understood as encoding what the agents *know* at each state (thus the name *knowledge* structures). In what follows we keep this terminology, but remain general and assume the relations in $\mathcal{R}$ are arbitrary binary relations on W. Thus, we are agnostic about the precise informational attitude being formalized (eg., knowledge or belief).

In a knowledge structure with states W, we can define a function for each agent i assigning to every *event*, or *proposition*, $X \subseteq W$ the event "agent i believes (or knows) that X is true". Formally, a function is associated with each relation in a knowledge structure:

Definition 2 (R-Operators). *Given a knowledge structure $\mathcal{M} = \langle W, N, \mathcal{R}, V\rangle$. An R-*operator *is a function $O_R : 2^W \to 2^W$ such that, for all $X \subseteq W$, $O_R(X) = \{w : R_i[w] \subseteq X\}$.*

When the relations R_i are equivalence relations, for instance, the operator[3] O_i gives for each "event" X the set of states where i "knows" that X is true. Since we do not assume that the R_i are equivalence relations, in what follows we use the terminology *belief operators* for the O_i.

Mutual and higher-order information in knowledge structures correspond to operations on the relations R_i, and their corresponding operators. The most extensively studied is *common knowledge* and *common beliefs*, which corresponds to taking reflexive-transitive closures of the individual's binary relations. The following definition and observation is well-known, but we include it here for completeness (see, for example, [12, p. 6]).

Definition 3 (Reflexive-transitive closure). *Let G be a non-empty subset of N, and let $R_G^{\cup} = \bigcup_{i\in G}\{R_i : i \in G\}$. The* reflexive-transitive closure *R_G^* of the set $\{R_i : i \in G\}$ is defined as follows:*

$$R_G^* = \bigcap\{R'_G : R'_G \textit{ is a reflexive and transitive relation on } W \textit{ s.t. } R'_G \supseteq R_G^{\cup}\}$$

Given a set $G \subseteq N$ of agents, we write O_G for the operator corresponding to the relation R_G^* (cf. Definition 2).

3 Universal Structures - The state of the art

In most of the results showing non-existence of universal knowledge structure, universality is conceived of in terms of "information-preserving mappings." These are functions between two knowledge structures that preserve both the ground facts and the (higher-order) information. The most widely used notion of such mapping are the so-called *knowledge morphisms.*

Definition 4 (Knowledge morphism - [5, 7]). *Given two knowledge structures $\mathcal{M}$ and $\mathcal{M}'$ with the same labels (modulo ') for the relations in $\mathcal{R}$ and $\mathcal{R}'$, a* knowledge morphism *is a function $f : W \to W'$ such that, for all $w \in W$, $R \in \mathcal{R}$ and $R' \in \mathcal{R}'$,*

1. *for all $p \in \textsc{prop}$, $p \in V(w)$ iff $p \in V'(f(w))$.*
2. *$f(R[w]) = R'[f(w)]$, with $f(R[w]) = \{f(w') : wRw'\}$.*

Intuitively, if there is no knowledge morphism from one structure to another, then either they disagree on some basic facts or, more interestingly, the agents have different information in each of them. That is, knowledge morphisms are mappings that preserve not only basic propositional truth, condition

[3] According to Definition 2, we should write O_{R_i} for the operator associated with R_i. For simplicity and since we assume each agent i is associated with exactly one relation R_i in a knowledge structure, we will write O_i.

1, but also the the informational structure. More precisely, a well-known observation is that if there is a knowledge morphism f from a knowledge structure $\mathcal{M} = \langle W, N, \{R_i\}_{i \in N}, V \rangle$ to a knowledge structure $\mathcal{M}' = \langle W', N, \{R'_i\}_{i \in N}, V' \rangle$, then for all states $w \in W$, all $i \in N$ and sets $X \subseteq W$,

$$w \in O_{R_i}(X) \text{ iff } f(w) \in O_{R'_i}(f(X)).$$

It is not difficult to see that a knowledge morphism is equivalent to the well-studied *bounded morphism*, or *p-morphism*, familiar in modal logic literature (see [12, Definition 2.10, pg. 59]). This equivalence assumes, as is done throughout the literature on universal type spaces, that a knowledge morphism is a *total* function. There are a number of well-known results about bounded morphisms and modal languages that describe knowledge structures (cf. [12]) which we will discuss in Section 5.

The first definition of a universal structure that we study is a knowledge structure that "contain" all other knowledge structures in the following sense.

Definition 5 (Universality as mapping - 1). *A structure $\mathcal{M}$ is* M-universal *iff for any other structure $\mathcal{M}'$ there is a knowledge morphism from f from $\mathcal{M}$ to $\mathcal{M}'$.*

A structure is M-universal whenever it is rich enough to contain the image of any possible structure, under knowledge morphism. In the type space literature there a different notion of universality has been used, boiling down to require that a universal structure should contain all consistent set of ground facts and information. This notion has a straightforward counterpart in knowledge structures, which we explore briefly in Section 5.3.

Theorem 1 ([7]). *There is no M-universal structure.*

This result is based on an earlier non-existence result of [5] for knowledge structures whose relations are partitions. Meier's result hold for arbitrary knowledge structures.

In this report we want to look at such non-existence results from an logical point of view, and in particular to revisit the notion of universality defined above. There is already some work in this direction, starting with the work of [13] and [6] to more recent work by [14] and [15], exploring the existence of universal structures from a coalgebraic point-of-view. We briefly comment on these below. Our primary concern in this report are conceptual issues,and so we start by stressing the significance of such results for the foundations of game theory and interactive epistemology.

4 Why Universal Structures?

The possibility of constructing a knowledge structure where one makes as less substantive assumptions as possible hinges on whether universal knowledge structures exist or not. The question of the existence of universal structures is thus not only mathematically interesting, it is also important for the foundation of game theory, and for interactive epistemology more generally.

Substantive assumptions are assumptions about how, and how much, information is imparted to the agents, over and above those that are intrinsic to the mathematical formulation of knowledge structures. The latter type of assumptions include, for instance, monotonicity and closure under intersection of the operators generated by the relations R. Substantive assumptions, on the other hand, concern what the agents know and believe about what the others know and believe, and in particular what is commonly known in a given situation. Given a countably infinite set of propositions or basic facts, for example, in finite structures is will always be common knowledge that some logically consistent combination of these basic facts are not realized, and *a fortiori* that for logically consistent configurations of information and higher-order information about these basic facts.

Extending the state space is the usual technique to relax substantive assumptions in a given knowledge structure. See, for example, the discussion already in [8] and the extended discussion in [1]. Substantive assumptions are nontrivial facts that are commonly known to some or all agents in a given situation. They can also be said to describe a specific *context* of the situation being modeled. By adding to a structure states where some of these agents are uncertain about these facts, and expanding the relations R_i accordingly, common knowledge of these facts disappear.

The question naturally arises, then, whether for any knowledge structure based on a set of basic facts PROP, there is a structure also based on PROP that makes strictly less substantive assumptions, or whether by sufficiently expanding the state space one can construct a structure where no, or at least as few substantive assumptions as possible are made. If no such structure exists, any choice of model for a given situation carries some substantive assumptions about what the agents know and believe about each other, and this might compromise the generality of the analysis done on such models. Universal structures, if they existed, would be such structures that minimize the substantive assumptions. M-universality indeed rules out by definition the possibility of finding another structure where the agents "know" strictly less.

The non-existence results presented above are thus important, from a foundational point of view, as they show that there is no ultimate safe retreat, so to speak, where one can be sure that no substantive assumptions are made. Whatever knowledge structure one builds, one can always point out to some substantive assumptions that could be relaxed or need explaining.

This can been seen as quite negative for those inclined to the use of knowledge structures to model strategic uncertainty. Non-existence results show that the epistemic analysis of an interactive situation in terms of a knowledge structure is always tinted by *some* substantive assumptions, that the analysis can never claim to be as neutral as possible regarding the agents' information.

In what follows, however, we argue that the situation is not as bad as it looks, once we give up the idea that universal structures should be rich enough to represent any other structure whatsoever, and instead require that universal structures should contain all that can be said about the agents' information and higher-order information in some natural logical language.

5 Structure-based and Language-Based Universality

In the pure semantic approach presented in Section 2, the set of events that the agents can reason about is determined by the size and the configuration of the knowledge structure itself. Agents can think about any set-theoretically definable event in a given knowledge structure, however complex, and the hierarchy of higher-order information can go as high in the set theoretical universe as the structure itself permits.

Arguably, it is then not surprising that, absent additional structure on the space of events that the players can think about, there does not exist a knowledge structure capable of representing *all possible* epistemic states (i.e., a universal knowledge structure). Recall that if there is a knowledge morphism f from $\mathcal{M}$ to $\mathcal{M}'$ then the image of everything that the agents knows or believe at any state w in $\mathcal{M}$ is also known or believed at $f(w)$. This means that for any subset X, however complex, knowledge morphisms commute with the set-theoretical operators O_R generated by the relations in $\mathcal{R}$.

In contrast with this fine-grained, structure-based perspective, *syntactic* approaches suggests an alternative view on information and informational equivalence. Instead of letting the structure alone determine what events the agents can know and believe in a given interactive situation, one *first* specifies this syntactically, and *then* looks at the resulting definable informational events in a given structure. This is common wisdom in modal logic, but it is of crucial importance to the question of the existence of universal knowledge structures.

Whether or not universal structures will come out of such an approach will of course depend on the expressive power of the language used to describe the models. The full set-theoretical description of knowledge structures presented above can be seen as an extreme case of that methodology. From this point of view the non-existence results become less surprising. There is, however, a plethora of well-studied, less expressive, but nevertheless intuitive languages at one's disposal, and as we now observe they provide straightforward, alternative notions of informational equivalence and universality.

5.1 Finitary epistemic languages

In this report we use standard multi-agent epistemic logic with a "common knowledge," here a reflexive-transitive closure modality [10, 11], as illustrative case. This language has a long tradition in philosophy [16], computer science [10], and epistemic game theory [9]. The observations we make in the following sections, however, carry over to more expressive languages such as Propositional Dynamic Logic [17] and the modal μ-calculus [18].

Let $\mathcal{S}$ be a *signature* for a class of knowledge structures: a finite set of agents N, a countable set of proposition PROP and binary relations R_i for each agent $i \in N$. Epistemic languages are built from this signature as follow:

Definition 6 (Finitary Epistemic Language). *Given a signature $\mathcal{S}$, a finitary epistemic language $\mathcal{L}_{EL}$ is recursively defined as follows:*

$$\phi := p \mid \neg\phi \mid \phi \wedge \phi \mid \Box_{R_i}\phi \mid \Box^*_{R_G}\phi$$

where i ranges over N, p over PROP*, and R_i over the $\mathcal{R}_i$ and $\emptyset \neq G \subseteq I$.*

Epistemic languages are interpreted on a knowledge structure as usual:

Definition 7. *We write $||\phi||^{\mathcal{M}}$ for $\{w \in |\mathcal{M}| \ : \ \mathcal{M}, w \Vdash \phi\}$. We omit $\mathcal{M}$ when it is clear from the context.*

$$\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & p \in V(w) \\
\mathcal{M}, w \Vdash \neg\phi & \text{iff} & \mathcal{M}, w \nVdash \phi \\
\mathcal{M}, w \Vdash \phi \wedge \psi & \text{iff} & \mathcal{M}, w \Vdash \phi \text{ and } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \Box_{R_i}\phi & \text{iff} & \forall v (\text{if } wR_iv \text{ then } \mathcal{M}, v \Vdash \phi) \\
\mathcal{M}, w \Vdash \Box^*_{R_G}\phi & \text{iff} & \forall v (\text{if } wR^*_Gv \text{ then } \mathcal{M}, v \Vdash \phi)
\end{array}$$

It is well-known that the set of modally definable events in a given knowledge structure can be strictly smaller than the set of all events. Furthermore, there are elegant model-theoretic characterizations of the classes of knowledge structures definable in this language[4]. By fixing explicitly what can be known and believed by the agents in a given knowledge structure, such a syntactic approach provides an alternative notion of informational invariance and of M-universality.

5.2 M-universality Revisited

The language specified above provides a straightforward notion of informational equivalence. Instead of saying that two structures are informationally equivalent whenever any event that is known or believed in one is known or believed in the other, and vice versa, one just requires the two structures to be informationally equivalent up to definability in $\mathcal{L}_{EL}$.

Definition 8 ($\mathcal{L}_{EL}$-morphism). *Given two knowledge structures $\mathcal{M}$ and $\mathcal{M}'$ of signature $\mathcal{S}$ and an epistemic language $\mathcal{L}_{EL}$ for this signature, a function $f : W \to W'$ if a $\mathcal{L}_{EL}$-*morphism *iff, for all $w \in W$ and all $\phi \in \mathcal{L}_{EL}$:*

$$\mathcal{M}, w \Vdash \phi \textit{ iff } \mathcal{M}', f(w) \Vdash \phi.$$

[4] For example, a class of pointed knowledge structures is definable by a (set of) modal formulas iff the class is closed under bisimulations and ultraproducts and its complement is closed under ultrapowers [12, Theorem 2.75, pg.107]. Bisimulations are defined in the next section (Definition 9), see [12] for the definitions of an ultrapower and ultraproduct of a class of knoweldge structures.

$\mathcal{L}_{EL}$-morphisms are language-based notions of informational equivalence, and they are of course weaker than knowledge morphisms. The information that $\mathcal{L}_{EL}$-morphisms preserve is not only dependent on the underlying knowledge structure, but also on the expressive power of $\mathcal{L}_{EL}$.

Observation 2 *Every knowledge morphism f from $\mathcal{M}$ to $\mathcal{M}'$ is a $\mathcal{L}_{EL}$-morphism, but not the other way around.*

The general situation is as follows: a knowledge morphism is a special case of a *bisimulation*:

Definition 9 (Bisimulation). *Given two knowledge structures $\mathcal{M}$ and $\mathcal{M}'$ based on signature $\mathcal{S}$, a* bisimulation *is a relation $\underline{\leftrightarrow}$ between W and W' such that, for all $w \in W$ and $v \in W'$, if $w \underline{\leftrightarrow} v$ then, for all $R_i \in \mathcal{R}$ and $R'_i \in \mathcal{R}'$,*

prop: for all $p \in \text{PROP}$, $p \in V(w)$ iff $p \in V'(v)$.
forth: if wR_iw' then there is a $v' \in W'$ such that vR'_iv' and $w' \underline{\leftrightarrow} v'$.
back: if vR'_iv' then there is a $w' \in W$ such that wR_iw' and $w' \underline{\leftrightarrow} v'$.

It is a simple but instructive proof to see that any knowledge morphism is a bisimulation (and not the other way around). Furthermore, it is well known that if two knowledge structures are bisimilar then they satisfy the same $\mathcal{L}_{EL}$ formulas, but the converse does not necessarily hold unless the structures are *image-finite* (i.e., for any relation R and state w, $R[w]$ is finite, see [12, pp. 68,69]).

Knowledge morphism are thus sensitive to differences that cannot be expressed in $\mathcal{L}_{EL}$. Up to what can be said in $\mathcal{L}_{EL}$ about the agents' information, two $\mathcal{L}_{EL}$-equivalent structures, even if there is no knowledge morphism between them, are identical.

$\mathcal{L}_{EL}$-morphism suggest a natural alternative to M-universality. Instead of requiring universal structures to be rich enough to embed any other structure, one requires only the embedding to preserve what can be defined in $\mathcal{L}_{EL}$.

Definition 10 (Universality as mapping - 2). *Given a signature $\mathcal{S}$, a structure $\mathcal{M}$ is* $\mathcal{L}_{EL}$-universal *iff for any other structure $\mathcal{M}'$ of the same signature there is a $\mathcal{L}_{EL}$-morphism $f : W' \to W$.*

Intuitively, a $\mathcal{L}_{EL}$-universal structure is thus one where on makes no substantive assumption, as far as these can be expressed in $\mathcal{L}_{EL}$. This restriction to the expressive power of $\mathcal{L}_{EL}$ is the key to the following observation.

Observation 3 *Let $\Lambda_{\mathcal{L}_{EL}}$ be a sound and complete axiomatization of a given class of knowledge structures $\mathcal{K}$, and let $\mathcal{M}^C$ be the canonical model for $\Lambda_{\mathcal{L}_{EL}}$, constructed in the standard way. See [12] for details. For any knowledge structure $\mathcal{M} \in \mathcal{K}$ there is $\mathcal{L}_{EL}$-morphism $f : W \to W^C$.*

Proof. Take $f(w) = \{\phi : \mathcal{M}, w \Vdash \phi\}$.

Corollary 1. *$\mathcal{M}^C$ is $\mathcal{L}_{EL}$-universal.*

The standard canonical model construction for an axiomatization in $\mathcal{L}_{EL}$ of a given class of knowledge structures is thus $\mathcal{L}_{EL}$-universal. As [19] observes, there are many structures that are $\mathcal{L}_{EL}$-equivalent to the canonical model, and some will of course make strictly less substantive assumptions in an absolute, set-theoretical sense. These assumptions go beyond the expressive power of $\mathcal{L}_{EL}$. These are difference that the formal language cannot express.

Observation 3 and corollary 1 are completely standard in the modal logic as well as the epistemic game theory communities; their import is rather the conceptual point that whether there are structures where one makes as less substantive assumption as possible depends on the expressive power of the language one uses to describe the agent's information. As Theorem 1 shows, if one uses the full resources of the language of set theory, then there is indeed no such structure. However, if one works with a less expressive language (and so reduces the number of details taken into account in an epistemic analysis of a given interactive situation) then there are structures which make no substantive assumptions.

Observation 3 can also be applied to the counter-example to the existence of a universal knowledge structure provided by [5].

Corollary 2. *Take any pointed structure W^α, for an arbitrary ordinal α and as defined in [5, p.267-268]. Then there is $\mathcal{L}_{EL}$-morphism $f : W^\alpha \to W^C$, from W^α the canonical model for $S5$ based on the set of basic fact Θ.*

This corollary is nothing but a preliminary observation; a full language-based analysis of Heifetz and Samet's construction should reveal that, at some point in the sequence of ever increasing structures, there should be no difference up to $\mathcal{L}_{EL}$-satisfaction. This can be formally proven using the well-studied notion of (n-)bisimulation. We only state one key observation here and leave a complete analysis for the full paper. We first give the formal definition of an *n-bisimulation* that characterizes definability up to modal depth n.

Definition 11 (n-Bisimulation). *Two pointed knowledge structures $\mathcal{M}, w$ and $\mathcal{M}', v$ are* n-bisimular *whenever there is a sequence $\underline{\leftrightarrow}_n \subseteq ... \subseteq \underline{\leftrightarrow}_0$ of relations such that, for all $i + 1 \leq n$ and $w' \in W$ and $v' \in W$:*

1. $w \,\underline{\leftrightarrow}_n v$;
2. If $w' \,\underline{\leftrightarrow}_0 v'$ *then* $V(w') = V'(v')$.
forth: If $w' \,\underline{\leftrightarrow}_{n+1} v'$ *if* $w' R_i w''$ *then there is a* $v'' \in W'$ *such that* $v' R'_i v''$ *and* $w'' \,\underline{\leftrightarrow}_n v''$.
back: Same from W' to W.

The key observation is that at the "ω stage" in the sequence of ever increasing structures constructed, states that are "equivalent" in the model are in fact bisimular: For a given ordinal α, let $P^\alpha(w)$ be as defined in [5, p.265-266]. Take any W^α. For all $n < \omega$, set $w \,\underline{\leftrightarrow}_n w'$ iff $w' \in P^n(w)$.

Claim. $\underline{\leftrightarrow}_n$ is a n-bisimulation.

Proof. $P^0(w)$ is the partition of W^α according to the propositional valuation, so we automatically get that if $w \underline{\leftrightarrow}_0 w'$ then $V(w) = V(w')$. Now take any w' such that $w \underline{\leftrightarrow}_n w'$, and suppose $w \sim_i v$. We have to find a v' such that $w' \sim_i v'$ and $v \underline{\leftrightarrow}_{n-1} v'$. Take $v' = (v^{\leq n-1}, w'^{\geq n})$. [5] show on page 271 that for all w and any ordinal β, $P^\beta(w) = \{w' : w^{<\beta} = w'^{<\beta}\}$. So we know that $v \underline{\leftrightarrow}_{n-1} v'$. But we also know that $w'^{\leq n-1} = w^{\leq n-1}$. Since $v'^{\leq n-1} = v^{\leq n-1}$ and $w \sim_i v$ we know that $w'^{\leq n-1} \sim_i v'^{\leq n-1}$ in W^{n-1}. But then by Lemma 3.2 in [5], we know that there is a $u \sim_i w'$ in W^α such that $u^{\leq n-1} = v'^{\leq n-1}$. But then $u \underline{\leftrightarrow}_{n-1} v' \underline{\leftrightarrow}_{n-1} v$, which completes the proof.

5.3 R-Universality

In the literature, a number of definitions of "universality" of a (type) structure have been studied (see the discussions in [20] and the survey [21]). One such definition of universality is not conceived directly in terms of mappings and informational equivalence, but rather in terms of realizability (see [4]).

One can of course define a similar notion for knowledge structures. Basic facts and the informational attitudes form a well-defined algebra on knowledge structures [22, Chapter 6] and this provide an explicit notion of consistency. We already mentioned monotonicity and closure under conjunction, but positive and negative introspections are other famous examples of such constraints.

Definition 12 (Universality as realizability). *Given a signature $\mathcal{S}$ and a sound and complete axiomatization $\Lambda_{\mathcal{L}_{EL}}$ of a class of knowledge structure $\mathcal{K}$, a structure $\mathcal{M} \in \mathcal{K}$ is* R-universal *iff for all $\Lambda_{\mathcal{L}_{EL}}$-consistent set Γ there is a state in W such that for all $\phi \in \Gamma$, $\mathcal{M}, w \Vdash \phi$.*

A R-universal structure is a structure that is rich enough to realize all consistent sets of statements about what the agents know or believe about each other, including the knowledge and beliefs of the others. This syntactical view on knowledge structures goes back to the very beginnings of epistemic logic [16], and lies behind the construction of canonical structures mentioned above:

Observation 4 *$\mathcal{M}^C$ is R-universal.*

It should be clear that a knowledge structure is R-universal iff it is $\mathcal{L}_{EL}$-universal, once one relativize the later to axiomatize classes of knowledge structures. [23] provide a model-theoretic version of R-universality, and indeed show that no such universal structures exist in general. The relation between M-universality and the results in that paper will also be investigated in the full version of this paper.

6 Language Dependency in Type Spaces and Unified Perspective

It is interesting to note this language-dependency seems also to hold for type spaces. [6] show, for instance, that if one moves from countable to finite additivity of the underlying probability functions, then universal type spaces cease to exist.

These results point towards an interesting question about knowledge structures, namely how robust are the non-existence results? Various results in the type space literature have shown that the existence of a universal type space is rather robust to relaxing the assumption on the underlying topology, but the result in [6] show that it is not robust to a simple variation in the language used to construct these structures. Is the situation similar for knowledge structures?

To put these results in perspective, a more general framework seems to be needed. Such an abstract framework would also help to situate the language-based perspective on knowledge structures with the current discussion on the relation between belief hierarchies and type morphisms [24]. The co-algebraic work cited above seems to provide the right tools for this but, at this point, the general picture still missing.

7 Conclusion

In this preliminary report, we put together a number of basic observations in order to argue that the question of whether universal knowledge structures exist depends on the language that one use to describe them. Coarse-grained languages yield universal structures, fine-grained languages do not.

At the conceptual level, the question is of course to which extend a fine-grained analysis is needed or, the other way around, why would one choose to ignore details and go for coarse-grained languages. Issues of computational complexity speak in favor of the second approach, while notions of behavioral equivalence seems to point towards the first. We do not take issue on this question here, but rather ask, with one of the founding fathers of analytic philosophy: Of what one cannot speak, must one pass over in silence?

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