Analysis of two fuzzy negotiation solutions by Knowledge Engineering to n-person cooperative games, from the point of view of imputation's axioms and Shapley Value's axiomatic

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Abstract. The aim of this paper is to prove the rationality of two new fuzzy solutions to cooperative n-person games, and its advantages over classic solutions. These solutions, called Fuzzy Negotiation Solution by Knowledge Engineering (FNSKE) and Compensatory Negotiation Solution by Knowledge Engineering (CNSKE), are based on bargaining over statements coming from experts. The proof of rationality consists in demonstrating that the elements of the FNSKE and CNSKE are imputations. The approach satisfies Efficiency, Symmetry, and Dummy Axiom of Shapley's axiom set. A counterexample is developed to show that the FNSKE and CNSKE don't satisfy Additivity, a property widely criticized in game theory literature. Though the FNSKE and CNSKE are based on the knowledge of experts in negotiation, these proofs of rationality show that the field of application of these solutions is not limited to bargaining and negotiation.

Keywords: fuzzy logic, cooperative n-person game, Shapley Value, imputation, rationality.

1 Introduction

The Shapley Value is a classic solution concept for coalitional n-person games with transferable utility. It is the unique function that satisfies the (Shapley's) axiom set: Efficiency or Pareto Optimality, Equal Treatment or Symmetry, Null Player or Dummy, and Additivity [2]. The first three axioms are considered of rationality in accordance with normative Decision Theory [6] and Game Theory [10]. However, the fourth axiom is polemical in Game Theory because of its lack of rationality [10]. The Shapley Value is a well accepted solution because of its

existence and uniqueness; yet other classic solutions the Core, for instance frequently do not satisfy these properties [10].
The Fuzzy Negotiation Solution by Knowledge Engineering (FNSKE) and the Compensatory Negotiation Solution by Knowledge Engineering (CNSKE) are recents, advantageous solutions in the theory of cooperative n-person games [5]. They are based on propositions coming from experiences in bargaining and negotiation among n parts. The propositions are provided by experts on negotiation in a non-mathematical way and represented in natural language. The solution is a set of vectors, one for every possible coalition among the players; each vector has the gains for the n players in the coalition, and in addition, the truth value of a membership function (e.g., [11]) representing the likelihood of reaching a deal in the respective coalition.
The aim of this paper is to prove that the FNSKE and the CNSKE satisfy the axioms of imputation and two of the Shapley's axiom set, i.e., Efficiency and Symmetry. On the other hand, one counterexample is examined to show that the fuzzy solutions do not satisfy the polemical Additivity Axiom. A proof is made to demonstrate that the fuzzy solutions satisfy the Dummy Axiom; this is a manner to support the rationality of the FNSKE and CNSKE in agreement with the Normative Approach to Decision Making [6].
The paper begins with an exposition of the fundamentals of cooperative n-person games and the Shapley's axiom set. The FNSKE and the CNSKE is further exposed in detail. Next, demonstrations that FNSKE and the CNSKE satisfy axioms of imputation, Efficiency, and Symmetry are developed. One counterexample is given to show that the FNSKE and the CNSKE do not satisfy Additivity. Finally, a proof that the FNSKE and the CNSKE satisfy Dummy Axiom is realized.

2 Basic concepts of cooperative n-person games and the Shapley Value

A *cooperative n-person game* is a pair (N, v), where $N = \{1, 2, \cdots, n\}$ is a non-empty, finite set of players. A coalition is an element of the power set of N, denoted by 2^N. v is the characteristic function of the game, which maps 2^N into the set of real numbers $\mathbb{R}$, satisfying the following properties:

(i) $\mathrm{v}(\emptyset) = 0$
(ii) $\mathrm{v}(S \cup T) \geq \mathrm{v}(S) + \mathrm{v}(T)$, where S and T are disjoints coalitions of 2^N.

The value $\mathrm{v}(S)$ is the worth of S (its members act together, as a single unit). Property ii is called *superadditivity*, it implies that the worth of a coalition is greater than or equal to the worth of any subcoalition (a subset of the coalition).
An imputation [10] is a payoff allocation $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ that satisfies:

(i) $\sum_{i=1}^{n} x_i = \mathrm{v}(N)$, then x is said to be *group rational* or *efficient.*
(ii) $x_i \geq \mathrm{v}(\{i\})$, and x is said to be *individually rational.*

An imputation is a division of worth among the players, such that the sum of their individual values is exactly the worth of the coalition of all players (or grand coalition). In a feasible coalition each player obtains a gain that is at least equal to their individual value.
The *Shapley Value* is one of the most important solution concepts in cooperative games. The Shapley Value of a game (N, v) is the unique n-tuple, $\phi(\mathrm{v}) = (\phi_1(\mathrm{v}), \phi_2(\mathrm{v}), \cdots, \phi_n(\mathrm{v}))$ that satisfies the four axiom set of rationality [2], that is:

1. Efficiency. $\sum_{i \in N} \phi_i(\mathrm{v}) = \mathrm{v}(N)$.
2. Symmetry. For all bijections $\pi : N \to N$ and $i \in N$, $\phi_{\pi(i)}(\mathrm{v}) = \phi_i(\mathrm{v} \circ \pi)$.
3. Dummy Axiom. If i is such that $\mathrm{v}(S) = \mathrm{v}(S \cup \{i\})$ for any coalition S not containing i, then $\phi_i(\mathrm{v}) = 0$.
4. Additivity. If u and v are characteristic functions, then $\phi(\mathrm{u}+\mathrm{v}) = \phi(\mathrm{u})+\phi(\mathrm{v})$.

Efficiency signifies that the set of all players profits exactly the worth of the grand coalition. The interpretation of Symmetry is that the Shapley Value solution is independent of any order assigned to the players. The meaning of the Dummy Axiom is that a player that does not contribute to the worth of any coalition obtains a null value solution. Additivity indicates that if a game is defined as the sum of two different games, then the solution of the resulting game is equal to the sum of the solutions of the two games, played separately or at different times. The last is, however, controversial because Additivity is just a fairly convincing simplification of the intrinsic complexity meant by playing two different games [3].
If $\mathrm{v}(N)$ is interpreted as a worth of S, if it is formed, one may argue that $(N, v + w)$ does not represent "properly", the aggregate of the games (N, v) and (N, w), because S may be "more likely" to form one of these games and less like in the other [8]. Additivity Axiom seems only reasonable if there are no complementary relations between the assets of the two enterprises [8].
One method, based on a different principle, is employed by an "enterprise" consisting of husband and wife, who want to decide on a divorce settlement. Wife takes back every item that she brought out the marriage, husband does the same and the rest is divided equally among the couple. This method, of sharing in proportion to the investments is not additive [8].
A proper theorem guaranties the existence and uniqueness of the Shapley Vector in regard to the axioms 1-4 [2].

3 A fuzzy approach to cooperative n-person games

A recent kind of fuzzy solution to cooperative n-person games [5] is a model of bargaining based upon expert knowledge, summarized in four propositions, extracted from the literature on negotiation, expressed in natural language; to know, [1]:

1. A negotiation part has bargaining capacity if and only if the following conditions hold:

   - The contribution of its institution to the settlement(agreement, or business) in discussion is important.
   - The part can find feasible alternatives and some advantages if no settlement is reached, or its contribution is essential.
2. Any increase in contribution to the business by one of the parts, or an increase in benefits due to its corresponding alternatives, increases its bargaining capacity.
3. The benefit obtained by each part is equal to the amount that it could obtain without he contribution of other parts, plus a portion of the additional benefit derived from the settlement. This increment obtained by each part is approximately proportional to its bargaining capacity.
4. A settlement is possible if and only if all the parts are important to the business and the corresponding benefit to every part is important to each of them.

This is an example of application of *Knowledge Engineering* outside the concept of rationality. Nevertheless, it will be proved in what follows that this model also satisfies rationality. "*Knowledge Engineering* is the discipline (of Artificial Intelligence) that allows to build intelligent systems by means of the deduction of knowledge, having as central processes the acquisition, representation, manipulation and validation of this knowledge" [9].

Definition 1. *The Good Deal Index (GDI) of negotiator i, from the point of view of negotiator j, in a given bargaining-set, is the benefit that i could obtain if the collective benefits estimated by j prevailed, and the whole set of negotiators including j had a similar performance in defense of their interests during the bargaining process.*

In this fuzzy model the institutions are represented by the players in N, the bargaining sets are represented by coalitions belonging to 2^N, and the characteristic function represents the benefits obtained in the bargaining-sets. A formula of the GDI is developed in [5]:

$$X(i,C) = \begin{cases} \mathrm{v}(\{i\}) + \frac{r(i,C)}{\sum_{j\in C} r(j,C)} \left[\mathrm{v}(C) - \sum_{j\in C} \mathrm{v}(\{j\})\right] & if \;\; i \in C, \\ 0 & if \;\; i \notin C \end{cases} \tag{1}$$

Where $r(i, C) \in [0\,, 1]$ is the truth value of the fuzzy predicate "player i has bargaining capacity in the bargaining set C".
The fuzzy logic system used in this model is the Probabilistic Fuzzy Logic. Operators' formulas of this system are:

- $\mathrm{u}(p \wedge q) = \mathrm{u}(p)\mathrm{u}(q)$. Conjunction.
- $\mathrm{u}(p \vee q) = \mathrm{u}(p) + \mathrm{u}(q) - \mathrm{u}(p)\mathrm{u}(q)$. Disjunction.
- $\mathrm{u}(\neg p) = 1 - \mathrm{u}(p)$. Negation.

The following equation is consequence of Proposition 2:

$$r(i,C) = p(i,C) \wedge (a(i,C) \vee p^2(i,C)) \tag{2}$$

Predicates $p(i,C)$ and $a(i,C)$ correspond to the statements "player i is important to coalition C", and "player i has feasible and advantageous alternatives to reach an agreement within C". $p(i,C)$ is squared in order to model the linguistic modifier related to the statement "very" [12].
The following expression is used as a strict fuzzy order:

$$O(C,D) = o(\mathrm{v}(C),\mathrm{v}(D)) = \frac{0.5}{\lambda}\left[\frac{\mathrm{v}(C)-\mathrm{v}(D)}{max_{E,F\in 2^N}|\mathrm{v}(E)-\mathrm{v}(F)|}\right] + 0.5,\ \lambda \geq 1. \tag{3}$$

$p(i,C) = O(C, C\backslash\{i\})$ and $q(i,C) = o(X(i,C), \mathrm{v}(\{i\}))$ correspond to the affirmations "coalition C obtains more benefits than coalition $C\backslash\{i\}$", and "player i obtains more benefits in coalition C than in coalition $\{i\}$", respectively. Thus, these affirmations correspond to the statements "the contribution of i is important for coalition C", and "coalition C is important for player i".
The predicate:

$$S(i,C,D) = \frac{0.5}{\lambda}\left[\frac{X(i,C)-X(i,D)}{max_{E,F\in 2^N}|\mathrm{v}(E)-\mathrm{v}(F)|}\right] + 0.5 \tag{4}$$

corresponds to the statement "i obtains more benefits in coalition C than in coalition D".
The formula of the predicate corresponding to the statement of Proposition 4: "it is possible to reach an agreement within coalition C", is the following:

$$f(C) = \wedge_{j\in C}(p(j,C) \wedge q(j,C)) \tag{5}$$

And the statement: "there are advantageous and possible alternatives for player i to reach an agreement within coalition C", is modeled by the expression:

$$a(i,C) = \vee_{B\neq C,\ \{i\}\subset B}(s(i,B,C) \wedge f(B))$$

The value of λ has been obtained experimentally so that $\lambda = 1.0494$. The GDI is then calculated by the recurrent formula:

$$X(i,C) = \mathrm{g}(X(i,C)) \tag{6}$$

in which g is the operator that transforms nxm matrices into nxm matrices. It generates the GDI of all players considering the possible coalitions of the game. The solution of the game is finally obtained by the expression:

$$S = \left\{(X(1,C_j), X(2,C_j), \cdots, X(n,C_j); f(C_j)) : j = 1,2,\cdots, card(2^N)\right\} \tag{7}$$

Expression (7) yields a set of vectors; each has n elements that is a division of worth among the players in the coalition, and the truth value of $f(C)$ according to (5).

The Counterpart Convenience Index 1 (CCI1) of the player i, in the coalition C of game v is denoted by $D_v(i,C)$, it is equal to:

$$D_v(i,C) = q_v(i,C) \wedge f_v(C) \tag{8}$$

This index allows to rank coalitions by order of convenience for each player i.

Definition 2. *([5]) Let v be an n-person game and β a partition of N. A subgame v_β of v and β is defined to be a game whose players are all the sets in β (coalitions). A subgame satisfies:* $\mathrm{v}_\beta(\{F_1, F_2, \cdots, F_n\}) = \mathrm{v}(\bigcup_{i=1}^n F_i)$.

An index, called Counterpart Convenience Index 2 (CCI2) is defined by:

$$d_v(i,C) = D_v(i,C) \wedge \{q_v^2(i,C) \vee \{\bigwedge_{\bar{C}\in\beta} \bigvee_{S\subset\beta; S\neq\{C\}} D_{v_\beta}(\bar{C},S)\}\} \tag{9}$$

This index models the statement: "The bargaining set C is convenient for player i whenever the following conditions are satisfied:

– it is possible to obtain and advantageous agreement in this set.
– if the agreement is not very advantageous for i then, no matter what structure of coalition had been presented, a convenience bargaining set S for the coalition C (as a player of v_β) exists"

Let v_β be a sub-game created after the negotiation with a structure of coalition β. The CCI2 indicates the convenience of renegotiating, where every $C \in \beta$ negotiates like a single player.
The joint of the solution set in formula (7), the CCI1 given by formula (8), and the CCI2 given by formula (9), is the Fuzzy Negotiation Solution by Knowledge Engineering (FNSKE). For any cooperative n-person game, a unique GDI exists [5].
Compensatory Negotiation Solution by Knowledge Engineering (CNSKE) is a new solution that improves the FNSKE. It consists in the substitution of logical system, the *Compensatory Fuzzy Logic* (CFL) by the Probabilistic Fuzzy Logic. *Compensatory Fuzzy Logic* is a new advantageous fuzzy logic system that satisfies the most important axioms of Decision Theory. Its operators of conjunction and disjunction are continuous and satisfy the idempotency [4]. Some of the main operators of this system are:

– c : $[0,1]^n \rightarrow [0,1]$, such that $\mathrm{c}(x_1, x_2, \cdots, x_n) = \sqrt[n]{\prod_{i=1}^n x_i}$. Conjunction.
– d : $[0,1]^n \rightarrow [0,1]$, such that $\mathrm{d}(x_1, x_2, \cdots, x_n) = 1 - \sqrt[n]{\prod_{i=1}^n (1-x_i)}$. Disjunction.
– n : $[0,1] \rightarrow [0,1]$, such that $\mathrm{n}(x) = 1 - x$. Negation.
– i : $[0,1]^2 \rightarrow [0,1]$, such that $\mathrm{i}(x,y) = d(n(x), c(x,y))$. Implication of Zadeh.

An inconvenience of this logical system is that, contrary to the probabilistic logic, if the number of players decreases, then no necessarily increases the bargaining capacity, this is because of the compensatory tendency of the system. For this reason, it is necessary to complement the previous propositions with a new one that takes into account the quantity of players in each coalition. Keeping this in mind, a change to the fourth proposition is as follows:

4. A settlement is possible if and only if the following conditions are satisfied:
   - All the parts are important to the business and the corresponding benefit to every part is important to each of them.
   - The number of parts associated to the settlement is not large.

The change to the fourth proposition implies a change in (5). The new formula of the "likelihood to reach an agreement in the coalition C" is the following:

$$f(C) = (\wedge_{j\in C}(p(j,C) \wedge q(j,C))) \wedge J(card(C)) \tag{10}$$

Where $J(card(C))$ is the membership function of the proposition: "the quantity of players in the coalition C is not large". Function J depends on the cardinal number of C, $card(C)$.
The function J was modelled using the negation of the sigmoidal membership function having the following formula:
$J(card(C)) = 1 - \frac{1}{1+e^{-\alpha(card(C)-\gamma)}}$, $\alpha = \frac{\ln(0.9)-\ln(0.1)}{\gamma-\beta}$, $\gamma, \beta \in \mathbb{N}$.
Evidently $J(\beta) = 0.9$ and $J(\gamma) = 0.5$. The function $sig(C) = \frac{1}{1+e^{-\alpha(card(C)-\gamma)}}$ models the proposition "the quantity of players in the coalition C is large".
The other formulas remain as stated.
Compensatory Model 2 [7], with parameters $\beta = 1$ and $\gamma = 4$, minimizes the relative error between GDI of the FNSKE and the GDI of the following formula:

$$X(i,C) = \begin{cases} \mathrm{v}(\{i\}) + \frac{r^2(i,C)}{\sum_{j\in C} r^2(j,C)}\left[\mathrm{v}(C) - \sum_{j\in C}\mathrm{v}(\{j\})\right] & if \;\; i \in C, \\ 0 & if \;\; i \notin C \end{cases} \tag{11}$$

This means that the players of each coalition have a much bargaining capacity, according to the linguistic modifiers or hedges used empirically in fuzzy logic. The relative error between both GDI is less than 5% with a probability of 0.95, according to the statistical comparison of both models. This solution exists and it is unique.

Definition 3. *Let (N, v) be a cooperative n-person game. An equivalent "crisp" solution to the FNSKE and the CNSKE for this game is:* $x \in S_c = \{y \in S$ such that its corresponding coalition C satisfies: $v(C) = v(N)\}$. *Where S is the solution set defined by (7), and the $f(C)$ of x is maximal among those corresponding to the elements of S_c.*

The crisp solution is comparable with the Shapley Value. This solution is not empty, because the vector solution corresponding to the grand coalition are contained in S_c.

4 The rationality of the fuzzy negotiation solutions by Knowledge Engineering

Theorem 1. *Every vector of the FNSKE and CNSKE, is an imputation.*

Proof. The fuzzy solutions obtained by (7) contains vectors having as their elements the values of the GDI for all the players in every coalition. The GDI can be calculated by (1) or (11), hence, $\forall i \in C$, the formula to calculate the elements of the vector is:

$$\mathbf{X}(i,C) = \mathrm{v}(\{i\}) + \frac{r^s(i,C)}{\sum_{j\in C} r^s(j,C)} \left[\mathrm{v}(C) - \sum_{j\in C} \mathrm{v}(\{j\}) \right] \tag{12}$$

Where $s = 1$ for the FNSKE, and $s = 2$ for the CNSKE.
$\lambda > 1$ implies that $O(C,D) > 0$ in (3). $O(C,D)$ is the function used to define $p(i,C)$ and $q(i,C)$, so they are strictly positive, and by using (5) and the fuzzy operators, $r^s(i,C) > 0 \; \forall i \in C$.
Thus, $0 < \frac{r^s(i,C)}{\sum_{j\in C} r^s(j,C)} \leq 1$ and $\mathrm{v}(C) - \sum_{j\in C} \mathrm{v}(\{j\}) \geq 0$, because v is superadditive.
Expression (12) and the last results imply that $X(i,C) \geq v(\{i\}), \forall i \in C$. Hence, $X(i,C)$ is individually rational.
Further, (12) implies that $\sum_{i\in C} \mathbf{X}(i,C) = \mathrm{v}(C)$, because $\sum_{i\in C} \frac{r^s(i,C)}{\sum_{j\in C} r^s(j,C)}$ $= 1$, then $X(i,C)$ is group rational. Also, if $i \notin C$, then $X(i,C) = 0$, hence, $\sum_{i\in N} \mathbf{X}(i,C) = \mathrm{v}(C)$.

□

Corollary 1. *The crisp solution equivalent to the FNSKE and CNSKE, is an imputation.*

Proof. It is a particular case of theorem 1.

□

Theorem 2. *Every vector of the FNSKE and CNSKE, satisfies the Efficiency and Symmetry of Shapley's axiomatic.*

Proof. Efficiency (group rational axiom) has been proved in Theorem 1 above. Now, for exploring Symmetry, let us prove that if $\pi : N \to N$ is a bijection and $i \in C$, then the solutions satisfy symmetry, this means: $\mathbf{X}_v(\pi(i), \pi(C)) = \mathbf{X}_{v\circ\pi}(i,C) \; \forall C \in 2^N$, where $\mathbf{X}_v$ and $\mathbf{X}_{v\circ\pi}$ are the GDI for n-person games with characteristic functions v and $\mathrm{v} \circ \pi$ respectively.
$\mathbf{X}^0_v(\pi(i), \pi(C)) =$

$$\begin{cases} \mathrm{v}(\{\pi(i)\}) + \eta_{i,C} \left[\mathrm{v}(\pi(C)) - \sum_{\pi(j)\in\pi(C)} \mathrm{v}(\{\pi(j)\}) \right] & if \;\; \pi(i) \in \pi(C), \\ 0 & if \;\; \pi(i) \notin \pi(C) \end{cases}$$

$$= \begin{cases} (\mathrm{v}\circ\pi)(\{i\}) + \eta_{i,C} \left[(\mathrm{v}\circ\pi)(C) - \sum_{j\in C} (\mathrm{v}\circ\pi)(\{j\}) \right] & if \;\; i \in C, \\ 0 & if \;\; i \notin C \end{cases}$$

$= \mathbf{X}^0_{v\circ\pi}(i,C)$
Where $\eta_{i,C}$ are random values such that $\eta_{i,C} \in [0 \;, 1]$ and $\mathbf{X}^0_v(\pi(i), \pi(C))$ is the initial value for the computation of the GDI by the recurrent equation:

$\mathbf{X} = \mathrm{g}(\mathbf{X})$.
Formula (3) implies that $O_{v\circ\pi}(C, D) = O_v(\pi(C), \pi(D))$. From definition of p_v and q_v, the last property and the property $\mathbf{X}_v^0(\pi(i), \pi(C)) = \mathbf{X}_{v\circ\pi}^0(i, C)$, the following equalities can be obtained: $p_v(\pi(i), \pi(C)) = p_{v\circ\pi}(i, C)$ and $q_v(\pi(i), \pi(C)) = q_{v\circ\pi}(i, C)$. Thus, from formula (5) it follows the formula: $f_v(\pi(B)) = f_{v\circ\pi}(B)$. Therefore, $s_v(\pi(i), \pi(C), \pi(D)) = s_{v\circ\pi}(i, C, D)$, $a_v(\pi(i), \pi(C)) = a_{v\circ\pi}(i, C)$, $r_v(\pi(i), \pi(C)) = r_{v\circ\pi}(i, C)$ and $\frac{r_v^s(\pi(i),\pi(C))}{\sum_{\pi(j)\in\pi(C)} r_v^s(\pi(j),\pi(C))} = \frac{r_{v\circ\pi}^s(i,C)}{\sum_{j\in C} r_{v\circ\pi}^s(j,C)}$.
Hence, the GDI obtained from the first iteration of the recurrent equation $\mathbf{X} = \mathrm{g}(\mathbf{X})$ is:
$\mathbf{X}_v^1(\pi(i), \pi(C)) =$

$$\begin{cases} \mathrm{v}(\{\pi(i)\}) + \frac{r_v^s(\pi(i),\pi(C))}{\sum_{\pi(j)\in\pi(C)} r_v^s(\pi(j),\pi(C))} \left[\mathrm{v}(\pi(C)) - \sum_{\pi(j)\in\pi(C)} \mathrm{v}(\{\pi(j)\})\right] & if \quad \pi(i) \in \pi(C), \\ 0 & if \quad \pi(i) \notin \pi(C) \end{cases}$$

$$= \begin{cases} (\mathrm{v}\circ\pi)(\{i\}) + \frac{r_v^s(\pi(i),\pi(C))}{\sum_{\pi(j)\in\pi(C)} r_v^s(\pi(j),\pi(C))} \left[(\mathrm{v}\circ\pi)(C) - \sum_{j\in C} (\mathrm{v}\circ\pi)(\{j\})\right] & if \quad i \in C, \\ 0 & if \quad i \notin C \end{cases}$$

$= \mathbf{X}_{v\circ\pi}^1(i, C)$
Following the same ideas, the m-th iteration in the calculation of the GDI satisfies: $\mathbf{X}_v^m(\pi(i), \pi(C)) = \mathbf{X}_{v\circ\pi}^m(i, C) \quad \forall m \in \mathbb{N}$, and, therefore, the limit of the last equality when $m \to \infty$ is $\mathbf{X}_v(\pi(i), \pi(C)) = \mathbf{X}_{v\circ\pi}(i, C) \quad \forall i \in N, \forall C \in 2^N$. This limit exists and it is unique, according to the results explained in section 3.

□

Corollary 2. *The crisp solution equivalent to the FNSKE and CNSKE, satisfies Efficiency and Symmetry.*

Proof. It is a particular case of theorem 2.

□

Remark 1. The FNSKE and the CNSKE don't satisfy Additivity.

This may be shown by a counterexample:
Let $(1, 2, 3, \mathrm{v})$ and $(1, 2, 3, \mathrm{v}')$ be two 3-person games.
Let us consider v and v′ as: $\mathrm{v}(\{1\}) = 1$, $\mathrm{v}(\{2\}) = 20$, $\mathrm{v}(\{1, 2\}) = 100$, $\mathrm{v}'(\{1\}) = 2$, $\mathrm{v}'(\{2\}) = 40$, $\mathrm{v}'(\{1, 2\}) = 80$.
So, $(\mathrm{v} + \mathrm{v}')(\{1\}) = 3$, $(\mathrm{v} + \mathrm{v}')(\{2\}) = 60$, $(\mathrm{v} + \mathrm{v}')(\{1, 2\}) = 180$.
Calculating the results of the GDI for the three examples in a MATLAB language's program, they are:
$\mathbf{X}_{\mathrm{v}} = \begin{bmatrix} 1 & 0 & 35.0852 \\ 0 & 20 & 64.9148 \end{bmatrix}$, $\mathbf{X}_{\mathrm{v}'} = \begin{bmatrix} 2 & 0 & 14.3027 \\ 0 & 40 & 65.6973 \end{bmatrix}$, $\mathbf{X}_{\mathrm{v}+\mathrm{v}'} = \begin{bmatrix} 3 & 0 & 47.9191 \\ 0 & 60 & 132.0809 \end{bmatrix}$ and $\mathbf{X}_{\mathrm{v}+\mathrm{v}'} \neq \mathbf{X}_{\mathrm{v}} + \mathbf{X}_{\mathrm{v}'}$.
The CNSKE doesn't satisfy either Additivity Axiom in this example above.
$\mathbf{X}_{\mathrm{v}} = \begin{bmatrix} 1 & 0 & 33.9993 \\ 0 & 20 & 66.0007 \end{bmatrix}$, $\mathbf{X}_{\mathrm{v}'} = \begin{bmatrix} 2 & 0 & 14.485 \\ 0 & 40 & 65.515 \end{bmatrix}$, $\mathbf{X}_{\mathrm{v}+\mathrm{v}'} = \begin{bmatrix} 3 & 0 & 46.9398 \\ 0 & 60 & 133.0602 \end{bmatrix}$.

5 Dummy Axiom and the fuzzy negotiation solutions by Knowledge Engineering

Theorem 3. *Let (N, v) be a cooperative n-person game, and $NP = \{i_1, i_2, \cdots, i_k\}$ the set of null players in N such that $NP \neq \emptyset$, then, the "crisp" solution satisfies the Dummy Axiom.*

Proof. Let us analyze the set CS of coalitions C, such that $C = N \backslash T$, where $T \subseteq NP$. $\tilde{S} = N \backslash NP$ is contained in every element of CS and $\tilde{S} \in CS$.
If $R \in CS$, then $\mathrm{v}(R) = \mathrm{v}(N)$, due to the definition of null player. Hence, $\tilde{S}$ is a set that contains only all the non-null players, and its characteristic function is equal to $\mathrm{v}(N)$.
$f(C) = \wedge_{j \in C}(p(j, C) \wedge q(j, C)) = \prod_{i \in C} p(i, C) \prod_{i \in C} q(i, C)$ because of the associativity and the conmutativity of the conjunction operator and the formula (5) of $f(C)$.
$\prod_{i \in C} p(i, C) \prod_{i \in C} q(i, C) = \prod_{i \in \tilde{S}} p(i, C) \prod_{i \in C \backslash \tilde{S}} p(i, C) \prod_{i \in C} q(i, C)$.
$\prod_{i \in C \backslash \tilde{S}} p(i, C) \leq 1$, hence, $f(C) \leq \prod_{i \in \tilde{S}} p(i, C) \prod_{i \in C} q(i, C)$.
$p(i, \tilde{S})$ depends on $\mathrm{v}(N) - \mathrm{v}(\tilde{S} \backslash \{i\})$ (see formula of $p(i, C)$ in section 3), and $\mathrm{v}(N) - \mathrm{v}(\tilde{S} \backslash \{i\}) \geq \mathrm{v}(N) - \mathrm{v}(C \backslash \{i\})$ because v is superadditive, hence, $p(i, \tilde{S}) \geq p(i, C)$ and $\prod_{i \in \tilde{S}} p(i, \tilde{S}) \geq \prod_{i \in \tilde{S}} p(i, C)$.
$\mathbf{X}(i, \tilde{S}) - \mathrm{v}(\{i\})$ in $q(i, \tilde{S})$ (see formula of $q(i, C)$ in section 3), is a division of the difference equal to $\mathrm{v}(N) - \sum_{j \in \tilde{S}} \mathrm{v}(\{j\})$ (see formula 1) and $\mathrm{v}(N) - \sum_{j \in \tilde{S}} \mathrm{v}(\{j\}) \geq \mathrm{v}(N) - \sum_{j \in C} \mathrm{v}(\{j\})$, where $C \in CS$, because $\tilde{S}$ is contained in the others members of CS and it is the smallest cardinality set in CS. Hence, $q(i, \tilde{S})$ is obtained from a division of the biggest remainder, by the least number of players, of all the coalitions in the CS set, in (1), and $\prod_{i \in \tilde{S}} q(i, \tilde{S})$ is the greatest value of those corresponding to the members of CS, then $f(C) \leq f(\tilde{S}) = \prod_{i \in \tilde{S}} p(i, \tilde{S}) \prod_{i \in \tilde{S}} q(i, \tilde{S})$.
$\tilde{S}$ is the set among those of CS with the greatest value of $f(C)$, that is, $\tilde{S}$ has a biggest likelihood to reach an agreement in the coalition, compared with the other members of CS.
If a coalition C is obtained by excluding a non-null player i in some member of CS, and the characteristic function value remains equal to $\mathrm{v}(N)$, then $C \backslash NP$ maintains the value of the characteristic function. Let us call $\hat{S}$ the coalition $C \backslash NP$.
Applying again the ideas exposed above, the new set $\hat{S}$ will have the biggest value of $f(C)$, compared with all members of CS, all of them without the player i. This reasoning is valid, even if more than one non-null player is excluded.
Hence, all the members of NP allways obtain the null value as the "crisp" solution equivalent to the FNSKE.
$p(i, C)$ is the same in the FNSKE and the CNSKE. $q(i, C)$ depends on the GDI, which is approximately same in both models, consequently $q(i, C)$ remains approximately equal. If the truth values of $f(C)$, according to formula (5) in the FNSKE, were ranked, then $\prod_{i \in C} q(i, C) \wedge p(i, C)$ in formula (10), maintain the order. Hence, coalitions with same number of players maintain the order of truth

values of $f(C)$ in CNSKE. If two coalitions have two different number of players, then, the coalition with the smallest number of players, which satisfies Dummy Axiom, obtains approximately the maximum of $f(C)$, because of the fitting realized to obtain the CNSKE. Also, a statistical test confirmed last proposition. □

6 An illustrative example

Example 1. Consider the game $(\{1,2,3,4\}, \mathrm{v})$, such that: $\mathrm{v}(\{1\}) = \mathrm{v}(\{2\}) = 0$, $\mathrm{v}(\{3\}) = \mathrm{v}(\{4\}) = \frac{1}{3}$, $\mathrm{v}(\{1,2\}) = 0$, $\mathrm{v}(\{1,3\}) = \mathrm{v}(\{1,4\}) = \mathrm{v}(\{2,3\}) = \mathrm{v}(\{2,4\}) = \frac{1}{3}$, $\mathrm{v}(\{3,4\}) = 1$, $\mathrm{v}(\{1,2,3\}) = \mathrm{v}(\{1,2,4\}) = \frac{1}{3}$, $\mathrm{v}(\{1,3,4\} = \mathrm{v}(\{2,3,4\}) = \mathrm{v}(\{1,2,3,4\}) = 1$.

The solution is the following set:
$S = \{(0,0,0,0;0.25), (0,0,\frac{1}{3},0;0.3294), (0,0,0,\frac{1}{3};0.3294), (0,0,0,0;0.0625),$
$(0,0,\frac{1}{3},0;0.0824), (0,0,0,\frac{1}{3};0.0824), (0,0,\frac{1}{2},\frac{1}{2},0.2244), (0,0,\frac{1}{3},0;0.0206),$
$(0,0,0,\frac{1}{3};0.0206), (0.0477,0,0.4761,0.4761;0.0564),$
$(0,0.0477,0.4761,0.4761;0.0564), (0.0424,0.0424,0.4576,0.4576;0.0141)\}$
The null players in this example are players 1 and 2. The best coalition with more than one player is $\{3,4\}$, corresponding to the biggest truth value of the likelihood to reach an agreement in the framework of the coalition (0.2244), except for the truth values of 1-person coalitions.
The best solution for players 1 and 2, if players 3 and 4 form a coalition, is not to negotiate with anybody. .
The biggest value of the conjunction of CCI2 is equal to 0.0019, corresponding to coalition $\{3,4\}$. To coalition $\{1,2\}$ corresponds a conjunction of CCI2 equal to 0.0001.
In this example, any increase in the number of players in a coalition implies a decrease in the likelihood to reach an agreement.
The CNSKE to this example is:
$S = \{(0,0,0,0;0.6708), (0,0,\frac{1}{3},0;0.7187), (0,0,0,\frac{1}{3};0.7187), (0,0,0,0;0.6373),$
$(0,0,\frac{1}{3},0;0.6596), (0,0,0,\frac{1}{3};0.6596), (0,0,\frac{1}{2},\frac{1}{2},0.7477), (0,0,\frac{1}{3},0;0.5946),$
$(0,0,0,\frac{1}{3};0.5946), (0.0571,0,0.4714,0.4714;0.6467),$
$(0,0.0571,0.4714,0.4714;0.6467), (0.0490,0.0490,0.4510,0.4510;0.5419)\}$
The truth values of the membership functions of the CNSKE can be interpreted semantically, this is the main advantage of the CNSKE over the FNSKE. The truth value 0.5419, which is the likelihood to obtain an agreement in grand coalition in CNSKE, has a cardinal meaning, interpreted as: 'slightly true'. But, the similar case in FNSKE, 0.0564, has only an ordinal interpretation and nothing can be said about this value by itself.

7 Concluding Remarks

The imputation axioms and the Shapley's axiom set, except Additivity, are considered rational in the classical approach to cooperative n-person games, accord-

ing to the concept of rationality in Decision Theory, particulary in Normative Decision Theory. The consensus about rationality concerning such axioms can be used to prove the rationality of other solutions to cooperative n-person games.
The FNSKE and CNSKE models based on experts' knowledge on bargaining are recent solutions to cooperative n-person games. In this paper, a demonstration has been developed to show that the elements of this solution are imputations. Hence, these elements are group rational and individually rational.
The elements of these solutions satisfy two Shapley's axioms: Efficiency and Symmetry. The fact that the elements of these solutions do not satisfy Additivity, has been exposed by a counterexample.
A crisp solution to the game equivalent to the FNSKE and the CNSKE is a solution for the coalition with the same value of the grand coalition characteristic function, having the biggest value of likelihood to reach an agreement. It has been proved that this crisp solution satisfies the Dummy Axiom of the Shapley's axiom set.
The rationality of the FNSKE and CNSKE, can be considered a proof that these new solutions can be used to solve any cooperative n-person game problem, not only negotiation and bargaining problems.

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