# Normal Modal Resolution: Preliminary Results

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***Abstract.*** *We present a clausal resolution-based method for normal modal logics. Differently from other approaches, where inference rules are based on the syntax of a particular set of axioms, we focus on the restrictions imposed on the binary accessibility relation for each particular normal logic.*

## 1. Introduction

Several aspects of complex computational systems can be modelled by means of normal modal logics. Some of them can be used as a natural way to model notions such as belief and knowledge [Rao and Georgeff 1991] and, also, in a variety of applications related to distributed systems [Fagin et al. 1995]. Given a logical specification, a theorem prover can then be used for verifying properties of the system. In order to model the different aspects of a particular situation, it may be necessary to combine different logical languages. When the combination is a *fusion* of logics [Gabbay et al. 2003], i.e., the components are independently axiomatisable, proofs can be obtained by combining the provers for each language. This combination may require special care such that all relevant information is correctly handled and exchanged between the different tools. Also, it may require the use of tools based on different implementations (e.g. different input languages) or on different approaches (e.g. partially based on translation to first-order language $\times$ partially based on the modal language, resolution $\times$ tableau, etc), making this task harder.

We are currently investigating a uniform approach to deal with theorem proving for several *propositional normal modal logics*. Modal logics are usually extensions of classical logics with two additional unary operators: '$\Box$' and '$\Diamond$', whose meanings depend on the framework in which they are defined. The usual interpretations are "is necessary" and "is possible", respectively. So, formulae '$\Box p$' and '$\Diamond p$' are to be read as "p is necessary" and "p is possible", respectively. Evaluation of a modal formula depends on a set of interpretations, also called set of *possible worlds*. Given a world $\omega$, a set of worlds, which are *accessible* from $\omega$, is defined. Different modal logics define different *accessibility relations* between worlds. Worlds and their accessibility relations define a structure known as a *Kripke structure*. The evaluation of a formula depends on this structure. Given a Kripke structure and a world $\omega$, a formula $\Box p$ is true at $\omega$, if $p$ is true at all worlds accessible from $\omega$; $\Diamond p$ is true at $\omega$, if $p$ is true at some world accessible from $\omega$.

In normal modal logics, the schema $\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$ (the axiom **K**), where $\varphi$ and $\psi$ are well-formed formulae, is valid. We are interested in multi-modal normal logics based on **K** and the axioms: **T** ($\Box\varphi \Rightarrow \varphi$), **D** ($\Box\varphi \Rightarrow \Diamond\varphi$), **4:**

($\Box\varphi \Rightarrow \Box\Box\varphi$), **5:** ($\Diamond\varphi \Rightarrow \Box\Diamond\varphi$), and **B:** ($\Diamond\Box\varphi \Rightarrow \varphi$). We formally introduce the weakest of these logics, $\mathsf{K}_{(n)}$, in Section 2. The addition of those axioms to $\mathsf{K}_{(n)}$, where only the axiom **K** holds, impose some restrictions on the class of models where formulae are valid. Thus, a formula valid in a logic containing **T** is valid only if it is valid in a **reflexive** model. The other axioms, **D**, **4**, **5**, and **B** restrict the models to be **serial**, **transitive**, **euclidean**, and **symmetric**, respectively. We have chosen clausal resolution as the proof method for each logic. Although it is well-known that resolution behaves badly *for certain classes of problems*, as shown in [Cook 1971] and [Urquhart 1987], this proof system is still the most widely implemented, tested, and used method in automatic theorem proving. Besides reliable implementations, we can also profit from several complete refinements of the method (e.g. complete strategies), which can be extended to deal with modal resolution. The clausal normal form for all logics is the same and it is presented in Subsection 2.1. The resolution rules for $\mathsf{K}_{(n)}$ are presented in Subsection 2.2. Correctness results are given in Subsection 2.3. We present the other modal systems and their inference rules in Section 3. Concluding remarks are given in Section 4.

## 2. The Normal Logic $\mathsf{K}_{(n)}$

The weakest of the normal modal systems, known as $\mathsf{K}_{(n)}$, is an extension of the classical propositional logic with the operators $\boxed{i}$, $1 \leq i \leq n$, where only the axiom **K** holds. There is no restriction on the accessibility relation over worlds. As the subscript in $\mathsf{K}_{(n)}$ indicates, we consider the multi-agent version, that is, the fusion of several copies of $\mathsf{K}_{(1)}$.

Formulae are constructed from a denumerable set of **propositional symbols**, $\mathcal{P} = \{p, q, p', q', p_1, q_1, \ldots\}$. The finite set of agents is defined as $\mathcal{A} = \{1, \ldots, n\}$. Besides the propositional connectives ($\neg, \wedge$), we introduce a set of unary modal operators $\boxed{1}, \ldots, \boxed{n}$, where $\boxed{i}\varphi$ is read as "agent $i$ considers $\varphi$ necessary". When $n = 1$, we may omit the index, that is, $\Box\varphi = \boxed{1}\varphi$. We do not define the operator $\Diamond$: the fact that an "agent $i$ considers $\varphi$ possible" is expressed by $\neg\boxed{i}\neg\varphi$. The language of $\mathsf{K}_{(n)}$ is defined as follows:

**Def. 2.1** *The* **set of well-formed formulae***,* $\mathsf{WFF}_{\mathsf{K}_{(n)}}$*, is given by:*

- *the propositional symbols are in* $\mathsf{WFF}_{\mathsf{K}_{(n)}}$*;*
- **true** *is in* $\mathsf{WFF}_{\mathsf{K}_{(n)}}$*;*
- *if* $\varphi$ *and* $\psi$ *are in* $\mathsf{WFF}_{\mathsf{K}_{(n)}}$*, then so are* $\neg\varphi$*,* $(\varphi \wedge \psi)$*, and* $\boxed{i}\varphi$ $(\forall i \in \mathcal{A})$.

A **literal** is either a proposition or its negation. $\mathcal{L}$ is the set of literals. A **modal literal** is either $\boxed{i}l$ or $\neg\boxed{i}l$, where $l \in \mathcal{L}$ and $i \in \mathcal{A}$.

Semantics of $\mathsf{K}_{(n)}$ is given, as usual, in terms of a Kripke structure.

**Def. 2.2** *A* **Kripke structure** *$M$ for $n$ agents over $\mathcal{P}$ is a tuple $M = \langle \mathcal{S}, \pi, \mathcal{R}_1, \ldots, \mathcal{R}_n \rangle$, where $\mathcal{S}$ is a set of possible* worlds *(or* states*) with a distinguished world $s_0$; the function $\pi(s) : \mathcal{P} \rightarrow \{true, false\}$, $s \in \mathcal{S}$, is an interpretation that associates with each state in $\mathcal{S}$ a truth assignment to propositions; and each $\mathcal{R}_i \subseteq S \times S$ is a binary relation on $\mathcal{S}$.*

The binary relation $\mathcal{R}_i$ captures the possibility relation according to agent $i$: a pair $(s, t)$ is in $\mathcal{R}_i$ if agent $i$ considers world $t$ possible, given her information in world $s$. We write $(M, s) \models \varphi$ to express that $\varphi$ is true at world $s$ in the Kripke structure $M$.

**Def. 2.3** *Truth of a formula is given as follows:*

- $(M, s) \models$ **true**
- $(M, s) \models p$ *if, and only if,* $\pi(s)(p) = true$*, where* $p \in \mathcal{P}$
- $(M, s) \models \neg\varphi$ *if, and only if,* $(M, s) \not\models \varphi$
- $(M, s) \models (\varphi \wedge \psi)$ *if, and only if,* $(M, s) \models \varphi$ *and* $(M, s) \models \psi$
- $(M, s) \models \boxed{i}\varphi$ *if, and only if, for all* $t$*, such that* $(s, t) \in \mathcal{R}_i$*,* $(M, t) \models \varphi$*.*

The formulae **false**, $(\varphi \vee \psi)$, and $(\varphi \Rightarrow \psi)$ are introduced as the usual abbreviations for $\neg$**true**, $\neg(\neg\varphi \wedge \neg\psi)$, and $(\neg\varphi \vee \psi)$, respectively. Formulae are interpreted with respect to the distinguished world $s_0$. Intuitively, $s_0$ is the world from which we start reasoning. Let $M = \langle \mathcal{S}, \pi, \mathcal{R}_1, \ldots, \mathcal{R}_n \rangle$ be a Kripke structure. Thus, a formula $\varphi$ is said to be **satisfiable in M** if $(M, s_0) \models \varphi$; it is said to be **satisfiable** if there is a model $M$ such that $(M, s_0) \models \varphi$; and it is said to be **valid** if for all models $M$ then $(M, s_0) \models \varphi$.

### 2.1. A Normal Form for $\mathsf{K}_{(n)}$

Formulae in the language of $\mathsf{K}_{(n)}$ can be transformed into a normal form called Separated Normal Form for Normal Logics ($\mathsf{SNF}_K$). We introduce a nullary connective **start**, in order to represent the world from which we start reasoning. Formally, we have that $(M, s) \models$ **start** if, and only if, $s = s_0$. A formula in $\mathsf{SNF}_K$ is represented by a conjunction of clauses, which are true at all states, that is, they have the general form

$$\Box^* \bigwedge_i A_i$$

where $A_i$ is a clause and $\Box^*$, the universal operator, is defined as: $(M, s) \models \Box^*\varphi$ if, and only if, $(M, s) \models \varphi$, and for all $s'$ such that $(s, s') \in \mathcal{R}_i$, for some $i \in \mathcal{A}$, $(M, s') \models \Box^*\varphi$. Observe that $\varphi$ holds at the actual world $s$ and at every world reachable from $s$, where reachability is defined in the usual way. That is, let $M$ be a model and $u$ and $u'$ be worlds in $M$. Then $u'$ is reachable from $u$ if, and only if, either (i) $(u, u') \in \mathcal{R}_i$ for some agent $i \in \mathcal{A}$; or (ii) there is a world $u''$ in $M$ such that $u''$ is reachable from $u$ and $u'$ is reachable from $u''$. The universal operator, which surrounds all clauses, ensures that the *translation* of a formula is true at all worlds. Clauses are in one of the following forms:

- Initial clause $\quad$ **start** $\Rightarrow \bigvee_{b=1}^{r} l_b$
- Literal clause $\quad$ **true** $\Rightarrow \bigvee_{b=1}^{r} l_b$
- $\boxed{i}$-clause $\quad l \Rightarrow m_i$

where $l, l_b \in \mathcal{L}$, and $m_i$ is a modal literal containing a $\boxed{i}$ or a $\neg\boxed{i}$ operator. We often say *modal clause* to refer to a $\boxed{i}$-clause. Transformation rules, which are based on classical rewriting and renaming, together with the proof that they are satisfiability preserving can be found in [Nalon and Dixon 2006a] and [Nalon and Dixon 2006b].

### 2.2. Inference Rules for $\mathsf{K}_{(n)}$

In the following, $l, l_i \in \mathcal{L}$; $m_i$ are modal literals; and $D$, $D'$ are disjunctions of literals.

**Literal Resolution.** This is classical resolution applied to the propositional portion of the combined logic. An initial clause may be resolved with either a literal clause or an initial clause (IRES1 and IRES2). Literal clauses can be resolved together (LRES):

$$
\text{[IRES1]}\ \begin{array}{rcl} \mathbf{true} & \Rightarrow & D \lor l \\ \mathbf{start} & \Rightarrow & D' \lor \neg l \\ \hline \mathbf{start} & \Rightarrow & D \lor D' \end{array}
\qquad
\text{[IRES2]}\ \begin{array}{rcl} \mathbf{start} & \Rightarrow & D \lor l \\ \mathbf{start} & \Rightarrow & D' \lor \neg l \\ \hline \mathbf{start} & \Rightarrow & D \lor D' \end{array}
\qquad
\text{[LRES]}\ \begin{array}{rcl} \mathbf{true} & \Rightarrow & D \lor l \\ \mathbf{true} & \Rightarrow & D' \lor \neg l \\ \hline \mathbf{true} & \Rightarrow & D \lor D' \end{array}
$$

**Modal Resolution.** These rules are applied between clauses which refer to the same context, that is, they must refer to the same agent. For instance, we can resolve two $\boxed{1}$-clauses; or several $\boxed{1}$-clauses and a literal clause. The modal inference rules are:

$$
\text{[MRES]}\ \begin{array}{rcl} l & \Rightarrow & m_i \\ l' & \Rightarrow & \neg m_i \\ \hline \mathbf{true} & \Rightarrow & \neg l \lor \neg l' \end{array}
\qquad
\text{[GEN]}\ \begin{array}{rcl} l'_1 & \Rightarrow & \boxed{i}\neg l_1 \\ & \vdots & \\ l'_m & \Rightarrow & \boxed{i}\neg l_m \\ l' & \Rightarrow & \neg\boxed{i}\neg l \\ \mathbf{true} & \Rightarrow & l_1 \lor \ldots \lor l_m \lor \neg l \\ \hline \mathbf{true} & \Rightarrow & \neg l' \lor \neg l'_1 \lor \ldots \lor \neg l'_m \end{array}
$$

$$
\text{[GEN2]}\ \begin{array}{rcl} l'_1 & \Rightarrow & \boxed{i} l_1 \\ l'_2 & \Rightarrow & \boxed{i}\neg l_1 \\ l'_3 & \Rightarrow & \neg\boxed{i}\neg l_2 \\ \hline \mathbf{true} & \Rightarrow & \neg l'_1 \lor \neg l'_2 \lor \neg l'_3 \end{array}
$$

MRES is equivalent to classical resolution, as a formula and its negation cannot be true at the same state. The GEN rule corresponds to generalisation and several applications of classical resolution: as clauses are true at every state, we can rewrite the literal clause as $\mathbf{true} \Rightarrow \boxed{i}(l_1 \lor \ldots \lor l_m \lor \neg l)$; by propositional reasoning and by the axiom **K**, we have $\mathbf{true} \Rightarrow \neg\boxed{i}\neg l_1 \lor \ldots \lor \neg\boxed{i}\neg l_m \lor \boxed{i}\neg l$; the modal literals in this formula can then be resolved with their complements in the other parent clauses. GEN2 is a special case of GEN, as the parent clauses can be resolved with tautologies as $\mathbf{true} \Rightarrow l_1 \lor \neg l_1 \lor l_2$.

### 2.3. Correctness Results

The proof method for $\mathsf{K}_{(n)}$ consists of rules IRES1, IRES2, LRES, MRES, and GEN. The proofs that the first three rules are sound can be easily obtained as they correspond to propositional resolution, but applied to the specific normal form presented here. MRES is also sound, as $m_i$ and $\neg m_i$ cannot be both true at the same state. The following is the soundness proof for the GEN rule. Soundness of GEN2 follows from soundness of GEN.

**Soundness of GEN.** The GEN rule corresponds to several applications of modal resolution. If $(M,t) \models \neg l'$, the resolvent is trivially true. Next, we assume that $(M,t) \models l'$. Suppose that $(M,t) \models l' \Rightarrow \neg\boxed{i}\neg l$. Hence, by semantics of implication, $(M,t) \models \neg\boxed{i}\neg l$. By the semantics of the necessity operator, there is a state $t'$, such that $(t,t') \in \mathcal{R}_i$ and $(M,t') \models l$. As $\mathbf{true} \Rightarrow l_1 \lor \ldots \lor l_m \lor \neg l$ holds everywhere, we have $(M,t') \models l_1 \lor \ldots \lor l_m \lor \neg l$. By the resolution principle, we have $(M,t') \models l_1 \lor \ldots \lor l_m$. By semantics of $\boxed{i}$, $(M,t) \models \neg\boxed{i}\neg(l_1 \lor \ldots \lor l_m)$. Equivalently, $(M,t) \models \neg\boxed{i}\neg l_1 \lor \ldots \lor \neg\boxed{i}\neg l_m$. Observe that this formula will contribute to finding a contradiction only in the case where all disjuncts can be resolved with their correspondent complementary modal literals by several applications of MRES. In order to avoid renaming of $\neg\boxed{i}\neg l_1 \lor \ldots \lor \neg\boxed{i}\neg l_m$, we use the first $m$ clauses to generate the resolvent.

**Termination** is guaranteed as no new propositional symbols need to be defined after translation. As we have a finite number of symbols, only a finite number of right-hand sides of clauses can be defined, so at some point either we derive a contradiction or no new clauses can be generated.

**Completeness** can be proved by induction on the number of nodes of a graph, built from a set of clauses. We prove that an empty graph corresponds to an unsatisfiable set of clauses and that, in this case, there is a refutation by the resolution method presented here.

Formally, the graph is a pair $\mathcal{G} = \langle \mathcal{N}, \mathcal{E} \rangle$, built from the set of $\mathsf{SNF}_K$ clauses $\mathcal{T}$, where $\mathcal{N}$ is a set of nodes and $\mathcal{E}$ is a set of labelled edges. Intuitively, $\mathcal{N}$ corresponds to states, i.e., a consistent set of literals and modal literals occurring in $\mathcal{T}$. There are $n$ types of edges representing the accessibility relations of each agent in $\mathcal{A}$. An edge labelled by $i \in \mathcal{A}$ is called an $i$-edge. Let $\eta$ and $\eta'$ be nodes. We say that $\eta'$ is $i$-reachable from $\eta$, if there is an $i$-edge between $\eta$ and $\eta'$. We say that $\eta$ is immediately reachable, if $\eta$ is $i$-reachable from any $\eta'$, for some $i \in \mathcal{A}$. First, we define truth of a formula with respect to a set of literals:

**Def. 2.4** *Let $\mathcal{V}$ be a consistent set of literals and modal literals. Let $\varphi$, $\psi$, and $\psi'$ be a boolean combinations of literals and modal literals. We say that $\mathcal{V}$ satisfies $\varphi$ (written $\mathcal{V} \models \varphi$), if, and only if:*

- *$\varphi \in \mathcal{V}$, if $\varphi$ is a literal or modal literal;*
- *$\varphi$ is of the form $\psi \wedge \psi'$ and $\mathcal{V} \models \psi$ and $\mathcal{V} \models \psi'$;*
- *$\varphi$ is of the form $\psi \vee \psi'$ and $\mathcal{V} \models \psi$ or $\mathcal{V} \models \psi'$;*
- *$\varphi$ is of the form $\neg\psi$ and $\mathcal{V}$ does not satisfy $\psi$ (written $\mathcal{V} \not\models \psi$).*

We also define satisfiability of a formula and a set of formulae with respect to a node:

**Def. 2.5** *Let $\mathcal{V}$ be a consistent set of literals and modal literals, $\eta$ be a node such that $\eta$ contains all literals in $\mathcal{V}$, $\varphi$ be a boolean combination of literals and modal literals, and $\chi = \{\varphi_1, \ldots, \varphi_m\}$ be a set of formulae, where each $\varphi_i$, $1 \leq i \leq m$, is a boolean combination of literals and modal literals. We say that $\eta$* **satisfies** *$\varphi$ (written $\eta \models \varphi$) if, and only if, $\mathcal{V} \models \varphi$. We say that $\eta$* **satisfies** *$\chi$ (written $\eta \models \chi$) if, and only if, $\eta \models \varphi_1 \wedge \ldots \wedge \varphi_m$.*

Let $\mathcal{T}$ be a set of clauses into $\mathsf{SNF}_K$. We construct a finite direct **graph** $\mathcal{G} = \langle \mathcal{N}, \mathcal{E} \rangle$ **for** $\mathcal{T}$, where $\mathcal{N}$ is a set of nodes and $\mathcal{E}$ is a set of labelled edges, as follows. A node $\eta \in \mathcal{N}$ is a maximal consistent set of literals and modal literals. Formally, we first construct all possible sets of consistent modal literals, $\mathcal{V}_i(l)$, for all $l \in \mathcal{L}$, for all $i \in \mathcal{A}$:

$$\mathcal{V}_i(l) = \{\{l, \boxed{i}\,l, \neg\boxed{i}\,\neg l\}, \{l, \boxed{i}\,l, \boxed{i}\,\neg l\}, \{l, \neg\boxed{i}\,l, \neg\boxed{i}\,\neg l\}, \{l, \neg\boxed{i}\,l, \boxed{i}\,\neg l\}\}$$

As $l \in \mathcal{L}$, $\mathcal{V}_i(l)$ contains a proposition or its negation. Besides, it contains $\boxed{i}\,l$, if, and only if, it does not contain $\neg\boxed{i}\,l$. Note that a node satisfies both $\boxed{i}\,l$ and $\boxed{i}\,\neg l$ if there is no $i$-edge to another node. The union of $\mathcal{V}_i(l)$ and $\mathcal{V}_j(l)$, $i \neq j$, is **consistent** if both sets satisfy the same propositional symbols, i.e. $\mathcal{V}_i(l) \models l$ if, and only if, $\mathcal{V}_j(l) \models l$. We take $\mathcal{V}(l)$ to be a consistent union of $\mathcal{V}_i(l)$ for every agent $i$. We take $\eta \in \mathcal{N}$ to be the union of $\mathcal{V}(l)$ for every $l \in \mathcal{L}$. Note that, by construction, there are no nodes containing the modal literal $m_i$ and its negation, which corresponds to applications of the inference rule MRES.

Next, we delete any nodes that do not satisfy the literal and modal clauses in $\mathcal{T}$, that is, if **true** $\Rightarrow l_1 \vee \ldots \vee l_m \in \mathcal{T}$, we delete the nodes $\eta \in \mathcal{N}$ such that $\eta \models \neg l_1 \wedge \ldots \wedge \neg l_m$. Also, if $l \Rightarrow m_i \in \mathcal{T}$, we delete nodes $\eta \in \mathcal{N}$ such that $\eta \models l \wedge \neg m_i$. This ensures that literal and/or modal clauses in $\mathcal{T}$ are satisfiable by nodes $\eta \in \mathcal{G}$.

Given a non-empty set of nodes, we construct the set of labelled edges, $\mathcal{E}$, as follows. First, for each agent $i \in \mathcal{A}$, for some $l \in \mathcal{L}$, there is an $i$-edge between two nodes $\eta$ and $\eta'$, if and only if, $\eta \models \neg\boxed{i}\,l$ and $\eta' \models \neg l$. Next, for every node $\eta$ that satisfies a set of modal literals $\{\boxed{i}\,l_1, \ldots, \boxed{i}\,l_m\}$, we delete $i$-edges to all nodes that do not satisfy $l_1 \wedge \ldots \wedge l_m$. If $\neg\boxed{i}\,l \in \eta$ and there is no $i$-edge to any node $\eta'$, we delete $\eta$. This corresponds to an application of the either the inference rule GEN or GEN2. We regard the initial state as part of the construction of the graph. Intuitively, the initial states are those which satisfy all the right-hand sides of initial clauses. If all initial states are deleted, the graph is empty.

If the resulting graph is empty, the set of clauses $\mathcal{T}$ is not satisfiable and there is a resolution proof corresponding to the deletion procedure, as described above. If the graph is not empty, it provides a model for the satisfiable set of clauses $\mathcal{T}$. $\blacksquare$

## 3. The Other Normal Modal Logics

There are 24 possible combinations between the axiom **K** and the axioms **T**, **D**, **4**, **5**, and **B**, but because **T** implies **D**; **B** and **5** implies **4**; **T** and **5** implies **B**; and **4**, **B**, and **D** implies **T**, there are only fifteen different normal modal logics [Chellas 1980]: K, T, KD, KB, K4, K5, KTB = B, KT4=S4, KT5=S5, K4B, K45, KDB, KD4, KD5, and KD45.

From correspondence theory, we know that formulae valid in normal logics which include the axiom **T** (resp. **D**, **4**, **5**, and **B**) are valid in the class of Kripke structures whose accessibility relations are **reflexive** (resp. **serial**, **transitive**, **euclidean**, and **symmetric**). In the following, we provide resolution-based inference rules that captures those restrictions on the accessibility relation. We indicate how soundness can be proved. Full proofs will be given in the extended version. The inference rules are applied together with IRES1, IRES2, LRES, MRES, GEN and GEN2 to a set of $\mathsf{SNF}_K$ clauses until a contradiction is found or no new clauses can be generated. The idea is that by combining those rules, we hope to achieve complete resolution-based systems for all the fifteen normal modal logics listed above. Completeness proof of those proof methods is ongoing work. In the following, $l$, $l_i \in \mathcal{L}$; $m_i$ are modal literals; and $D$, $D'$ are disjunctions of literals.

### 3.1. Inference Rules for Reflexive Logics

$$\text{[REF1]}\quad \begin{array}{rcl} l_1 & \Rightarrow & \boxed{i}\, l \\ \mathbf{true} & \Rightarrow & D \vee \neg l \\ \hline \mathbf{true} & \Rightarrow & \neg l_1 \vee D \end{array} \qquad \text{[REF2]}\quad \begin{array}{rcl} l_1 & \Rightarrow & \boxed{i}\, l \\ l_2 & \Rightarrow & \boxed{j}\, \neg l \\ \hline \mathbf{true} & \Rightarrow & \neg l_1 \vee \neg l_2 \end{array}$$

Firstly, as $l_1 \Rightarrow \boxed{i}\, l$ implies $\neg l_1 \vee l$, by resolution, we obtain the resolvent in REF1. Secondly, as $l_2 \Rightarrow \boxed{j}\, \neg l$ implies $\neg l_2 \vee \neg l$, by REF1, we obtain the resolvent in REF2.

### 3.2. Inference Rules for Serial Logics

$$\text{[SER]}\quad \begin{array}{rcl} l_1 & \Rightarrow & \boxed{i}\, l \\ l_2 & \Rightarrow & \boxed{i}\, \neg l \\ \hline \mathbf{true} & \Rightarrow & \neg l_1 \vee \neg l_2 \end{array}$$

Because of **D**, $l$ and $\neg l$ cannot be both true in all accessible worlds; thus, at least one of the antecedents in the parents is false.

### 3.3. Inference Rules for Transitive Logics

$$\text{[TRANS1]}\quad \begin{array}{rcl} l_1 & \Rightarrow & \boxed{i}\, l \\ l_2 & \Rightarrow & \neg \boxed{i}\, l \\ \hline l_1 & \Rightarrow & \boxed{i}\, \neg l_2 \end{array} \qquad \text{[TRANS2]}\quad \begin{array}{rcl} l_1 & \Rightarrow & \neg \boxed{i}\, \neg l_2 \\ l_2 & \Rightarrow & \neg \boxed{i}\, l \\ \hline l_1 & \Rightarrow & \neg \boxed{i}\, l \end{array}$$

If $l_1 \Rightarrow \boxed{i}\, l$ is satisfied, because of **4**, then so is $l_1 \Rightarrow \boxed{i}\,\boxed{i}\, l$. If any immediately reachable world satisfies $l_2$, it also satisfies $\neg \boxed{i}\, l$. TRANS1 ensures that if $l_1$ is satisfied, then $l_2$ is not satisfied in any immediately reachable world. Note that the parent clauses are the same then those in MRES, but that the resolvent in this case is much stronger. TRANS2 is justified by the converse of **4**, that is, $\neg \boxed{i} \neg \neg \boxed{i} \neg \varphi \Rightarrow \neg \boxed{i} \neg \varphi$, for any formula $\varphi$.

### 3.4. Inference Rules for Euclidean Logics

$$
\begin{array}{lrcl}
\text{[EUC1]} & l_1 & \Rightarrow & \neg\boxed{i}l \\
 & l_2 & \Rightarrow & \boxed{i}l \\
\hline
 & l_1 & \Rightarrow & \boxed{i}\neg l_2
\end{array}
\qquad
\begin{array}{lrcl}
\text{[EUC2]} & l_1 & \Rightarrow & \neg\boxed{i}\neg l_2 \\
 & l_2 & \Rightarrow & \boxed{i}l \\
\hline
 & l_1 & \Rightarrow & \boxed{i}l
\end{array}
$$

Because of **5**, if $l_1 \Rightarrow \neg\boxed{i}l$ is satisfied, then $\neg\boxed{i}l$ is satisfied at all immediately reachable worlds. The resolvent in EUC1 says that $l_2$ is not satisfied in any of those worlds. EUC2 is justified by the converse of **5**, that is, $\neg\boxed{i}\neg\boxed{i}\varphi \Rightarrow \boxed{i}\varphi$, for any formula $\varphi$.

### 3.5. Inference Rules for Symmetric Logics

$$
\begin{array}{lrcl}
\text{[SYM1]} & l_1 & \Rightarrow & \boxed{i}\neg l \\
\hline
 & l & \Rightarrow & \boxed{i}\neg l_1
\end{array}
\qquad
\begin{array}{lrcl}
\text{[SYM2]} & l_1 & \Rightarrow & \neg\boxed{i}\neg l \\
\hline
 & l & \Rightarrow & \neg\boxed{i}\neg l_1
\end{array}
$$

If $l_1$ is satisfied, any immediately reachable world satisfies $\neg l$. If $l$ is satisfied, by symmetry, $l_1$ should not be satisfiable in any immediately reachable world, as in the resolvent in SYM1. SYM2 says that if $l_1$ is satisfied in a world, then $l$ is satisfied in some immediately reachable world. To ensure symmetry, if $l$ is satisfied, so it is $\neg\boxed{i}\neg l_1$.

### 3.6. A Short Example

We prove that the formula $\neg\boxed{i}\neg\boxed{i}p \Rightarrow p$, an instance of the axiom **B**, can be proved in $\mathsf{S5}_{(n)}$, where the accessibility relation is euclidean and reflexive (which implies that it is also symmetric). The proof method for $\mathsf{S5}_{(n)}$ applies the rules of inference in Subsections 2.2, 3.1, and 3.4. The translation of the negated formula into $\mathsf{SNF}_K$ is given below:

$$
\begin{array}{lrcl}
1. & \textbf{start} & \Rightarrow & t_1 \\
2. & t_1 & \Rightarrow & \neg\boxed{i}\neg t_2 \\
3. & t_2 & \Rightarrow & \boxed{i}p \\
4. & \textbf{true} & \Rightarrow & \neg t_1 \vee \neg p
\end{array}
$$

where clause 1 corresponds to the whole formula, anchored to the initial state; clauses 2 and 3 correspond to the antecedent; and clause 4 corresponds to the negation of the consequent. The refutation proceeds as follows:

$$
\begin{array}{lrcll}
5. & t_1 & \Rightarrow & \boxed{i}p & \text{[2, 3, EUC2]} \\
6. & \textbf{true} & \Rightarrow & \neg t_1 & \text{[5, 4, REF1]} \\
7. & \textbf{start} & \Rightarrow & \textbf{false} & \text{[6, 1, IRES1]}
\end{array}
$$

## 4. Conclusions

As in the propositional case, there is a variety of proof methods for modal logics. Non-clausal methods have been proposed by [Abadi and Manna 1986, Fitting 89, Areces et al. 1999], but they do not apply to all normal systems. Also, a survey on translated-based approaches for non-transitive modal logics (i.e. modal logics that do not include the axiom **4**) can be found in [de Nivelle et al. 2000]. Clausal resolution (e.g. [Mints 1990, Dixon and Fisher 2000]) are applied only to few modal systems.

We have shown a sound, complete and terminating resolution-based proof method for $\mathsf{K}_{(n)}$, the weakest normal logic. We have also shown rules for normal systems based on the axioms **T**, **D**, **4**, **5**, and **B** and have indicated how those rules can be proved sound. We are currently working on the completeness proofs for those proof methods. Once completeness is proved, we will investigate the complexity of the method for each logic. Implementations as well model generation for satisfiable sets of clauses are future work.


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