Improved Undecidability Results on the Emptiness Problem of Probabilistic and Quantum Cut-Point Languages

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Preliminaries

- *Stochastic* (*Markov*) matrix: Each column is a probability distribution.
- *Adjoint* matrix: $(M^*)_{ij} = (M_{ji})^*$.
- *Unitary* matrix: $U^*U = UU^* = I$.
- *Alphabet* $\Sigma = \{a_1, \ldots, a_k\}$ is a finite set.
- *Words* over Σ: Σ^*.

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- An n-state *probabilistic automaton* over Σ: $P = (\boldsymbol{x}, \{M_a \mid a \in \Sigma\}, \boldsymbol{y})$. $\boldsymbol{y} \in \mathbb{R}^n$ is the *initial distribution*, $\boldsymbol{x} \in \{0,1\}^n$ is the *final state vector*, and each M_a is an $n \times n$ stochastic matrix.

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- An n-state *quantum automaton* over Σ: $Q = (P, \{U_a \mid a \in \Sigma\}, \boldsymbol{y})$. $\boldsymbol{y} \in \mathbb{C}^n$ is the *initial amplitude vector* with $||\boldsymbol{y}|| = 1$, P is the *measurement projection*, and each U_a is an $n \times n$ unitary matrix.

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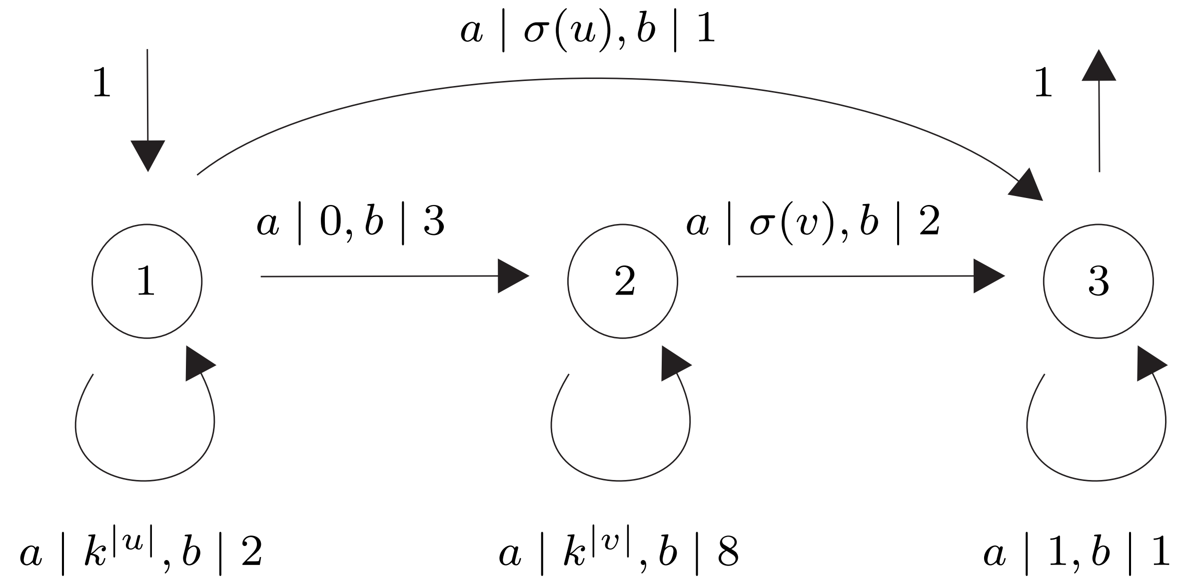
- An n-state *quantum automaton* over Σ: $Q = (P, \{U_a \mid a \in \Sigma\}, \boldsymbol{y})$. $\boldsymbol{y} \in \mathbb{C}^n$ is the *initial amplitude vector* with $||\boldsymbol{y}|| = 1$, P is the *measurement projection*, and each U_a is an $n \times n$ unitary matrix.
- A $\mathbb{Z}$*-automaton*: matrices and vectors with integer entries.

Graph Representation



$$M_a = \begin{pmatrix} k^{|u|} & 0 & 0 \\ 0 & k^{|v|} & 0 \\ \sigma(u) & \sigma(v) & 1 \end{pmatrix}, \quad M_b = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 8 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

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- Let $Q = (P, \{U_a \mid a \in \Sigma\}, \boldsymbol{y})$ be a quantum automaton over Σ.
- For $w = a_1 \ldots a_r \in \Sigma^*$, $f_Q(w)$ is defined as $f_Q(w) = ||PU_{a_r} \cdot \ldots \cdot U_{a_1} \boldsymbol{y}||^2$.

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The problems studied: given a binary automaton A and λ, is $L_{\geq\lambda}(A) = \emptyset$? Is $L_{>\lambda}(A) = \emptyset$?

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- Undecidable if all minimal solutions are of form $i_1 = 1$, $i_n = k$, and $i_2 \ldots i_{n-1} \in \{2, \ldots, k-1\}^+$. (V. Claus 1980).

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$$\gamma(u,v) = \begin{pmatrix} 2^{2|u|} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2^{|uv|} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^{2|v|} & 0 & 0 & 0 \\ \sigma(u)2^{|u|} & \sigma(v)2^{|u|} & 0 & 2^{|u|} & 0 & 0 \\ 0 & \sigma(u)2^{|v|} & \sigma(v)2^{|v|} & 0 & 2^{|v|} & 0 \\ \sigma(u)^2 & 2\sigma(u)\sigma(v) & \sigma(v)^2 & 2\sigma(u) & 2\sigma(v) & 1 \end{pmatrix}$$

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- γ is an embedding $\Sigma^* \times \Sigma^* \to \mathbb{N}^{6\times 6}$.
- $\boldsymbol{x}_1^T \gamma(u,v) \boldsymbol{y}_1 = 1 - (\sigma(u) - \sigma(v))^2$ for $\boldsymbol{x}_1 = (0,0,0,0,0,1)^T$ and $\boldsymbol{y}_1 = (-1,1,-1,0,0,1)^T$.

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- $\boldsymbol{x}_1^T \gamma(u,v) \boldsymbol{y}_1 > 0$ if and only if $u = v$.

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- $f_A(w) > 0$ for some $w \in \Sigma^+$ undecidable for $\mathbb{Z}$-automata with 6 states and $k = 7$ alphabet symbols.
- A shorthand notation: For $w = i_1 \ldots i_n$, let $A_w = A_{i_1} \ldots A_{i_n}$.

Reducing the number of matrices

Reducing the number of matrices

- $\boldsymbol{x}_2 = (\boldsymbol{x}_1^T A_1)^T$, $\boldsymbol{y}_2 = A_k \boldsymbol{y}_1$, $B_1 = A_2$, $\ldots$, $B_{k-2} = A_{k-1}$.

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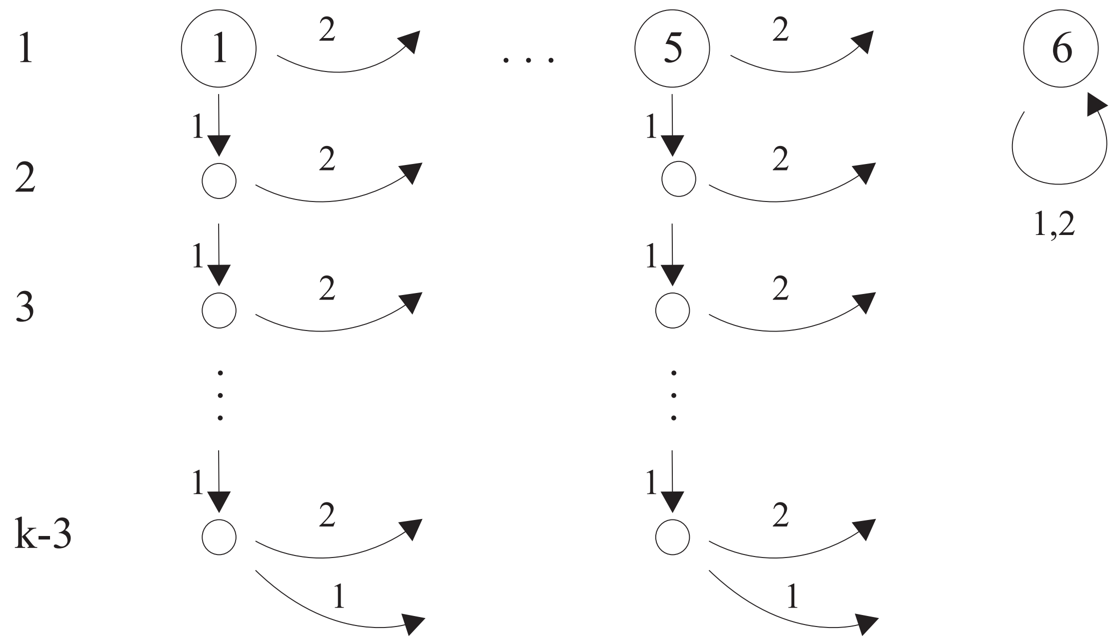
- $f_A(w) > 0$ for some $w \in \Sigma^+$ undecidable for $\mathbb{Z}$-automata with 6 states and $k-2=5$ alphabet symbols.
- For $k-2=5$ alphabet symbols we define $\psi(1) = 2$, $\psi(2) = 12$, $\psi(3) = 112$, $\psi(4) = 1112$, $\psi(5) = 1111$.

An observation

$$B_{i-1} = \gamma(u_i, v_i)$$

$$= \begin{pmatrix} 2^{2|u_i|} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2^{|u_i v_i|} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^{2|v_i|} & 0 & 0 & 0 \\ \sigma(u_i)2^{|u_i|} & \sigma(v_i)2^{|u_i|} & 0 & 2^{|u_i|} & 0 & 0 \\ 0 & \sigma(u_i)2^{|v_i|} & \sigma(v_i)2^{|v_i|} & 0 & 2^{|v_i|} & 0 \\ \sigma(u_i)^2 & 2\sigma(u_i)\sigma(v_i) & \sigma(v_i)^2 & 2\sigma(u_i) & 2\sigma(v_i) & \color{red}{1} \end{pmatrix}$$

Binary $\mathbb{Z}$-automaton



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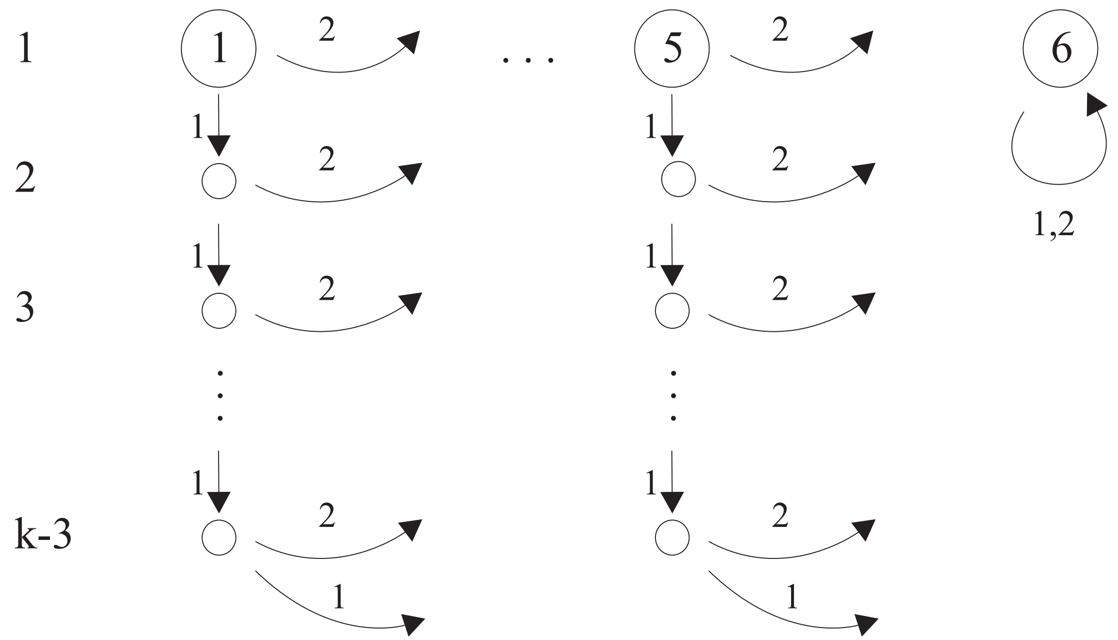

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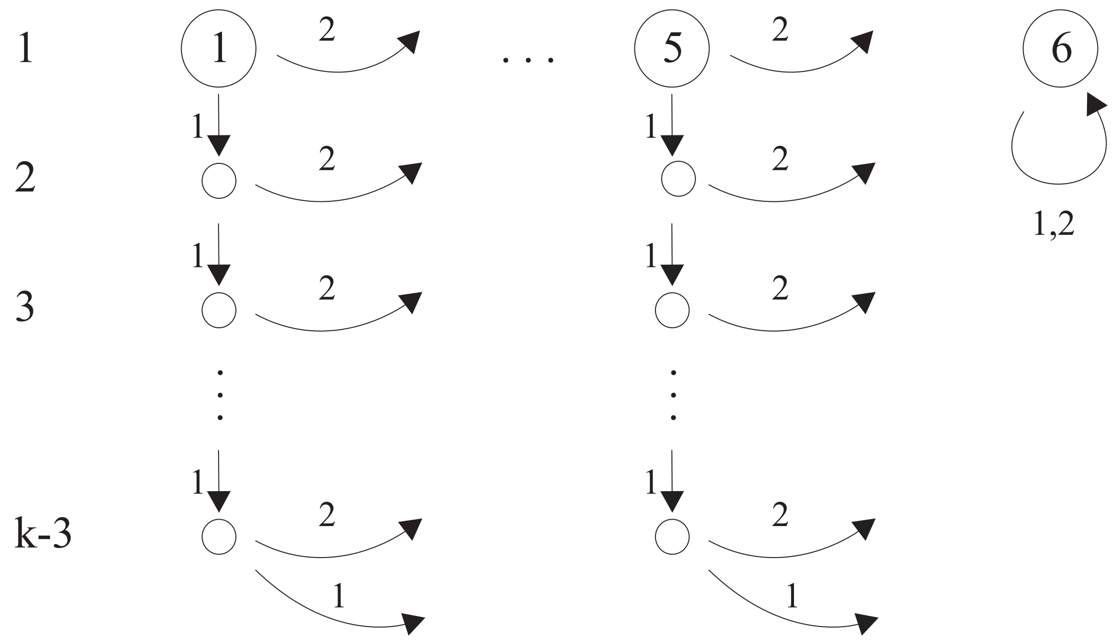

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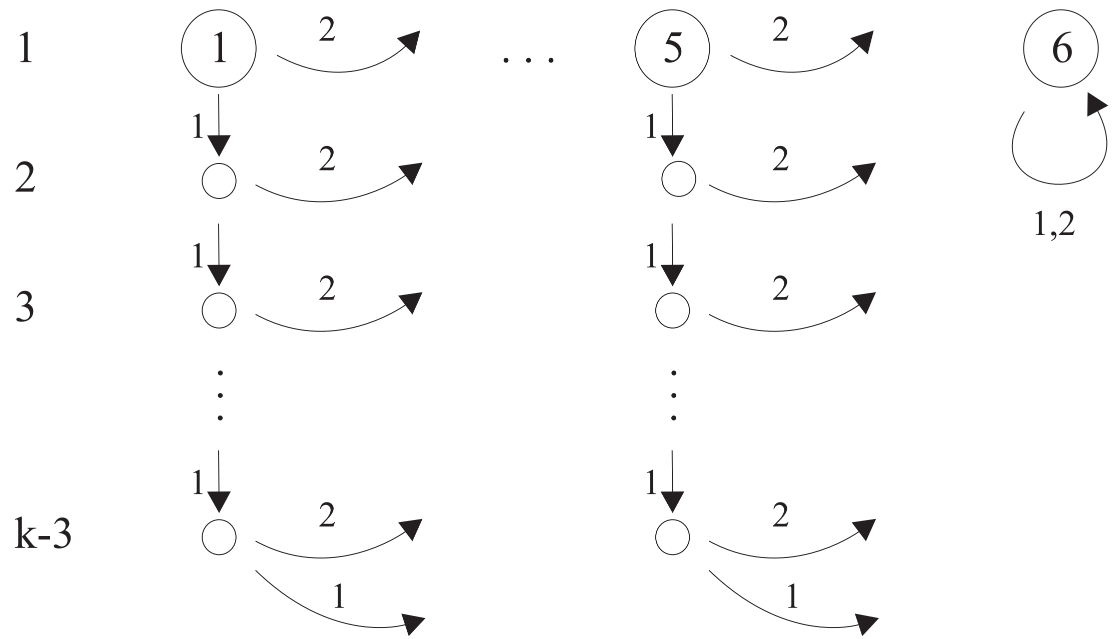

- New automaton has $5(k-3)+1 = 5k-14 = 21$ states.
- $\mathcal{I}$ has a solution iff $\boldsymbol{x}_3^T C_w \boldsymbol{y}_3 > 0$ for some $w \in \{1,2\}^*$.

From $\mathbb{Z}$-automaton to probabilistic

- Procedure by P. Turakainen (1969)

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1. $$D_i = \begin{pmatrix} 0 & 0 & 0 \\ C_i \boldsymbol{y}_3 & C_i & 0 \\ \boldsymbol{x}_3^T C_i \boldsymbol{y}_3 & \boldsymbol{x}_3^T C_i & 0 \end{pmatrix}, \boldsymbol{x}_4 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

$$\boldsymbol{y}_4 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \boldsymbol{x}_4^T D_w \boldsymbol{y}_4 = \boldsymbol{x}_3^T C_w \boldsymbol{y}_3; 21 + 2 = 23$$

states.

From $\mathbb{Z}$-automaton to probabilistic

- Procedure by P. Turakainen (1969)

2. $E_i = \begin{pmatrix} 0 & 0 & 0 \\ \boldsymbol{t}_i & D_i & 0 \\ s_i & \boldsymbol{r}_i^T & 0 \end{pmatrix}$,

$\boldsymbol{x}_5 = (0, \boldsymbol{x}_4^T, 0)^T$, $\boldsymbol{y}_5 = (0, \boldsymbol{y}_4^T, 0)^T$;
$\boldsymbol{x}_5^T E_w \boldsymbol{y}_5 = \boldsymbol{x}_4^T D_w \boldsymbol{y}_4$; $23 + 2 = 25$ states.

From $\mathbb{Z}$-automaton to probabilistic

3. $F_i = E_i + c\mathbf{1}$. Note that $E_i\mathbf{1} = \mathbf{1}E_i = \mathbf{0}$, $\mathbf{1}^k = 25^{k-1}\mathbf{1}$; $F_w = E_w + c^{|w|}25^{|w|-1}\mathbf{1}$.

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4. $G_i = \frac{1}{25c}F_i$. Now $G_w = \frac{1}{(25c)^{|w|}}E_w + \frac{1}{25}\mathbf{1}$, and $\boldsymbol{x}_5^T G_w \boldsymbol{y}_5 = \frac{1}{(25c)^{|w|}}\boldsymbol{x}_5^T E_w \boldsymbol{y}_5 + \frac{1}{25}$. Notice also that G_1 and G_2 are (doubly) stochastic matrices.

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Theorem: For a 25-state probabilistic automaton $(\boldsymbol{x}_5, \{G_1, G_2\}, \boldsymbol{y}_5)$, $\boldsymbol{x}_5^T G_w \boldsymbol{y}_5 > \frac{1}{25}$ for some $w \in \Sigma^*$ if and only if $\mathcal{I}$ has a solution.

Quantum automata

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Lemma: For each u, $v \in \Sigma^* = \{1, 2\}^*$, equality $U_u\boldsymbol{y} = U_v\boldsymbol{y}$ implies $u = v$.

Quantum automata

Define $\gamma(u,v) = \frac{1}{2}\begin{pmatrix} U_u + U_v & U_u - U_v \\ U_u - U_v & U_u + U_v \end{pmatrix}$,

$\boldsymbol{y}_1 = (\boldsymbol{y}, \boldsymbol{0})^T$, and $P_1 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$.

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$$P_1\gamma(u,v)\boldsymbol{y}_1 = \begin{pmatrix} \mathbf{0} \\ U_u\boldsymbol{y} - U_v\boldsymbol{y} \end{pmatrix}.$$

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$$P_1\gamma(u,v)\boldsymbol{y}_1 = \begin{pmatrix} \mathbf{0} \\ U_u\boldsymbol{y} - U_v\boldsymbol{y} \end{pmatrix},$$

so $||P_1\gamma(u,v)\boldsymbol{y}_1||^2 = 0$ iff $u = v$.

From PCP to quantum automata

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- $\Rightarrow f_Q(w) = ||P_1 A_w \boldsymbol{y}_1||^2 = 0$ for some nonempty w is undecidable for quantum automata with 4 states and 7 alphabet symbols.

Reducing the number of matrices

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- Let $B_1 = A_2, \ldots, B_{k-2} = A_{k-1}$, $\boldsymbol{y}_2 = A_k \boldsymbol{y}_1$, and $P_2 = A_1^{-1} P_1 A_1$.

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- $||P_2 B_w \boldsymbol{y}_2||^2 = ||A_1^{-1} P_1 A_1 B_w A_k \boldsymbol{y}_1||^2 = ||P_1 A_1 B_w A_k \boldsymbol{y}_1||^2$

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- $\Rightarrow f_Q(w) = 0$ for some w is undecidable for quantum automata with 4 states and 5 alphabet symbols.

Reducing the number of matrices

$$C_1 = \begin{pmatrix} B_1 & 0 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 & 0 \\ 0 & 0 & B_3 & 0 & 0 \\ 0 & 0 & 0 & B_4 & 0 \\ 0 & 0 & 0 & 0 & B_5 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ I & 0 & 0 & 0 & 0 \end{pmatrix}.$$

C_1 and C_2 are unitary 20×20-matrices. Let also

$$P_3 = \begin{pmatrix} P_2 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_2 \end{pmatrix}, \boldsymbol{y}_3 = \begin{pmatrix} \boldsymbol{y}_2 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}$$

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- For each $w \in \{1,2\}^*$ $||P_3 C_w \boldsymbol{y}_3||^2 = ||P_2 B_{w'} \boldsymbol{y}_2||^2$ for some $w' \in \{1,\dots,5\}^*$.

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- $C_2 C_1 C_2^{-1} = Diag(B_2, B_3, \ldots, B_5, B_1)$

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- For each $w \in \{1,2\}^* \ ||P_3 C_w \boldsymbol{y}_3||^2 = ||P_2 B_{w'} \boldsymbol{y}_2||^2$ for some $w' \in \{1,\ldots,5\}^*$.
- $C_2 C_1 C_2^{-1} = Diag(B_2, B_3, \ldots, B_5, B_1)$
- $\Rightarrow \forall w \in \{1,\ldots,5\}^* \ ||P_2 B_w \boldsymbol{y}_2||^2 = ||P_3 C_{w'} \boldsymbol{y}_3||^2$ for some $w' \in \{1,2\}^*$.

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- $\Rightarrow$ $||P_3 C_w \boldsymbol{y}_3||^2 = 0$ for some $w \in \{1,2\}^*$ if and only if $\mathcal{I}$ has a solution.

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- $\Rightarrow ||P_3 C_w \boldsymbol{y}_3||^2 = 0$ for some $w \in \{1,2\}^*$ if and only if $\mathcal{I}$ has a solution.
- $\Rightarrow f_Q(w) = 0$ for some w is undecidable for binary quantum automata with 20 states.

Setting the threshold

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- $1 = ||C_w \boldsymbol{y}_3||^2 = ||(I - P_3)C_w \boldsymbol{y}_3||^2 + ||P_3 C_w \boldsymbol{y}_3||^2$, hence $||(I - P_3)C_w \boldsymbol{y}_3||^2 \geq 1$ if and only if $\mathcal{I}$ has a solution.

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- Let $D_i = \begin{pmatrix} C_i & 0 \\ 0 & 1 \end{pmatrix}$, $P_4 = \begin{pmatrix} I - P_3 & 0 \\ 0 & 0 \end{pmatrix}$, and $\boldsymbol{y}_4 = (\sqrt{\lambda}\boldsymbol{y}_3^T, \sqrt{1-\lambda})^T$.

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- $||P_4 D_w \boldsymbol{y}_4||^2 = ||\sqrt{\lambda}(I - P_3)C_w \boldsymbol{y}_4||^2 = \lambda(1 - ||P_3 C_w \boldsymbol{y}_3||^2)$.

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- $||P_4 D_w \boldsymbol{y}_4||^2 = ||\sqrt{\lambda}(I - P_3)C_w \boldsymbol{y}_4||^2 = \lambda(1 - ||P_3 C_w \boldsymbol{y}_3||^2)$.
- $\Rightarrow f_Q(w) \geq \lambda$ for some some w is undecidable for binary quantum automata with 21 states.

Thank You!