Pushing Anderson’s Envelope: The Modal Logic of Ascription

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Abstract. The paper proposes a formal analysis of the ascriptive view of norms as resulting from pulling together Anderson’s reductionist approach, the analysis of counts-as, and a novel modal approach to the formal representation of languages in logic. This unifying attempt results in the definition of a new form of reduction of deontic logic based on counts-as statements. Such result is discussed also in the light of Jørgensen’s dilemma.

Key words: Modal logic, Anderson’s reduction, counts-as, ascription, Jørgensen’s dilemma.

1 Introduction

The present paper intends to pull together independent threads which have been thus far followed by the (formal) studies of norms. Such threads are the reductionist approach to norms started with [1–3, 19], the study of counts-as initiated in [26, 27] and first pursued with formal means in [17], and the ascriptive view of norms first put forth in [24] and more recently developed, among others, in [16]. According to this latter, norms are actually ascriptions of deontic properties to actions or states of affairs. In short, to state norms means to create new properties, which are somehow inexistent in reality (e.g., Anderson’s “violation”), to create new words to name them, and consequently to predicate them of the relevant states of affairs or actions.

[... ] as the original manner of producing physical entities is creation, there is hardly a better way to describe the production of moral entities than by the word ‘imposition’ [impositio]. For moral entities do not arise from the intrinsic substantial principles of things but are superadded to things already existent and physically complete [24, pp. 100-101].

The paper proposes a formal analysis of this view of norms which builds, in the first place, on Anderson’s reduction, in the second place, on the formal analysis of counts-as developed in [10–14] and, in the third place, on a formal characterization of the language creation aspect of the ascriptive view of norms.
As a result, a comprehensive formal theory of norms is presented and formalized in modal logic.

The paper is structured as follows. Section 2 summarizes Anderson’s reduction approach and provides a contextual version of it. Part of the section consists of a summary of the results presented in [10–14] and provides the ground for a counts-as based view of Anderson’s reduction. At the end of the section the ascriptive view of norms is exposed in more details to introduce Section 3. There, a language-based notion of indistinguishability between propositional models is introduced and a modal logic, first studied in [20, 21], is exposed for reasoning about it. This language-based notion of indistinguishability will be the key for capturing the phenomenon of language creation inherent in the ascriptive view of norms. A simple example is used throughout the exposition of the formalism. Section 4 applies the formalisms presented in Sections 2 and 3, providing a formal characterization of the ascriptive view of norms in the guise of a new notion of counts-as. The section ends with some remarks concerning the relation between the formal analysis presented and Jørgensen’s dilemma. Finally, Section 5 draws some conclusions and sketches future research lines.

2 Anderson’s reduction revisited

By “Anderson’s reduction” the present paper intends, in general, the approach to deontic logic which is based on the reduction of deontic notions to evaluative ones (e.g., ‘good’, ‘ideal’, ‘bad’, ‘violation’). Such approach was first systematically developed in Anderson’s work [1–3]. In that work, the reduction of deontic statements to alethic ones is based on the intuition according to which the fact that \( \phi \) is obligatory means that \( \neg \phi \) “necessarily” implies a violation, in symbols: \( \Box (\neg \phi \to V) \), where \( V \) is a specific atom for which it is valid that \( \neg V \), i.e., that the violation is not “necessary”. The nature of the reduction lies in how this reference to a “necessity” is formally modeled. In the original proposal of Anderson the system chosen for the reduction was \( K^1 \). Various alternative versions of Anderson’s reduction are studied, for instance, in [7, 20, 22].

2.1 Terminological necessities

We start considering the form of reduction based on system \( S5 \) such as the ones studied in [7, 20]. By interpreting the \( \Box \) operator occurring in the reduction expression as an \( S5 \) necessity, formulae \( \Box (\neg \phi \to V) \) could be soundly rephrased as: the negation of \( \phi \) unconditionally implies a violation. Notice that the \( S5 \)-based interpretation of the reduction is in line with Anderson’s intuition [4] that the occurrence of a violation follows analytically from the fact that an obligation is not fulfilled.

\(^1\) It might be instructive to recall that Kanger independently developed an analogous reduction based on a constant \( Q \) denoting normative ideality, or the absence of violation [19]. In this case, the fact that \( \phi \) is obligatory means that \( \phi \) “necessarily” follows from ideality, in symbols: \( \Box (Q \to \phi) \).
It is well-known that S5 is the modal logic of universal quantification since the so-called universal modality (i.e., the modality interpreted on the $W \times W$, where $W$ is the model’s domain) is an S5 modality [6]. Now, viewing the $\Box$ modality in Anderson’s reduction as the universal modality, which we denote by $[u]$, conveys a key semantic hint:

$$M, w \models [u](\neg \phi \rightarrow V) \iff \forall w' \in W : M, w' \models \neg \phi \rightarrow V \quad (1)$$

$$I(\neg \phi) \subseteq I(V) \quad (2)$$

where $M$ is a model for the modal language with universal modality $[u]$, $W$ is its domain and $I$ its evaluation function. Formulae 1 and 2 show a very precise interpretation of Anderson’s reduction: $\phi$ is obligatory means that all states (i.e., possible worlds) are such that either $\phi$ is true or, if $\phi$ is false, then a violation is also true. In this view, deontic statements amount to set-theoretic relations concerning the interpretation $I(V)$ of the atom $V$.

We mention two of the most obvious of such relations. The first one, set theoretic inclusion, expresses an ought-to-be form of prohibition: states satisfying $\phi$ are violation states. The second one, non-empty intersection, expresses an ought-to-be form of permission: there exist states which satisfy $\phi$ and are not violation states.

$$M, w \models [u](\phi \rightarrow V) \iff I(\phi) \subseteq I(V) \quad (3)$$

$$M, w \models \neg [u](\phi \rightarrow V) \iff I(\phi) \cap I(\neg V) \neq \emptyset \quad (4)$$

Obligation statements, like the one expressed in Formula 1, are instances of Formula 3 where $\phi$ is substituted by $\neg \phi$. As Formulae 3 and 4 very well show, the type of necessity involved in Anderson’s reduction is of a terminological sort in the sense that it solely concerns the interpretation of the term violation.

If the deontic statements of a normative system can be represented by modal formulae involving the universal modality and the violation atom, what happens if we want to consider, under the same formalism, deontic statements belonging to several different normative systems? Technically speaking, we then look for operators that can “locally” behave like a universal modality, but that can “globally” behave in a weaker way allowing for the representation of different and possibly inconsistent deontic statements at the same time. We should find a multi-modal logic such that: a) the logic enables as many modalities as the normative systems we intend to represent; b) these modalities retain as many characteristics of $[u]$ as possible; c) the logic allows for the satisfiability of expressions such as: $[i](\neg \phi \rightarrow V) \land [j](\neg \phi \rightarrow V)$. To put it roughly, we look for a modal logic by means of which to express contextual terminological necessity.

2.2 A modal logic of context

In logic, contexts have been studied as sets of models [9]. Now, if the models considered are models of propositional languages, then contexts can be studied

$^2$ The reader is referred to [10] for a more comprehensive exposition of the ideas just sketched.
as sets of possible worlds [28]. The present section exposes a logic based on this view. The result is a contextual version of Anderson’s reduction.

Syntax of Cxt

The syntax of Cxt is the syntax of a multi-modal language \( L_{Cxt} \) [6] where \( n \) is the cardinality of the set \( C \) of contexts and \( u \) the index of the universal modality. The alphabet of \( L_{Cxt} \) contains: an at most countable set \( P \) of propositional atoms \( p \); the set of boolean connectives \( \{\neg, \land, \lor, \rightarrow\} \); a finite non-empty set \( C \) of context indexes containing the context index \( u \). Metavariables \( i, j, \ldots \) are used to denote elements of \( C \). The set of well-formed formulae \( \phi \) of \( L_{Cxt} \) is defined by the following BNF:

\[
\phi ::= \top \mid p \mid \neg \phi \mid \phi_1 \land \phi_2 \mid \phi_1 \lor \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid [i] \phi \mid \langle i \rangle \phi.
\]

where \( i \) denotes elements in \( C \).

Semantics of Cxt

Languages \( L_{Cxt} \) are given a semantics via the class of \( C_{\top} \) frames \( F = (W, \{W_i\}_{i \in C}) \) such that \( W \in \{W_i\}_{i \in C} \). Leaving technicalities aside, these frames consist of the domain \( W \) and of a finite number \( n = |C| \) of subsets of \( W \) among which \( W \) itself[^4]. Such subsets straightforwardly model the notion of context sketched above. Notice that the domain \( W \) represents the global, or universal, context. Models are, as usual, structures \( (\mathcal{F}, \mathcal{I}) \) where \( \mathcal{F} \) belongs to the class \( C_{\top} \) and \( \mathcal{I} \) is the valuation function \( I : P \rightarrow P(W) \).

**Definition 1.** (Satisfaction based on \( C_{\top} \) frames)

Let \( \mathcal{M} \) be a model built on a \( C_{\top} \) frame.

\[
\mathcal{M}, w \models [i] \phi \iff \forall w' \in W_i : \mathcal{M}, w' \models \phi
\]

where \( u \) is the universal context index, \( W_u = W \) and \( i \) ranges on the context indexes in \( C \). The obvious clauses for the Boolean connectives and the dual of \( [i] \) are omitted.

Notice that the \( [u] \) is the universal operator. Notice also that, while in standard modal logic the truth of \( [i] \) and \( \langle i \rangle \) formulae depends on the evaluation state, the truth of such formulae interpreted within \( C_{\top} \) frames abstracts therefrom: in other words truth implies validity. This is what we would intuitively expect for the contexts of normative systems: what holds in the context of a given normative system is not determined by the point of evaluation but just by the system as such, i.e., by its own norms.

[^3]: Readers are again referred to [10] for a more detailed exposition.

[^4]: Notice that such structures are multi-sets, or bags, rather than frames. However, it is proven that they represent secondarily universal frames which also contain one universal relation [10, Ch. 4]. An alternative more general semantics to Cxt can be given via the class \( 2\mathcal{E}^\sim \) of frames satisfying the following properties: they are i-j transitive (if \( wR_i w' \) and \( w'R_j w'' \) then \( w'R_j w'' \)), i-j euclidean (if \( wR_i w' \) and \( w'R_j w'' \) then \( w'R_j w'' \)), and they contain an equivalence relation \( R_u \) such that for all \( i \in C \), \( R_i \subseteq R_u \). It is proven, however, that classes \( C_{\top} \) and \( 2\mathcal{E}^\sim \) are modally equivalent (see [10, Appendix]).
Axiomatics of Cxt

Logic Cxt results from the union of the modal logic K45, which axiomatizes contexts [10], with an S5 logic axiomatizing the behavior of the global context u, plus the interaction axiom \( \subseteq .ui \), which just states that u is the biggest context.

\[ \begin{align*}
\text{(P)} & \quad \text{all tautologies of propositional calculus} \\
\text{(K')} & \quad [i](\phi_1 \to \phi_2) \to ([i]\phi_1 \to [i]\phi_2) \\
\text{(4') } & \quad [i]\phi \to [j][i]\phi \\
\text{(5')} & \quad \neg [i]\phi \to [j]\neg [i]\phi \\
\text{(T')} & \quad [u]\phi \to \phi \\
\text{(\subseteq .ii)} & \quad [u]\phi \to [i]\phi \\
\text{(Dual)} & \quad \langle i \rangle \phi \leftrightarrow \neg [i]\neg \phi \\
\text{(MP)} & \quad \text{If } \vdash \phi_1 \text{ and } \vdash \phi_1 \to \phi_2 \text{ then } \vdash \phi_2 \\
\text{(N')} & \quad \text{If } \vdash \phi \text{ then } \vdash [i]\phi
\end{align*} \]

where i, j denote elements of the set of indexes C and u denotes the universal context index in C. The interaction axiom \( \subseteq .ui \) states something quite intuitive concerning the interaction of the \([u]\) operator with all other context operators: what holds in the global context, holds in every context. Soundness and completeness of this axiomatization w.r.t. Cxt frames are proven in [10].

### 2.3 Anderson’s reduction contextualized

Everything has been put into place to provide a contextualization of the version of Anderson’s reduction sketched in Section 2.1. The fact that \( \phi \) is ideally the case in context \( i \) can be formalized as \([i](\neg \phi \to V)\) and read as: the negation of \( \phi \) necessarily implies a violation within context \( i \). It becomes thus possible to express that \( \phi \) is obligatory in the context \( i \) of a given normative system, while \( \neg \phi \) is permitted in the context \( j \) of a different normative system: \([i](\neg \phi \to V) \land \langle j \rangle (\neg \phi \land \neg V)\).

In [10] such reduction has been called a “counts-as reduction of deontic logic”. Counts-as is the problematic locution introduced in [26,27], and formally investigated for the first time in [17], which Searle takes as the basic syntax of constitutive rules, that is, of the building blocks of social reality. From a semantic point of view, such locution can acquire several different meanings, some of which have been systematically analyzed in [10–14]. One of these senses —the classificatory counts-as— is there formalized as the strict implication in Cxt:

\[ \neg \phi \Rightarrow_i^d V := [i](\neg \phi \to V) \]  

Intuitively, the negation of \( \phi \) counts as a violation in context \( i \), meaning that the negation of \( \phi \) is classified as a violation in context \( i \).

Such reduction can be straightforwardly strengthened by considering stronger senses of counts-as. One of these is the proper classificatory counts-as, also formalizable in Cxt:

\[ \neg \phi \Rightarrow_i^{d+} V := [i](\neg \phi \to V) \land \neg [u](\neg \phi \to V) \]  

where \( i \), \( j \) denote elements of the set of indexes C and u denotes the universal context index in C. The interaction axiom \( \subseteq .ui \) states something quite intuitive concerning the interaction of the \([u]\) operator with all other context operators: what holds in the global context, holds in every context. Soundness and completeness of this axiomatization w.r.t. Cxt frames are proven in [10].
Intuitively, the negation of $\phi$ counts as a violation in context $i$, meaning that the negation of $\phi$ is classified as a violation in context $i$ (first conjunct of the right-hand side of Formula 6), but the negation of $\phi$ is not always classified as a violation (second conjunct of the right-hand side of Formula 6).

### 2.4 Norms as ascriptions

The reduction of deontic to counts-as statements of the type displayed in Formula 6 stresses that a state of affairs properly determines a violation only within a context, since outside the context that would not necessarily be the case. In [12] and [10], the rationale behind this formal characterization was taken from Searle’s words themselves:

> [. . . ] where the rule (or system of rules) is constitutive, behaviour which is in accordance with the rule can receive specifications or descriptions which it could not receive if the rule did not exist [26, p. 35].

Constitutive rules add something to what is already the case and proper contextual classification is a way to capture this intuition. However, there is also another way to look at the novelty introduced by constitutive rules. In a sense, what they do is to literally introduce new concepts, rather than just validating classifications which would otherwise not be valid. They create new terms to be used for a further conceptualization of reality. Such view of rules as ascriptions has a long history, starting with Pufendorf’s notion of “impositio” [24, pp. 100–101] and have been advanced in more recent times for instance in [16] or [?]. As a matter of fact, Searle’s thesis according to which institutional facts are construed upon brute ones [27] is an instance of this ascriptive view of social reality.

Now, the central aspect of ascription is language creation. In order for an ascription to take place, a new term needs to be created, which can then be used for denoting the desired property. If we take an ascriptive view of Anderson’s reduction, this means that the term “violation” is introduced in order to separate desired or ideal actions or states of affairs from their undesired or sub-ideal counterparts. Interestingly enough, this exact view is neatly formulated in Jørgensen’s paper which introduced his dilemma [18]:

How is a sentence of the form “Such and such is to be so and so” to be verified? How is it for instance to be verified that all promises are to be kept? To this question I know of no other answer than the following: The phrase “is to be etc.” describes not a property which an action or a state of affairs either has or not, but a kind of quasi-property which is ascribed to an action or a state of affairs when a person is willing or commanding the action to be performed, resp. the state of affairs to be produced [18, pp. 292–293]

The following sections develop a formal analysis of this ascriptive view of norms. The primary technical difficulty resides in providing a suitable formal ground for the representation of language creation. From a propositional point
of view, language creation means that new propositional atoms are somehow introduced in the language and consequently evaluated in the models. Therefore, in order to model language creation in logic, we should first be able to model, within the same logical framework, different languages. This is an aspect which, at first, might look hard to capture in a standard logical framework since evaluation functions are typically not partial, i.e. they evaluate all the atoms in the language.

3 “In the beginning was the Word”

The present section shows how modal logic offers an elegant way to represent different languages within one same formalism, without resorting to non-standard tools such as partial evaluation functions.

3.1 Adam & Eve

Consider the propositional language $L$ built from the alphabet $P$ of propositional atoms: $\text{eat\_apple}$ (“the apple has been eaten”), $V$ (“a violation has occurred”). We have of course four possible models such that: $w_1 \models \text{eat\_apple} \land V$, $w_2 \models \text{eat\_apple} \land \neg V$, $w_3 \models \neg \text{eat\_apple} \land V$ and $w_4 \models \neg \text{eat\_apple} \land \neg V$. That is, we have the state in which the apple is eaten and there is a violation ($w_1$), the state in which the apple is eaten but there is no violation ($w_2$), the state where the apple is not eaten and there is a violation ($w_3$), and finally the state where no apple is eaten nor there is a violation ($w_4$).

Obviously, all these states can be distinguished from each other. But suppose now to compare the models ignoring atom $V$. Models $w_1$ and $w_2$ would not be distinguishable any more, nor would states $w_3$ and $w_4$. Which is just another way to say that, had we used a sublanguage $L_i$ of $L$ containing only atom $\text{eat\_apple}$, we would have been able to distinguish only states $w_1$ from $w_3$ and $w_2$ from $w_4$. This latter can be considered to be the language at disposal of Adam & Eve in their pre-moral stage, before hearing God commanding “you shall not eat of the fruit of the tree that is in the middle of the garden”—rather than before actually eating the apple. In fact, after hearing God’s command they were already endowed with the possibility to discern good ($\neg \text{eat\_apple}$) from evil ($\text{eat\_apple}$), that is, their language was enriched and they got to distinguish also states $w_1$ from $w_2$ and $w_3$ from $w_4$, thanks to the newly introduced notion of violation ($V$).

3.2 Propositional sublanguage equivalence

The intuitions sketched in the previous section are here made formal. Take two propositional models $m$ and $m'$ for a propositional language $L$. Models $m$ and $m'$ are equivalent if they satisfy the same formulae expressible in $L$: $m \models \phi$ iff $m' \models \phi$. If $m$ and $m'$ are equivalent ($m \sim m'$) then there is no set $\Phi$ of formulae of $L$ whose models contain $m$ but not $m'$, or vice versa. That
is to say, the two models are indistinguishable for $\mathcal{L}$. However, two models which are not equivalent with respect to a given alphabet (a given set of atomic propositions), may become equivalent if only a sub-alphabet (a subset of the atomic propositions) is considered.

**Definition 2. (Propositional sublanguage equivalence)**

Two models $m$ and $m'$ for a propositional language $\mathcal{L}$ are equivalent w.r.t. sublanguage $\mathcal{L}_i$ if they satisfy the same set of formulae expressible using the alphabet of $\mathcal{L}_i$. For any $\phi \in \mathcal{L}_i$: $m \models \phi$ iff $m' \models \phi$. If $m$ and $m'$ are equivalent w.r.t. $\mathcal{L}_i$ (i.e., $m \sim_i m'$) then they cannot be distinguished by any set $\Phi$ of formulae of $\mathcal{L}_i$.

The definition makes precise the idea of two propositional models agreeing up to what is expressible on a given alphabet. To put it another way, it formalizes the idea that two models $m$ and $m'$ are equivalent modulo the alphabet in the complement $\mathcal{L} \setminus \mathcal{L}_i$ (i.e., $L \setminus L_i$) of the sublanguage considered: $m$ is indistinguishable from $m'$ if we disregard the alphabet of $\mathcal{L}_i$. Notice that if $m \sim_i m'$ and $L_i = L$ (i.e., the maximal element in $\mathcal{Sub}(\mathcal{L})$) then $\sim_i = \sim$, that is, $\sim_i$ is the standard equivalence between propositional models.

**Proposition 1. (Properties of $\sim_i$)**

Let $m$ and $m'$ be two models for the propositional language $\mathcal{L}$. The following holds:

1. For every sublanguage $\mathcal{L}_i$ of $\mathcal{L}$, relation $\sim_i$ is an equivalence relation on the set of all models of language $\mathcal{L}$.
2. For all sublanguages $\mathcal{L}_i$ and $\mathcal{L}_j$ of $\mathcal{L}$: if $\mathcal{L}_i \subseteq \mathcal{L}_j$ then $\sim_i \subseteq \sim_j$. It follows that for every sublanguage $\mathcal{L}_i$ of $\mathcal{L}$: $\sim \subseteq \sim_i$, that is, standard equivalence implies sublanguage equivalence.

**Proof.** Claim (1) is straightforwardly proven. It is easy to see that: identity is a subrelation of $\sim_i$ for any sublanguage $\mathcal{L}_i$; and that $\sim \circ \sim_i$ and $\sim_i^{-1}$ are subrelations of $\sim_i$ for any sublanguage $\mathcal{L}_i$. Claim (2) is proven by considering that, if $\mathcal{L}_i$ is a sublanguage of $\mathcal{L}_j$ and $m \sim_j m'$, then for all propositions $\phi \in \mathcal{L}_i$: $m \models \phi$ iff $m' \models \phi$. Hence, $m \sim_i m'$.

### 3.3 Release logic

Propositional release logics (PRL) have been first introduced and studied in [20, 21] in order to provide a modal logic characterization of the notion of irrelevancy. Irrelevancies are, in short, those aspects which we can choose to ignore. Irrelevancy is represented via modal release operators, specifying what is relevant to the current situation and what can instead be ignored. Release operators are indexed by an abstract ‘issue’ denoting what is considered to be irrelevant for evaluating the formula in the scope of the operator: $\Delta I \phi$ means ‘formula $\phi$ holds in all states where issue $I$ is irrelevant’, or ‘$\phi$ holds in all states modulo issue $I$’; $\nabla I \phi$ means ‘formula $\phi$ holds in at least one of the states where issue $I$ is irrelevant’, or ‘$\phi$ possibly holds while releasing issue $I$’.
Issues can be in principle anything, but their essential feature is that they yield equivalence relations which cluster the states in the model. An issue $I$ is conceived as something that determines a partition of the domain in clusters of states which agree on everything but $I$, or which are equivalent modulo $I$. Release operators are interpreted on these equivalence relations. As such, propositional release logic can be thought of as a “logic of controlled ignorance” [20]. They represent what we would know, and what we would ignore, by choosing to disregard some issues.

Syntax of PRL The syntax of PRL is the syntax of a standard multi-modal language $L^{Prl}$ [6] where $n$ is the cardinality of the set $\text{Iss}$ of releasable issues. The alphabet of $L^{Prl}$ contains: an at most countable set $P$ of propositional atoms $p$; the set of boolean connectives $\{\neg, \land, \lor, \rightarrow\}$; a finite non-empty set $\text{Iss}$ of issues. Metavariables $I, J, ...$ are used for denoting elements of $\text{Iss}$. The set of well-formed formulae $\phi$ of $L^{Prl}$ is defined by the usual BNF:

$$\phi ::= \top | p | \neg \phi | \phi_1 \land \phi_2 | \phi_1 \lor \phi_2 | \phi_1 \rightarrow \phi_2 | \Delta_I \phi | \nabla_I \phi.$$  

where $I$ denotes elements in $\text{Iss}$.

One last important feature of PRL should be addressed before getting to the semantics. We have seen that modal operators are indexed by an issue denoting what is disregarded when evaluating the formula in the scope of the operator. The finite set $\text{Iss}$ of these issues is structured as a partial order, that is to say, $\langle \text{Iss}, \leq \rangle$ is a structure on the non-empty set $\text{Iss}$, where $\leq$ (“being a sub-issue of”) is a binary relation on $\text{Iss}$ which is reflexive, transitive and antisymmetric. The aim of the partial order is to induce a structure on the equivalence relations denoting the release of each issue in $\text{Iss}$: if $I \leq J$ then the clusters of states obtained by releasing $J$ contain the clusters of states obtained by releasing $I$. Intuitively, if $I$ is a sub-issue of $J$ then by disregarding $J$, $I$ is also disregarded. This aspect is made explicit in the models which, for the rest, are just Kripke models.

Semantics of PRL The semantics of PRL is given via the class $\mathcal{F}t$ of frames $\mathcal{F} = \langle W, \{R_I\}_{\text{Iss}} \rangle$ such that $W$ is a non-empty set of states and $\{R_I\}_{\text{Iss}}$ is a family of equivalence relations such that: if $I \leq J$ then $R_I \subseteq R_J$. Models are, as usual, structures $\mathcal{M} = \langle \mathcal{F}, I \rangle$ where $I$ is an evaluation function $I : P \rightarrow P(W)$ associating to each atom the set of states which make it true. PRL models are therefore just $\text{S5}_n$ models with the further constraint that the granularity of the equivalence relations follows the partial order defined on the set of issues: the $\leq$-smaller is the issue released, the more granular is the partition obtained via the associated equivalence relation. The satisfaction relation is standard. Boolean clauses are omitted.
Definition 3. (Satisfaction for PRL models)
Let $\mathcal{M}$ be a PRL model.

$\mathcal{M}, w \models \Delta \phi \iff \forall w', w R w' : \mathcal{M}, w' \models \phi$

$\mathcal{M}, w \models \forall \phi \iff \exists w', w R w' : \mathcal{M}, w' \models \phi$.

where $I \in \text{Iss}$. As usual, a formula $\phi$ is said to be valid in a model $\mathcal{M}$, in symbols $\mathcal{M} \models \phi$, iff for all $w$ in $W$, $\mathcal{M}, w \models \phi$. It is said to be valid in a frame $F$ ($F \models \phi$) if it is valid in all models based on that frame. Finally, it is said to be valid on a class of frames $F$ ($F \models \phi$) if it is valid in every frame $F$ in $F$.

Axiomatics of PRL
Finally, the axiomatics amounts to a multi-modal $S5$ plus the PO (partial order) axiom:

(P) all tautologies of propositional calculus
(K) $\Delta_1(\phi_1 \rightarrow \phi_2) \rightarrow (\Delta_1 \phi_1 \rightarrow \Delta_1 \phi_2)$
(T) $\Delta_1 \phi \rightarrow \phi$
(4) $\Delta_1 \phi \rightarrow \Delta_1 \Delta_1 \phi$
(5) $\forall_1 \phi \rightarrow \Delta_1 \forall_1 \phi$
(P0) $\Delta_1 \phi \rightarrow \Delta_1 \phi$ if $I \leq I$
(Dual) $\forall_1 \phi \leftrightarrow \neg \Delta_1 \neg \phi$
(MP) If $\vdash \phi_1$ and $\vdash \phi_1 \rightarrow \phi_2$ then $\vdash \phi_2$
(N$'$) If $\vdash \phi$ then $\vdash \Delta_1 \phi$

where $I, J \in \text{Iss}$. A proof of the soundness and completeness of this axiomatics w.r.t. to the semantics presented in Definition 3 is exposed in [21].

4 Modal aspects of ascriptivism

This section puts logics $\text{Cxt}^\omega$ and PRL at work together. Their fusion $\text{Cxt}^\omega \otimes \text{PRL}$ on language $L^\text{Cxt} \otimes L^\text{Prl}$ is all we need to get the axiomatics and semantics we are interested in. Notice that completeness will be preserved by the fusion of the axiom systems exposed in Sections 2.2 and 3.3 w.r.t. to the fusion $\mathcal{E}^- \otimes \Psi$ of their classes of frames$^5$.

4.1 Propositional sublanguage equivalence as release

Reasoning about propositional sublanguage equivalence is an instance of reasoning in release logic.

$^5$ Notice that the fusion $\mathcal{E}^- \otimes \Psi$ considers the semantics of $\text{Cxt}^\omega$ given in terms of $i$-$j$ transitive and $i$-$j$ euclidean frames containing an equivalence relation including all contexts (see Footnote 4). This is necessary because $\text{Cxt}^\omega$ frames are not closed under disjoint unions, which is a prerequisite for preserving Kripke completeness in fusions. See [8, Ch. 4].
Proposition 2. (Sublanguage equivalence models)
Consider a propositional language $L$ on the set of atoms $P$, and a set of states $W$. Any evaluation function $I : P \rightarrow P(W)$ determines a PRL model $m = \langle \langle W, \sim \rangle, I \rangle$.

Proof. It follows from the properties of $\sim$ proven in Proposition 1.

Notice that the release issues $\text{Iss}$ are the complements $\neg L_i$ of the sublanguages in $\Xi L$. In fact, what is released is just what cannot be expressed. The accessibility relations should therefore be taken to be the sublanguage-equivalence relations $\sim_{-i}$. Notice also that the set $\text{Iss}$ is ordered by set-theoretic inclusion $\subseteq$ between sublanguages of $L$.

To put it roughly, what the theorem says is that PRL is the logic to reason about scenarios like the Adam & Eve one sketched in Section 3.1. Let us get back to that example. Now it is possible to represent both the pre- and post-God’s commandment situations, within the same formalism, by making use of the release operators of PRL. Suppose Adam & Eve to be at state $w_1$ in the model with domain $W = \{w_1, w_2, w_3, w_4\}$ and evaluation $I$ as in Section 3.1. Recall that the language was built on atoms $P = \{\text{eat\_apple}, V\}$. So let us denote with $[V]$ and $[\text{eat\_apple}]$ the sublanguages containing only atom $V$ and, respectively, atom $\text{eat\_apple}$. These sublanguages represent the releasable issues together with the empty language $0$ and the full language $1 = P$. Let $M = \langle W, \sim_{[V]}, \sim_{[\text{eat\_apple}]}, \sim_0, \sim_1, I \rangle$ be the resulting release model. We have that:

$$M, w_1 \models \text{eat\_apple} \wedge V$$
$$M, w_1 \models \Delta_0(\text{eat\_apple} \wedge V)$$
$$M, w_1 \models \Delta_{[V]}\text{eat\_apple} \wedge \neg\Delta_{[V]}V$$

So Formula 7 just states what holds in $w_1$, which is the actual state where Adam & Eve eat the apple committing a violation. Formula 8 does the same by saying that, if you evaluate $\text{eat\_apple}$ and $V$ after releasing nothing, i.e., by using the full descriptive power of the language, then both $\text{eat\_apple}$ and $V$ necessarily hold. In fact, in the model at issue the set of states reachable from $w_1$ via $\sim_0$ coincides with $w_1$ itself, since there are no other states in $W$ which are equivalent with $w_1$ if all available atoms are used in the comparison. Hence, in the model at issue, $\Delta_0$ refers to the current evaluation state, i.e., $w_1$. Formula 9 shows what the effects of releasing atom $V$ are. In fact, by abstracting from $V$, state $w_1$ is not distinguishable any more from state $w_2$: $w_1 \sim V w_2$. Hence there exists a state $w_2 \in W$ such that $M, w_2 \models \text{eat\_apple} \wedge \neg V$.

Formulæ 9 and 8 represent Adam & Eve’s situation after and, respectively, before God’s commandment “you shall not eat of the fruit of the tree that is release issues $\text{Iss}$ are the complements $\neg L_i$ of the sublanguages in $\Xi L$. In fact, what is released is just what cannot be expressed. The accessibility relations should therefore be taken to be the sublanguage-equivalence relations $\sim_{-i}$. Notice also that the set $\text{Iss}$ is ordered by set-theoretic inclusion $\subseteq$ between sublanguages of $L$.

It is instructive to notice that although all models based on sublanguage equivalence relations are PRL models, the reverse does not hold. In a sense the characterization in terms of PRL is too liberal. Future work will try to find axiomatizations for characterizing exactly the models based on sublanguage equivalence relations (see Section 5).
in the middle of the garden”. Such commandment introduces a further characterization of reality, exemplified here by the notion of violation, which was not available to Adam & Eve before the commandment was uttered.

4.2 Ascription formalized

God’s commandment not to eat the apple is a statement \( \text{eat\_apple} \rightarrow V \). Let us now suppose the set of all God’s commandments to be \( \Gamma \). Such set naturally defines a context \( i \) whose extension \( W_i \) is just the set of states satisfying \( \Gamma \). Since \( \Gamma \) contains \( \text{eat\_apple} \rightarrow V \), such statement can be studied as a classificatory counts-as statement pertaining to the context \( i \) of divine commands. It corresponds to the validity of strict implication \( [i](\text{eat\_apple} \rightarrow V) \) in the model.

To represent this, we should add contexts to the PRL model introduced in the previous section. Let it be \( M' = \langle W, \{W_i\}, \{
eg \phi \}, \Delta \{\phi\}, \{0, \sim \}, I \rangle \).

Clearly, \( [i](\text{eat\_apple} \rightarrow V) \) will be valid in \( M' \) only if \( W_i \) does not contain state \( w_2 \), since \( M', w_2 \models \text{eat\_apple} \land \neg V \). Leaving technicalities aside, stating \( [i](\text{eat\_apple} \rightarrow V) \) in the Adam & Eve scenario modeled in \( M' \) corresponds to setting the boundaries of the context \( i \) of divine norms \( \Gamma \) in such a way to rule out states in which eating the apple is compatible with the non occurrence of a violation.

We hope the simple example of Adam & Eve to have conveyed the basic ideas behind the study of norms presented here, which builds on Anderson’s reductionist tradition, on the analysis of counts-as presented in [10], and on the notion of propositional sublanguage equivalence. If we now pull these threads together within logic \( \text{Cxt}^* \otimes \text{PRL} \), a new form of reduction can be defined which is based on a sense of counts-as taking its ascriptive aspect into account.

Definition 4. (Ascription of violation: \( \Rightarrow_{\text{As}}^i \))

“\( V \) is ascribed to \( \neg \phi \) in context \( i \)” is formalized in the logic \( \text{Cxt}^* \otimes \text{PRL} \), on a multi-modal language \( \mathcal{L}^{\text{Cxt}} \otimes \mathcal{L}^{\text{PRL}} \) containing atom \( V \) and the set of issues \( \text{Iss} = \text{Sub}(\mathcal{L}) \), with \( \mathcal{L} \) being the non-modal fragment of \( \mathcal{L}^{\text{Cxt}} \otimes \mathcal{L}^{\text{PRL}} \), as follows:

\[
\neg \phi \Rightarrow_{\text{As}}^i V := [i](\neg \phi \rightarrow V) \land \neg [i] \Delta \{\neg \phi \rightarrow V\} \quad (10)
\]

Intuitively, the ascription of violation amounts to a classificatory counts-as\(^8\) (first conjunct of the right-hand side of Formula 10) with the further condition (second conjunct) that the predicated implication does not hold in context \( i \) any more if it is evaluated releasing its consequent (in this case the violation atom \( V \)). It goes without saying that Definition 4 can easily be generalized to cover a notion of ascriptive counts-as \( \phi_1 \Rightarrow_{\text{As}} \phi_2 \) between any two formulae \( \phi_1 \) and \( \phi_2 \), where what is released in the second conjunct of the definition is the alphabet of \( \phi_2 \). The ascription of atom \( V \) is just a special case of ascriptive counts-as. In the

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\(^7\) The definition of contexts by sets of norms has been thoroughly investigated in [10,14] in relation with constitutive rules and with the warning raised in [23]: “no logic of norms without attention to a system of which they form part”.

\(^8\) Note that the stronger form of proper classificatory counts-as could also be used.
next section we briefly sketch some properties of this counts-as operator which adds on the formal analysis of counts-as developed in [10–14].

Definition 4 represents a strengthening of Anderson’s reduction along the line of Formula 5 and 6. It is worth spending a few more words on the right-hand side of Formula 10. Its dual version better displays the key idea behind it: 

\[ \neg \langle i \rangle (\neg \phi \land \neg \forall) \land \neg \forall \langle i \rangle \forall \forall (\neg \phi \land \neg \forall). \]

By releasing the consequent \( V \) of the ascription, it becomes impossible to distinguish states which satisfy \( V \) from states which falsify \( V \). Now, the definition says that, in order for an ascription to hold, there is a state belonging to context \( i \) from which another state \( w' \) outside of context \( i \) can be reached which is indistinguishable from \( w \) once \( V \) is released, and which falsifies the implicative content of the counts-as \((\neg \phi \land \neg \forall)^9\).

Getting back to our example, God’s commandment \( \text{eat}_\text{apple} \rightarrow V \) in Adam & Eve’s scenario can be qualified as an ascription of \( V \) to \( \text{eat}_\text{apple} \) in the context \( i \) denoting states \( w_1, w_3, w_4: \text{eat}_\text{apple} \Rightarrow_{As}^i V \). Notice that it follows from Definition 4 that commandments ascribing the same atom \( V \) to different formulae are all ascriptions since they all equally contribute to the (partial) definition of \( V \) in a given context. Suppose there are two commandments in the set \( \Gamma \) defining the context \( i \) of divine norms: “you shall not eat of the fruit that is in the middle of the garden” and then also “you shall not bare your bodies”. To model such situation we would need 8 states evaluating the three atoms \( \text{eat}_\text{apple}, \text{bare}_\text{bodies}, V \). Applying Definition 4 it is easy to check that \( \text{eat}_\text{apple} \Rightarrow_{As}^i V \) and \( \text{bare}_\text{bodies} \Rightarrow_{As}^i V \) would both be validities of the model.

It is important to stress that Definition 4 captures only a static notion of ascription. Commandments and norms are often added and removed from a set of norms \( \Gamma \) with a certain order: e.g. first it is commanded “you shall not eat of the fruit that is in the middle of the garden” and then also “you shall not bare your bodies”. In this case the first commandment introduces a notion of violation which the second commandment refines. Such phenomenon of norm dynamics fall outside the scope of this paper. However, we are convinced that the static formal framework presented here provides a solid ground also for formal investigations in the dynamics of norms. This is one of our future research priorities.

4.3 On the properties of ascriptive counts-as

There is no space to exhibit a full structural analysis of the syntax of the new counts-as connective \( \Rightarrow_{As}^i \). However, it is worth noticing that it is very similar, structurally speaking, to proper classificatory counts-as \( \Rightarrow_{Cl}^{+i} \) [10, 12]. Like proper classificatory counts-as, it satisfies the core of the structural properties of counts-as isolated in [17] (i.e., left and right logical equivalence, disjunction of the antecedents and conjunction of the consequents) and it falsifies transitivity. However, there are also two essential differences.

9 Typically, state \( w \) satisfies \( \neg \phi \), that is, the antecedent of the counts-as since \( w \) and \( w' \) differ only in the interpretation of atom \( V \). In the Adam & Eve scenario, for instance, \( w = w_1 \) and \( w' = w_2 \).

10 The countermodel of transitivity for \( \Rightarrow_{Cl}^{+i} \) (see [10,12]) works also for \( \Rightarrow_{As}^i \).
First of all, ascriptive counts-as requires non-empty contexts: \([i] \vdash \neg(\phi_1 \Rightarrow^A \phi_2)\). The validity of the property is easily checked semantically. None of the senses of counts-as analyzed in [10–14] enjoys this property. This is not surprising since the ascription of a property to something should presuppose the existence of that something. Secondly, contraposition, i.e., \((\phi_1 \Rightarrow^A \phi_2) \rightarrow (\neg \phi_2 \Rightarrow^A \neg \phi_1)\), is not valid. It fails in all models where a state in context \(i\) can be found which falsifies \(\phi_1 \rightarrow \phi_2\) by releasing \(\phi_2\) but no state in \(i\) can be found which falsifies \(\phi_1 \rightarrow \phi_2\) by releasing \(\phi_1\). This typically happens in models where \(i\) validates also \(\phi_1 \lor \phi_2\). The failure of contraposition is an interesting aspect of \(\Rightarrow^A\) since contraposition was one of the problematic properties of the classificatory view of counts-as. Ascription seems therefore to be a fruitful development of the classificatory perspective pursued in the series of works [10–14]. Further investigations in the structure of \(\Rightarrow^A\) and in its logical relationships with the other senses of counts-as is left for future research.

### 4.4 An ascriptive glance at Jørgensen's dilemma

The first of the ten philosophical problems urging today’s deontic logic according to [15] was the problem, already formulated in [23], concerning a suitable foundation of deontic logic in the face of Jørgensen’s dilemma:

How can deontic logic be reconstructed in accord with the philosophical position that norms are neither true nor false? [15, p. 3]

It is our claim that the ascriptive view of norms can provide the ground for such a reconstruction. Let us sketch how this would work in the case of Adam & Eve scenario. There, God’s commandment does three different things at the same time. First, the commandment defines the context \(i\) of divine norms. As such, formula \(\text{eat}_{\text{apple}} \rightarrow V\) defines the “logical space” [25, p. 6] of the normative system at issue, i.e., the context of the system (states \(w_1, w_3, w_4\)). Notice that, as such, \(\text{eat}_{\text{apple}} \rightarrow V\) is properly speaking neither true nor false, but it is rather taken or assumed to be true, exactly like an axiom. Second, the commandment teaches Adam & Eve how to recognize, to say it with Searle [27], states with a certain “institutional” property (‘violation’) on the ground of a “brute” property (‘eating the apple’). Third, the commandment increases the granularity of Adam & Eve’s language so that they can distinguish state \(w_1\) from state \(w_2\) (and \(w_3\) from \(w_4\)) by making use of suitable “institutional” terms. This is the aspect of language creation proper of the ascriptive view of norms. To sum up, a norm \(\phi \rightarrow V\) in a set of norms \(\Gamma\) works like an axiom defining the context \(i\) of the normative system \(\Gamma\), and defines the violation term(s) \(V\) by ascribing it to term(s) \(\phi\) built from some “brute” language.

With respect to the third point, notice that the statement \(\text{eat}_{\text{apple}} \rightarrow V\) is neither true nor false if a “brute” language is spoken, where the “institutional” term \(V\) is not used. In fact, in the scenario there are states in the model where neither \(\Delta_v(\text{eat}_{\text{apple}} \rightarrow V)\) nor \(\Delta_v(\text{eat}_{\text{apple}} \rightarrow V)\) are true. That is why, to say it with Jørgensen, norms correspond to “quasi-properties” of reality [18,
Properties, or to use Searle’s terminology again, “brute facts” hold independently of the human ascriptive activity, while “quasi-properties” or “institutional facts” hold only as a result of ascription, and in this sense they are in a way less true. Notice, however, that this notion of truth is not the technical one used in Kripke semantics: the notion of truth in Jørgensen’s dilemma (i.e., truth as what is evaluated as true given the brute language) is not the Kripke notion of truth (i.e., truth as what is evaluated as true given the whole language). The logic presented here generalizes this distinction to any possible partition besides the “brute” vs. “institutional” one.

5 Conclusions and future work

By providing Anderson’s reduction with sufficient modal means for supporting a notion of context and of linguistic indistinguishability, the paper has provided an original view of deontic statements as forms of ascriptions (Definition 4). This has been claimed to be a sound perspective for grounding a reduction-based deontic logic in the face of Jørgensen’s dilemma (Section 4.4).

Future work will focus on three aspects: first, a more accurate axiomatic characterization of $\Psi t$ frames with sublanguage equivalence relations will be pursued (see Footnote 6); second, the logical relations between ascriptive counts-as and the other forms of counts-as characterized in [10,14] will be investigated; finally, the dynamic aspect of ascription will be studied making use of some form of update logic in the spirit of, for instance, [5].

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References


