Boolean Role Inclusions in DL-Lite With and Without Time

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Abstract
Traditionally, description logic has focused on representing and reasoning about classes rather than relations (roles), which has been justified by the deterioration of the computational properties if expressive role inclusions are added. The situation is even worse in the temporalised setting, where monodicity is viewed as an almost necessary condition for decidability. We take a fresh look at the description logic DL-Lite with expressive role inclusions, both with and without a temporal dimension. While we confirm that full Boolean expressive power on roles leads to FO²-like behaviour in the atemporal case and undecidability in the temporal case, we show that, rather surprisingly, the restriction to Krom and Horn role inclusions leads to much lower complexity in the atemporal case and to decidability (and ExpSPACE-completeness) in the temporal case, even if one admits full Boolean on concepts. The latter result is one of very few instances breaking the monodicity barrier in temporal FO. This is also reflected on the data complexity level, where we obtain new rewritability results into FO with unary divisibility predicates.

1 Introduction
Description logics (DLs) have often been described as decidable fragments of first-order logic (FO) that model a domain by introducing complex concept descriptions and subsumptions between them. In fact, the main syntactic difference between DLs and FO is that, in the former, one can construct new, complex, concept descriptions from atomic concepts using concept constructors without the explicit use of individual variables. The subsumption relationship between complex concepts is then expressed using concept inclusions (CIs). Interestingly, corresponding role (binary relation) constructors taking as input atomic roles and describing complex roles have never become mainstream except for role composition, thus admitting role inclusions (RIs) of the form $R_1 \circ \cdots \circ R_n \sqsubseteq R$, with appropriate restrictions (Baader et al. 2017). The advantages of even a very limited form of Boolean expressive power on roles is well known (Hustadt and Schmidt 1998; Lutz and Sattler 2000b; Rudolph, Krötzsch, and Hitzler 2008a; Rudolph, Krötzsch, and Hitzler 2008b), so one can only speculate about the reasons for them not becoming more popular. The main issue appears to be that, from a computational perspective, adding Boolean operators on roles leads to expressivity similar to that of the two-variable fragment FO² of FO (Lutz and Sattler 2000a; Lutz, Sattler, and Wolter 2001), which, while still decidable, is significantly more challenging for automated reasoning than typical DL fragments of FO with some form of the tree model property (Grädel, Kolaitis, and Vardi 1997; Vardi 1996). In temporal DLs, the addition of expressivity for roles is even more problematic: just declaring a role to remain constant in time often leads to undecidability (Lutz, Wolter, and Zakharyaschev 2008; Gabbay et al. 2003). Again, the reason is well understood: if one goes beyond the monodic fragment of first-order temporal logic and is thus able to represent how relations change in time, one typically can encode the halting problem for Turing machines by using the relations to represent the tape and time to encode the computation (Gabbay et al. 2003).

Our aim here is to revisit Boolean RIs in the context of (temporal) DL-Lite and introduce logics with new expressivity for roles, for which the knowledge base (KB) satisfiability problem is decidable in the temporal case and of significantly lower complexity than FO² in the atemporal one. Recall that in DL-Lite}$^{\text{g-bool}}$ (Calvanese et al. 2007), also denoted DL-Lite}$^{\text{g-bool}}$, in the classification of (Artale et al. 2009), CIs and RIs take the form of binary Horn (aka core) inclusions $\vartheta_1 \sqsubseteq \vartheta_2$ or $\vartheta_1 \sqcap \vartheta_2 \sqsubseteq$, where the $\vartheta_i$ are either both concepts (that is, concept names or of the form $\exists R$) or roles. The DL-Lite languages we consider extend this schema by allowing CIs and RIs of the form

$$\vartheta_1 \sqcap \cdots \sqcap \vartheta_k \sqsubseteq \vartheta_{k+1} \sqcup \cdots \sqcup \vartheta_{k+m},$$

(1)

where the $\vartheta_i$ are all concepts or, respectively, roles. We classify ontologies by the form of their inclusions. Let $c, r \in \{\text{bool}, \text{g-bool}, \text{horn}, \text{krom}, \text{core}\}$. Then DL-Lite}$^{c}$ is the DL whose ontologies contain CIs and RIs of the form (1) satisfying the following conditions for $c$ and $r$, respectively:

- (horn) $m \leq 1$,
- (core) $k + m \leq 2$ and $m \leq 1$,
- (krom) $k + m \leq 2$,
- (bool) any $k \geq 0$ and $m \geq 0$,
- (g-bool) any $k \geq 1$ and $m \geq 0$.

It follows that core is included in both krom and horn, which are in bool (g-bool stands for guarded bool). The resulting languages provide a new way of classifying ontologies. While the languages DL-Lite}$^{\text{bool}}$ and DL-Lite}$^{\text{g-bool}}$ all have essentially the same expressivity as FO² and inherit NExpTime-completeness of KB satisfiability, the DL-Lite}$^{\text{krom}}$ provide a...
way of introducing ‘covering’ RIs \( \top \sqsubseteq R_1 \sqcup R_2 \) and also the complement of a role via disjointness and covering. Rather surprisingly, these disjunctions come for free as far as the complexity of KB satisfiability is concerned: even combined with Boolean CIs, satisfiability is still in NP, and combined with Krom RIs, it is even in NL. The full table of our complexity results is given below:

<table>
<thead>
<tr>
<th>RIs \ CI</th>
<th>(g-)bool</th>
<th>krom</th>
<th>horn</th>
<th>core</th>
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</thead>
<tbody>
<tr>
<td>bool</td>
<td>NEXPTIME</td>
<td></td>
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<tr>
<td>g-bool</td>
<td>EXPSPACE</td>
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<tr>
<td>krom</td>
<td>NP</td>
<td>NL</td>
<td>NP</td>
<td>NL</td>
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<tr>
<td>horn</td>
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<td>P</td>
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<td>core</td>
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Our main aim in this paper is to investigate extensions of these DL-Lite languages with the standard linear temporal logic (LTL) operators \( \Box_p, \Diamond_p \) (always in the future/past) and \( \Box_p \lor \Diamond_p \) (in the next/previous moment) interpreted over the timeline \( (\mathbb{Z}, \prec) \). The temporal DLs have an additional parameter \( o \in \{\Box, \Diamond, \Diamond^\text{core} \} \):

\[
\text{DL-Lite}^{\text{core}}_{\text{g-bool }} \text{allows ontologies whose axioms (1) may contain operators from } o \text{ (e.g., } o = \Box \text{ permits } \Box_p \lor \Diamond_p \text{ only) and comply with } c \text{ for CIs and } r \text{ for RIs. A CI or RI is satisfied in a model if it holds globally, at all time points in } \mathbb{Z}. \text{ Even in the minimal language } \text{DL-Lite}^{\text{core}}_{\text{g-bool}} \text{, we can state in FO with relational primitive recursion, which entails NC}^1 \text{ completeness for data complexity.}

The inevitable fly in the ointment is that there is a \( \text{DL-Lite}^{\text{core}}_{\text{g-bool/g-bool }} \text{ ontology for which consistency with a given input data is undecidable.}

### 2 Preliminaries

We use the standard DL syntax and semantics. Let \( a_i, i < \omega \), be individual names, \( A_i \) concept names, and \( P_i \) role names. We define roles \( S \), basic concepts \( B \), temporalised roles \( R \) and temporalised concepts \( C \) by the following grammar:

\[
\begin{align*}
S &::= P_i \mid P_i^c, \\
B &::= A_i \mid \exists S, \\
R &::= S \mid \Box_p R \mid \Diamond_p R \mid \Diamond^\text{core}_p R \mid \Diamond^\text{g-bool}_p R, \\
C &::= B \mid \Box_p C \mid \Diamond_p C \mid \Diamond^\text{core}_p C \mid \Diamond^\text{g-bool}_p C.
\end{align*}
\]

A concept or role inclusion (CI or RI) takes the form (1), where the \( \theta_i \) are all temporalised concepts or, respectively, all temporalised roles. (The empty \( \Box \) and the empty \( \Diamond \) are \( \bot \)). A TBox \( T \) and an RBox \( R \) are finite sets of CIs and, respectively, RIs; their union \( O = T \cup R \) is an ontology. The temporal \( \text{DL-Lite}^{\text{core}}_{\text{g-bool/g-bool }} \) were defined in the introduction. We also set \( \text{DL-Lite}^{\text{core}}_{\text{g-bool/g-bool }} = \text{DL-Lite}^{\text{core}}_{\text{g-bool}} \).

To illustrate, imagine an estate agency describing properties by their proximity to various amenities, using roles \( \text{wd} \) for ‘walking distance’ and \( dd \) for ‘driving distance’. Then we can state in \( \text{DL-Lite}^{\text{core}}_{\text{g-bool/g-bool }} \) that \( \Box \text{wd} \sqsubseteq \Diamond \text{dd} \), and that these roles are disjoint (\( \text{wd} \sqcap \text{dd} \sqsubseteq \bot \)) and symmetric (e.g., \( \text{wd} \sqsubseteq \Diamond \text{dd} \)), and describe locations using CIs such as \( \text{FamilyLocation} \sqsubseteq \exists \text{wd}.\text{School} \sqcap \exists \text{dd}.\text{Pub} \) (which requires fresh auxiliary role names). In \( \text{DL-Lite}^{\text{core}}_{\text{g-bool/g-bool }} \), we can further say that \( \Box \text{wd}.\text{WellConnected} \) (see Theorem 1). In \( \text{DL-Lite}^{\text{core}}_{\text{g-bool/g-bool }} \), we can also express that, over the past three years, there has been a pub within walking distance: \( \Box \text{wd}.\text{Pub} \sqsubseteq \exists \text{wd}.\text{Pub} \sqcap \exists \text{dd}.\text{Pub} \).

An ABox, \( A \), is a finite set of atoms of the form \( A_i (a, b, \ell) \) and \( P_i (a, b, \ell) \), where \( a, b \) are individual names and \( \ell \in \mathbb{Z} \). We denote by \( \text{ind}(A) \) the set of individual names in \( A \), by \( \text{min}A \) and \( \text{max} A \) the minimal and maximal integers in \( A \), and set \( \text{ten}(A) \) for some \( n \in \mathbb{Z} \), \( \text{min} A \leq n \leq \text{max} A \).

For simplicity, we assume that \( \text{min} A = 0 \). A \( \text{DL-Lite}^{\text{core}}_{\text{g-bool/g-bool }} \) knowledge base (KB) is a pair \( \langle O, A \rangle \), where \( O \) is a \( \text{DL-Lite}^{\text{core}}_{\text{g-bool/g-bool }} \) ontology and \( A \) an ABox. The size \( |O| \) of \( O \) is the number of occurrences of symbols in it; the size of a TBox, RBox, ABox and KB is defined in the same way, with unary encoding of numbers in ABoxes.

A (temporal) interpretation is a pair \( \mathcal{I} = (\Delta^\mathcal{I}, I^\mathcal{I}(n)) \), where \( \Delta^\mathcal{I} \neq \emptyset \) and, for each \( n \in \mathbb{Z} \),

\[
I^\mathcal{I}(n) = (\Delta^\mathcal{I}, a_0^\mathcal{I} \ldots, a_n^\mathcal{I}, \ldots, P_0^\mathcal{I}(n), \ldots)
\]

is a standard DL interpretation with \( a_i^\mathcal{I} \in \Delta^\mathcal{I} \), \( A_i^\mathcal{I}(n) \subseteq \Delta^\mathcal{I} \) and \( P_i^\mathcal{I}(n) \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I} \). The DL constructs and temporal operators are interpreted in \( I^\mathcal{I}(n) \) as usual:

\[
\begin{align*}
\neg P_i^\mathcal{I}(n) &\equiv \{ (u, v) \mid (v, u) \in P_i^\mathcal{I}(n) \}, \\
\exists S &\subseteq I^\mathcal{I}(n) \equiv \{ u \mid (u, v) \in S^\mathcal{I}(n) \}, \text{ for some } v, \\
\Box p &\subseteq I^\mathcal{I}(n) = \bigcap_{k>n} I^\mathcal{I}(k), \\
\Diamond_p I^\mathcal{I}(n) &\subseteq I^\mathcal{I}(n+1), \\
\Diamond^\text{core}_p I^\mathcal{I}(n) &\subseteq I^\mathcal{I}(n+1), \\
\Diamond^\text{g-bool}_p I^\mathcal{I}(n) &\subseteq I^\mathcal{I}(n+1).
\end{align*}
\]
Cl and RLs are interpreted in $I$ globally in the sense that inclusion (1) is true in $I$ if
\[ \psi(n) \cup \psi(k) \subseteq \psi(k+1) \cup \psi(k+2), \quad \text{for all } n \in \mathbb{Z}. \]

For an inclusion $\alpha$, we write $I \models \alpha$ if $\alpha$ is true in $I$. We call $I$ a model of $(O, A)$ and write $I \models (O, A)$ if $I \models \alpha$ for all $\alpha \in \mathcal{O}$, $a^2 \in A^2(t)$ for $(a, t) \in A$, and $(a^2, b) \in P^2(t)$ for $P(a, b) \in A$. A KB has a satisfiable model if it has a model.

It is to be noted that the LTL operators $\bigcirc_p$ (eventually), $\mathcal{U}$ (until) and their past counterparts can be expressed in $\text{bool}$ using $\bigcirc_p \mathcal{O}_p$ and $\bigcirc \mathcal{O}_p$ (Fisher, Dixon, and Peim 2001; Artale et al. 2013). In many cases, one does not need full Boolean: $\bigcirc_p R \in Q$ is equivalent to $R \in \mathcal{Q}_p$, which can be expressed in $DL-Lite^\omega$ as $R \in \bigcirc_p S, S \in \mathcal{Q}_p$, where $S$ is fresh. Immediately it follows that convexity of $R$ (that is, $\bigcirc_p R \cap \bigcirc_p R \in R$) can be expressed in $DL-Lite^\omega$ and $DL-Lite^\omega$. Then $R \in \bigcirc_p S$ can be simulated in $DL-Lite^\omega$ using $\bigcirc_p R \in \mathcal{Q}_p$ and $\bigcirc_p \mathcal{Q}_p \in \mathcal{Q}_p$. That is, the lifespan of $R$ is bounded can be expressed in $DL-Lite^\omega$ using $\bigcirc_p R \in \mathcal{Q}_p$ and $\bigcirc_p \mathcal{Q}_p \in \mathcal{Q}_p$.

We are interested in the combined and data complexities of the satisfiability problem for KBs: for the former, both the ontology and the ABox of a KB are regarded as input, while for the latter, the ontology is fixed. We assume that $\#\mathcal{O}(\mathcal{A}) \geq \#\mathcal{A}(\mathcal{A})$ in any input ABox $A$ (if this is not so, we add the required number of dummies with the missing timestamps to $A$). Let $\#\mathcal{A}(\mathcal{A}) = \{a_1, \ldots, a_m\}$. We encode $A$ as a structure $\mathcal{S}_A$ with domain $\#\mathcal{O}(\mathcal{A})$ ordered by $\prec$ such that $\mathcal{S}_A \models A(k, \ell) \iff A(a_k, \ell) \in A$ and $\mathcal{S}_A \models P(k, \ell, \ell^\prime) \iff P(a_k, a_{\ell^\prime}, \ell^\prime) \in A$.

We establish our data complexity results by ‘rewriting’ ontologies to FO-sentences ‘accepting’ or ‘rejecting’ the input ABoxes. Let $L$ be a class of FO-sentences interpreted over $\mathcal{S}_A$. Say that $\Phi \in L$ is an $L$-rewriting of $O$ if, for any ABox $A$, the KB $(O, A)$ is satisfiable iff $\mathcal{S}_A \models \Phi$. Here, we need three classes $L$: (i) $\text{FO}(\prec)$ with binary and ternary predicates of the form $A(x, t)$ and $P(x, y, t)$ as well as $\prec$ and $\equiv$; (ii) $\text{FO}(\prec, \equiv)$ with extra unary predicates $t \equiv 0 (mod n)$, for any $n > 1$, and (iii) $\text{FO}(\text{RPR})$ that extends FO with relational primitive recursion, which allows one to construct formulas such as

\[
\begin{align*}
Q_1(z, t, x) &\equiv \theta_1(z_1, z, t, Q_1(z_1, z, t-1), \ldots, Q_n(z_n, t-1)) \\
& \vdots \\
Q_n(z_n, t, x) &\equiv \theta_n(z_1, z, t, Q_1(z_1, z, t-1), \ldots, Q_n(z_n, t-1))
\end{align*}
\]

where [\ldots] defines recursively, via the formulas $\theta_i$, the interpretations of the predicates $Q_i$ in $\Psi$. For data complexity, evaluation of FO$(\prec, \equiv)$-sentences over $\mathcal{S}_A$ is known to be in LOGTIME-uniform $AC^0$ (Immerman 1999) and evaluation of FO(RPR)-sentences is in $NC^1$ (Compton and Laflamme 1990).

3 Reasoning with Atemporal DL-Lite

To begin with, we establish the complexity of reasoning with the plain DLs underlying the temporal DL-Lite$^\omega_{c/r}$ introduced above. We denote them by DL-Lite$^\omega_{c/r}$ where as before $c, r \in \{\text{bool}, g\text{-bool}, \text{horn}, krom, \text{core}\}$. The satisfiability problem for DLs of the form DL-Lite$^\omega_{c/r}$ was studied by (Calvanese et al. 2007; Artale et al. 2009): it is NP-complete for DL-Lite$^\omega_{bool_{c/r}}$, P-complete for DL-Lite$^\omega_{horn_{c/r}}$, and NL-complete for DL-Lite$^\omega_{horn_{bool_{c/r}}}$ and DL-Lite$^\omega_{horn_{bool_{c/r}}}$ KBs.

We show that DL-Lite$^\omega_{bool_{c/r}}$ can be regarded as a notational variant of the extension $\text{ACCT}^{\omega_1}$ of AC with inverse roles and Boolean operators on roles. This logic has, in turn, almost the same expressive power as FO$^2$, except that the identity role has to be added. In detail, let $\text{ACCT}^{\omega_1}$ be the DL with roles $S$ and concepts $C$ defined by

\[
S, S' \equiv T | P_i | S \sqcap S' | \neg S | S', \\
C, C' \equiv T | A_i | \exists S.C | C \sqcap C' | \neg C.
\]

An $\text{ACCT}^{\omega_1}$ CI takes the form $C \subseteq C'$ (Lutz, Sattler, and Wolter 2001; Lutz and Sattler 2000a; Gavrog and Passy 1990). We say that a KB $K$ is a model conservative extension of a KB $K'$ if $K \models K'$, the signature of $K$ contains the signature of $K'$, and every model of $K'$ can be extended to a model of $K$ by providing interpretations of the fresh symbols of $K$ and leaving the domain and the interpretation of the symbols in $K'$ unchanged.

Theorem 1. (i) For every DL-Lite$^\omega_{bool_{c/r}}$ KB, one can compute in logarithmic space an equivalent $\text{ACCT}^{\omega_1}$ KB.

(ii) For every $\text{ACCT}^{\omega_1}$ KB, one can compute in log-space a model conservative extension in DL-Lite$^\omega_{bool_{c/r}}$.

Proof. (i) Clearly, any CI in $\text{ACCT}^{\omega_1}$ is an $\text{ACCT}^{\omega_1}$ CI $(\exists R = \exists R.T \mathcal{T})$. Any RI $S_1 \sqcap \cdots \sqcap S_k \sqcap S_{k+1} \sqcup \cdots \sqcup S_{k+m}$ in $\text{ACCT}^{\omega_1}$ is equivalent to the $\text{ACCT}^{\omega_1}$ CI $\exists R.T \sqcup \cdots \sqcup \exists R.T$, where $R$ abbreviates $S_1 \sqcap \cdots \sqcap S_k \sqcap \neg S_{k+1} \sqcup \cdots \sqcup \neg S_{k+m}$.

(ii) For any $\text{ACCT}^{\omega_1}$ KB $K$, we construct a model conservative extension of $K$ in $\text{ACCT}^{\omega_1}$ with CI in normal form:

\[
A \subseteq \forall S.B, \forall S.B \subseteq A, A \sqcap A_1 \subseteq B, A \subseteq B, A_1 \subseteq B, A_2 \subseteq B, A \sqcap B, A \sqcap B \subseteq A, A_1 \sqcap A_2 \subseteq B, A \subseteq B, A_1 \subseteq B, A_2 \subseteq B,
\]

where $A, A_1, A_2$ range over concept names and $\mathcal{T}$. Next, we replace CI $A \subseteq \forall S.B$ and $\forall S.B \subseteq A$ by $S \sqcap Q \sqcup R$, $\exists Q \subseteq B, B \subseteq \neg A$, and, respectively, $\neg A \subseteq \exists R, R \subseteq S, \exists R \subseteq \neg B$, with fresh role names $Q$ and $R$. Finally, RSs with a Boolean $S$ are transformed into normal form (1) to obtain a model conservative extension of $K$ in DL-Lite$^\omega_{bool_{c/r}}$.

The $\text{NEXPTime}$-completeness of $\text{ACCT}^{\omega_1}$ KB satisfiability (Lutz, Sattler, and Wolter 2001) implies that DL-Lite$^\omega_{bool_{c/r}}$ KB satisfiability is also $\text{NEXPTime}$-complete. To bring down the complexity to $\text{ExpTime}$, it suffices to avoid unguarded quantification by admitting only RSs with a non-empty left-hand side, as in the $\text{g\text{-bool}}$ RS. Then, for any DL-Lite$^\omega_{bool_{c/r}}$ KB, it is straightforward to compute in linear time an equivalent KB in the guarded two-variable fragment GF$^2$ of FO. Using the fact that KB satisfiability for the latter logic is in $\text{ExpTime}$ (Grädel 1999), we obtain the following:

Theorem 2. KB satisfiability is $\text{NEXPTime}$-complete for DL-Lite$^\omega_{bool_{c/r}}$ and $\text{ExpTime}$-complete for DL-Lite$^\omega_{bool_{c/r}}$.

We now show that the DL-Lite logics with Horn and Krom RS are reducible to propositional logic. For an ontology $O$, let $\text{role}^\omega(O) = \{P, P^- | P \text{ a role in } O\}$ and let $O = T \cup R$. We assume that $R$ is closed under taking the inverses of roles in $R$. Denote by $\sub T$ the set
of concepts in $T$ and their negations. A concept type $\tau$ for $T$ is a maximal subset of $\tau$ of sub-$\tau$ that is ‘propositionally’ consistent with $T$: if $B_1, \ldots, B_k \in \tau$ and $T$ contains $B_1 \sqcap \cdots \sqcap B_k \subseteq B_{k+1} \sqcup \cdots \sqcup B_{k+m}$, then one of $B_{k+1}, \ldots, B_{k+m}$ is also in $\tau$ (note, however, that $\tau$ does not have to be consistent with $T$ as it can contain $\exists P$ even if $\exists P \not\subseteq \bot$ is in $T$). Clearly, for an interpretation $J$ and $u \in \Delta^J$, the set comprising all $B \in \sub\tau$ with $u \in B^\wedge$ and all $\neg B \in \sub\tau$ with $u \notin B^\wedge$ is a concept type for $T$; it is denoted by $\tau^J_u$ and called the concept type of $u$ in $J$. Similarly, let $\sub\tau$ be the set of roles in $\tau$ and their negations. A role type $r$ for $\tau$ is a maximal subset of $\sub\tau$ propositionally consistent with $\tau$. For $(u, v) \in \Delta^J$, the set comprising all $S \in \sub\tau$ with $(u, v) \in S^\wedge$ and all $\neg S \in \sub\tau$ with $(u, v) \notin S^\wedge$ is a role type for $\tau$; it is denoted by $\rho^J_{u,v}$ and called the role type of $(u, v)$ in $J$. For a set of role literals (roles and their negations), let $\cl_{\tau^J}(\Xi)$ be the set of all role literals $L'$ such that $\tau \models \bigwedge_{i \in S} L \equiv L'$. The following lemma plays a key role in the reduction.

**Lemma 3.** For any satisfiable DL-Lite$^\text{krom}$ KB $K = (O, A)$, $O = T \cup R$, there is a model $\I = (\Delta^\I, \tau^\I)$ of $K$ such that

$$\Delta^\I = \mathfrak{ind}(O) \cup \{w^i_s \mid S \in \role^+(O) \text{ and } 0 \leq i < 3\}$$

and $(u, v) \in S^\wedge$, for every $u \rightarrow S v$ with $u \in (\exists S)^T$, where

$$\rightarrow_S = \{(a, w^0_s) \mid a \in \mathfrak{ind}(A)\} \cup \{(w^i_{R_i}, w^i_{S^+_i}) \mid w^i_R \in \Delta^\I\}$$

and $\oplus$ is addition modulo 3. In particular, DL-Lite$^\text{krom}$ has the linear model property: $|\Delta^\I| = |\mathfrak{ind}(A)| + |\sub\tau|$.

**Proof.** Given a model $\I = (\Delta^\I, \tau^\I)$ of $K$, we construct $\I$ as follows. For any $S \in \role^+(O)$, if $S^\wedge \neq \emptyset$, then we pick $w_S \in (\exists S)^T$; otherwise, we pick any $w_S \in S^\wedge$. We assume that the $w_S$ are distinct. Let $\Delta^\I$ comprise $\mathfrak{ind}(A)$ and three copies $w^0_s, w^1_s, w^2_s$ of each $w_S$; cf. (Börger, Grädel, and Gurevich 1997, Proposition 8.1.4). This also fixes the $\rightarrow_S$. Define $f : \Delta^\I \rightarrow \Delta^\I$ by taking $f(a) = a$, for all $a \in \mathfrak{ind}(A)$, and $f(w^i_s) = w^i_s$, for all $S$ and $i$. We then set $\tau_u = \tau^I_u$, for all $u \in \Delta^\I$. To define $\rho_{u,v}$ for $u, v \in \Delta^\I$, we consider the following three cases.

- If $u, v \in \mathfrak{ind}(A)$, then we take $\Xi = \{S \mid S(a, b) \in A\}$, assuming $\mathfrak{ind}(A) \subseteq A$ whenever $\mathfrak{ind}(b, a) \in A$.

- If $\exists S \in \tau_u$ and $u \rightarrow_S v$, then we take $\Xi = \{S\}$.

- Otherwise, we take $\Xi = \emptyset$.

We begin with $\rho_{u,v} = \cl_{\tau^J}(\Xi)$ and perform the following procedure for each RI $T \subseteq S_1 \sqcup S_2$ in $R$ such that none of $S_1$ and $\neg S_1$ is in $\rho_{u,v}$. As $\I = R$, either $S_1$ or $S_2$ is in $\rho^J_{u,v}$. So $\rho_{u,v}$ is extended with the respective $\cl_{\tau^J}(\{S_1\})$. Since any contradiction derivable from Krom formulas is derivable from two literals, the resulting $\rho_{u,v}$ is consistent with $\tau$ and both $\tau^J$ and $\tau_{u,v}$-compatible: that is, $\exists R \in \tau_u$ and $\forall R \in \tau_{u,v}$, for all $R \in \rho_{u,v}$. One can check that the constructed $\tau_u$ and $\rho_{u,v}$, for $u, v \in \Delta^\I$, are types for $\tau$ and $R$, respectively, and give rise to a model of $K$. $\square$

The existence of a model $\I$ from Lemma 3 can be encoded by a propositional formula $\varphi_K$ whose propositional variables take the form $B^I(u)$ and $P^I(u, v)$, for $u, v \in \Delta^\I$, assuming that $(P^J)^{\sim}(u, v) = P^I(u, v)$. The formula $\varphi_K$ is a conjunction of the following, for all $u, v \in \Delta^\I$:

$$B^I_1(u) \land \cdots \land B^I_k(u) \rightarrow B^I_{k+1}(u) \land \cdots \land B^I_{k+m}(u),$$

for $\forall B_1 \sqcap \cdots \sqcap B_k \subseteq B_{k+1} \sqcup \cdots \sqcup B_{k+m}$ in $T$, $S^I_1(u, v) \rightarrow S^I_2(u, v)$, for $\forall S_1 \sqsubseteq S_2$ in $R$, $\neg S^I_1(u, v) \lor \neg S^I_2(u, v)$, for $\forall S_1 \sqsubseteq S_2 \subseteq \bot$ in $R$, $S^I_1(u) \lor S^I_2(u)$, for $\forall S_1 \sqsubseteq S_2$ in $R$, $\forall A^I(a) \in A$, and $P^I(a, b)$, for $P(a, b) \in A$, $(\exists S)^I(u) \rightarrow S^I_1(u, v)$, for each $S$ with $u \rightarrow_S v$, $S^I_1(u) \lor S^I_2(u) \rightarrow (\exists S)^I(u)$, for each $S$.

Clearly, $K$ is satisfiable iff $\varphi_K$ is satisfiable. Also, if $K$ is in DL-Lite$^\text{krom}$, then $\varphi_K$ is a Krom formula constructed by a logspace transducer. Now, since DL-Lite$^\text{horn}$ can express DL-Lite$^\text{krom}$ (Krom RIs can simulate Krom CIs, and the latter can express the complement of concepts), we obtain:

**Theorem 4.** Satisfiability is NP-complete for DL-Lite$^\text{krom}$ and DL-Lite$^\text{horn}$ KBs, and NL-complete for DL-Lite$^\text{krom}$.

The next theorem is proved by a similar argument. However, for DL-Lite$^\text{krom}$ we use a polynomial (rather than logspace) reduction into Krom propositional logic.

**Theorem 5.** Satisfiability is NP-complete for DL-Lite$^\text{horn}$ KBs, and P-complete for DL-Lite$^\text{krom}$ and DL-Lite$^\text{krom}$ KBs.

### 4 Satisfiability of Temporal KBs

We now consider extensions DL-Lite$^\omega$ of DL-Lite$^\omega$ with temporal operators in $\mathfrak{o} \subseteq \{\land, \lor, \Box, \Diamond\}$ that can be applied to concepts and roles. Our first observation is negative:

**Theorem 6.** Satisfiability in DL-Lite$^\omega$ is undecidable.

**Proof.** The proof is by reduction of the undecidable $\mathbb{N} \times \mathbb{N}$-tiling problem (Berger 1966). Given a set $\Xi = \{1, \ldots, m\}$ of tile types, with the colours on the four edges of tile type $i$ denoted by $up(i), down(i), left(i)$, and $right(i)$, we define the following DL-Lite$^\omega$ ontology $O$, where $R_i$ is a role name associated with the tile type $i \in \Xi$:

$$I \subseteq \bigcup_{x \in \Xi} \exists R_i, \quad R_i \subseteq \bigcup_{right(i) = left(j)} \Box_p R_j,$$

$$R_i \subseteq \bigcup_{up(i) = down(j)} \exists R_j, \quad R_i \cap \exists R_j \subseteq \bot,$$

for $i \neq j$.

Then $(O, \{I(a, 0)\})$ is satisfiable iff $\Xi$ can tile $\mathbb{N} \times \mathbb{N}$. $\square$

Fortunately, the temporal DL-Lite languages with Krom, Horn and core RIs turn out to be less naughty. In the remainder of this section, we develop reductions of these languages to propositional and first-order LTL with one variable.

Given a DL-Lite$^\omega$ KB $K = (T \cup R, A)$, we construct a first-order temporal sentence $\Phi_K$ with one free variable $x$. We assume that $K$ has no nested temporal operators and that, in RIs of the form $T \subseteq R_1 \sqcup R_2$ from $R$, both $R_1$
are plain (atemporal) roles; also, $R$ is closed under taking
the inverses of roles in $R$. First, we set $\Phi_\text{K} = \bot$ if $(R, A)$
is unsatisfiable. Otherwise, we treat concept names and basic
concepts in $K$ as unary predicates and define $\Phi_\text{K}$ as a
conjunction of the following sentences, where $\square = \Box_p \neg p$:
\[
\forall x [C_1(x) \land \cdots \land C_k(x) \rightarrow C_{k+1}(x) \lor \cdots \lor C_{k+m}(x)],
\]
for $C_1 \cap \cdots \cap C_k \subseteq C_{k+1} \cup \cdots \cup C_{k+m}$ in $T$, \n$
\forall x [\exists S_1(x) \lor \exists S_2(x)]$ and \n$\exists S_1(x) \lor \exists S_2(x)$, for $R \models S_1 \subseteq S_2$ in $R$, \n$\Box_p A(a)$, for $A(a, \ell) \in A$, \n$\Box_p \exists P(a)$ and $\Box_p \exists P^- (b)$, for $P(a, b, \ell) \in A$, \n$[\Box x P(x) \leftrightarrow \Box x P^-(x)]$, for role name $P$ in $T$, \nand, for every $R \models \exists \tau_1 \exists \tau_2$ with $R \models \tau_1 \subseteq \tau_2$, where each $\tau_j$ is $\Box_p$, $\Box_i$, or blank, and $\tau_1 \subseteq \tau_2$ can be $\top$ and \n$\tau_2 \subseteq \tau_1$ can be $\bot$, the sentence
\[
\forall x [\Box_\tau \exists \tau_1(x) \rightarrow \Box_\tau \exists \tau_2(x)].
\]
We observe that $R \models \exists \tau_1 \subseteq \tau_2$ can be checked in $P$ (Artale et al. 2014, Lemma 5.3), and so $\Phi_\text{K}$ is constructed in polynomial time.

**Lemma 7.** A DL-Lite$^\text{bool}/\text{krom}$ KB is satisfiable iff $\Phi_\text{K}$ is satisfiable.

**Theorem 8.** The satisfiability problem for DL-Lite$^\text{bool}/\text{krom}$ KBs is ExpSpace-complete.

**Proof.** The upper bound follows from Lemma 7 since the one-variable fragment of first-order LTL is known to be ExpSpace-complete (Halpern and Vardi 1989; Gabbay et al. 2003); hardness is proved by reduction of the $(2^n - 1)$-corridor tiling problem (Van Emde Boas 1997): given a finite set $\Sigma$ of tile types $\{1, \ldots, m\}$ with four colours $up(i), down(i), left(i)$ and $right(i)$ and a distinguished colour $W$, decide whether $\Sigma$ can tile the grid $\mathbb{N} \times \{s \, | \, 1 \leq s < 2^n\}$ so that $(b_1)1$ tile 0 is placed at $((0), 1)$, $(b_2)$ every tile $i$ placed at every $(c, 1)$ has $down(i) = W$, and $(b_3)$ every tile $i$ placed at every $(c, 2^n - 1)$ has $up(i) = W$.

Let $A = \{A(a, 0)\}$ such that $\Sigma \subseteq \Box_p D, D \subseteq \Box_p D, A \subseteq \Box_p \exists P, \exists P^- \subseteq \Box_T$, \n$T_i \subseteq \Box_p \exists T_j, T_i \cap \exists S_j \subseteq \bot, T \subseteq S_1 \cup S_2$, for $i \in \Sigma$, \n$Q_j \models \exists Q_i \subseteq \bot$, for $i, j \in \Sigma$ with $up(i) \neq down(j)$.

Observe that $(\mathcal{O}, A)$ is satisfiable iff there is a placement of tiles on the grid: each of the $(2^n - 1)$-successors of $a$ created at moments $1, \ldots, 2^n$ represents a corridor column. However, the size of the CIs is exponential in $n$. We now describe how they can be replaced by polynomial-size CIs.

Consider a CI $A \subseteq \Box_p D$. We express it using the following CIs, for $k \in \mathbb{N}$ and $j < k$:
\[
A \subseteq \Box_p (\neg B_{n-1} \land \cdots \land \neg B_0) \land B_{n-1} \land \cdots \land B_0 \subseteq D, \neg B_k \land B_{k-1} \land \cdots \land B_0 \subseteq \Box_p (B_k \land \neg B_{k-1} \land \cdots \land \neg B_0), \neg B_j \land \neg B_k \subseteq \Box_p \neg B_j, \text{ and } B_j \land \neg B_k \subseteq \Box_p B_j,
\]
which have to be converted into normal form (1). Intuitively, they encode a binary counter from 0 to $2^n - 1$, where $\neg B_j$ and $B_k$ stand for the $j$th bit of the counter is 0 and, respectively, 1. The CIs of the form $C_1 \subseteq \Box_p C_2$ are handled similarly. For $A \subseteq \Box_\tau \exists \tau_1 \exists \tau_2$, we use the $B_k \subseteq \exists \tau_2$, for $0 \leq k < n$, instead of $B_{n-1} \cdots \land B_0 \subseteq \exists \tau_2$.

To ensure that $(\mathcal{O}, A)$ is satisfiable, we add to $\mathcal{O}$ the CIs
\[
A \land \Box_p \exists \tau_1 \subseteq \bot, \text{ for } i \in \mathcal{T} \setminus \{0\}, \nD \land \Box_p \exists \tau_1 \subseteq \bot, \text{ for } \downarrow(i) \neq W, \n\Box_p D \land \Box_p \exists \tau_1 \subseteq \bot, \text{ for } up(i) \neq W.
\]
One can show that $(\mathcal{T}, A)$ is as required. \n
Let $K = (T \cup R, A)$ be a DL-Lite$^\text{bool}/\text{horn}$ KB. We assume that $R$ is closed under taking the inverses of roles in $R$s and contains all roles in $\mathcal{T}$. A *beam* $b$ for $\mathcal{T}$ is a function from $Z$ to the set of concept types for $T$ such that, for all $n \in Z$,
\[
\Box_p B \models b(n) \iff c \models b(n + 1), \n\Box_p \exists P \models b(n) \iff c \models b(k), \text{ for all } k > n,
\]
and symmetric conditions for the past-time operators. The function $b^*_n: n \mapsto \{c \in \text{sub}_{\mathcal{T}} \mid u \in \mathcal{T}(n))\}$ (we specify only the positive component of types) is a beam, for any $\mathcal{T}$ and $u \in \Delta^\mathcal{T}$; we will refer to it as the *beam* of $u$ in $\mathcal{T}$.

A *rod* $r$ for $\mathcal{A}$ is a function from $Z$ to the set of role types for $\mathcal{A}$ such that $(2)$-$(3)$ and their past-time counterparts hold for all $n \in Z$ with replaced by $r$ and $c$ by temporalised roles $S$. For any $\mathcal{A}$ and any $u, v \in \Delta^\mathcal{A}$, the function $r^*_u: n \mapsto \{r \in \text{sub}_{\mathcal{A}} \mid (u, v) \in \mathcal{A}(n))\}$ is a rod for $\mathcal{A}$. Fix individual names $d, e$. Since the CIs in $\mathcal{A}$ are Horn, given any $\mathcal{A}, b$ with atoms of the form $S(d, e, f, \ell)$, define the $\mathcal{R}$-canonical rod $r_{\mathcal{A}}$ for $\mathcal{A}$ (consistent with $\mathcal{R}$):
\[
r_{\mathcal{A}}: n \mapsto \{r \in \text{sub}_{\mathcal{A}} \mid r_{\mathcal{A}} \models d(e, n))\}. \text{ In other words, } \mathcal{R} \text{-canonical rods are the minimal rods for } \mathcal{R} \text{ containing all }\}
\]

Given a rod $r$, a rod $c$ is said to be $c$-compatible if $\exists S \in \mathcal{B}(n)$ $\forall S \in \mathcal{B}(n)$, for $n \in Z$ and a basic concept $S$. We are now fully equipped to prove the following characterisation of DL-Lite$^\text{bool}/\text{horn}$ KBs satisfiability, where beams can be ‘shifted’ in (4) to achieve a finite representation.

**Lemma 9.** Let $K = (T \cup R, A)$ be a DL-Lite$^\text{bool}/\text{horn}$ KB. Let $\Delta = \text{ind}(A) \cup \{w_S \mid S \in \text{role}^3(R)\}$. Then $K$ is satisfiable iff there are beams $b_\mathcal{A}, w_\mathcal{A}, \Delta \subseteq \Delta$ for $\mathcal{T}$ such that
\[
A \models b_\mathcal{A}(l), \text{ for all } A(a, \ell) \in A, \nW_\mathcal{A} \models b_\mathcal{A}(l), \text{ for all } W_\mathcal{A}(a, \ell) \in A, \n\exists S \models b_\mathcal{A}(n) \iff \exists S \models b_\mathcal{A}(n), \text{ for some } k \in Z, \n\forall a, b \in \text{ind}(A), \text{ there is a } b_\mathcal{A}\text{-compatible rod } r \text{ for } \mathcal{R} \text{ with } S \models r(\ell), \text{ for all } S(a, b, \ell) \in A, \n\exists S \models b_\mathcal{A}(n) \iff \exists S \models b_\mathcal{A}(n), \text{ with } S \models r(n).
\]
We illustrate the construction by the following example.
Example 1. Let $\mathcal{K} = (\mathcal{O}, \{Q(a, b, 0)\})$, where $\mathcal{O}$ consists of $\exists Q \sqcap \exists P$. Beams and rods in Lemma 9 are depicted below:

Beams $b_u$, $b_o$ and $b_{ua}$ are shown by horizontal lines: the concept type contains $\exists P$ or $\exists Q$ whenever the large node is grey; similarly, the type contains $\exists P^\perp$ or $\exists Q^\perp$ whenever the large node is white (the label of the arrow specifies the role); we omit $A$ to avoid clutter. The rods are the arrows between the pairs of horizontal lines. For example, the rod in (5) for $a$ and $b$ is labelled by $r_{a,b}$: it contains only $Q$ at 0 (only the positive components of types are given); the rod in (5) for $b$ and $a$ is labelled by $r_{b,a}$: and in this case, it is the mirror image of $r_{a,b}$. In fact, if we choose $\mathcal{R}$-canonical rods in (5), then the rod for any $a$ and $b$ will be the mirror image of the rod for $a$, $b$. The rod $r_{p,2}$ required by (6) for $\exists P$ on $b_u$ at moment 2 is depicted between $b_u$ and $g_{p,2}$: it contains $P$ at 2 and $Q$ at 3. In fact, it should be clear that, if we choose canonical $\mathcal{R}$-rods in (6), then they will all be isomorphic copies of at most $|\mathcal{R}|$-many rods: more precisely, they will be of the form $r_{S(e, j, n)}$, for a role $S$ from $\mathcal{R}$.

In the proof of Lemma 9, we show how this collection of beams and $\mathcal{R}$-canonical rods can be used to obtain a model $I$ of $\mathcal{K}$ shown below (again, $A$ is omitted):

We now reduce the existence of the required collection of beams to the satisfiability problem for the one-variable first-order $LTL$ and thus establish decidability and the upper complexity bound for $DL$-$DL$-$DL$-$DL$-$DL$, which turns out to be tight.


Proof. We first show the upper bound. Let $\mathcal{K} = (\mathcal{O}, \mathcal{A})$ be a $DL$-$DL$-$DL$-$DL$-$DL$ KB with $\mathcal{T} = T \cup \mathcal{R}$. We assume that $\mathcal{R}$ is closed under taking the inverses of roles in $\mathcal{R}$.

We define a translation $\psi_\mathcal{K}$ of $\mathcal{K}$ into first-order $LTL$ with a single individual variable $x$. We treat elements of $\Delta$ as constants in the first-order language, basic concepts $B$ as unary predicates and roles $P$ as binary predicates, assuming that $P'(u, x) = P_i(x, u)$, and let $\psi_\mathcal{K}$ be a conjunction of the following sentences, for all constants $u$ in $\Delta$:

$$\Box \forall x (R_1(u, x) \land \cdots \land R_k(u, x) \rightarrow R_i(u, x)).$$

for RI $R_1 \cap \cdots \cap R_k \subseteq R$ in $\mathcal{R}$, and similarly with $\Box$ for RI $R_1 \cap \cdots \cap R_k \subseteq \Box$ in $\mathcal{R}$.

It can be seen that each collection of beams $b_u$, $u \in \Delta$, for $\mathcal{T}$ gives rise to a model $\mathfrak{M}$ of $\psi_\mathcal{K}$: the domain of $\mathfrak{M}$ comprises $\Delta$ and the $g_{s,m}$, for a role $S$ and $m \in \mathbb{Z}$. Then, take $\mathcal{R}$-canonical rods $r_{a,b}$ for $\{S(d, e, f) \mid S(a, b, f) \in \mathcal{A}\}$, which exist by (5), and $\mathcal{R}$-canonical rods $r_{s,m}$ for $\{S(d, e, m)\}$ for every $S$ and $m \in \mathbb{Z}$ with $\mathfrak{M}$ be the mirror image of $a$. We omit $A$ since $\mathcal{A}$ is a model.

In fact, it should be clear that, if we choose canonical $\mathcal{R}$-rods in (6), then they will all be isomorphic copies of at most $|\mathcal{R}|$-many rods: more precisely, they will be of the form $r_{S(e, j, n)}$, for a role $S$ from $\mathcal{R}$.


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We define a translation $\psi_\mathcal{K}$ of $\mathcal{K}$ into first-order $LTL$ with a single individual variable $x$. We treat elements of $\Delta$ as constants in the first-order language, basic concepts $B$ as unary predicates and roles $P$ as binary predicates, assuming that $P'(u, x) = P_i(x, u)$, and let $\psi_\mathcal{K}$ be a conjunction of the following sentences, for all constants $u$ in $\Delta$:

$$\Box \forall x (R_1(u, x) \land \cdots \land R_k(u, x) \rightarrow R_i(u, x)).$$

for RI $R_1 \cap \cdots \cap R_k \subseteq R$ in $\mathcal{R}$, and similarly with $\Box$ for RI $R_1 \cap \cdots \cap R_k \subseteq \Box$ in $\mathcal{R}$.
Proof. We encode $K$ in LTL following the proof of Theorem 10 and representing (7)–(8) as LTL-formulas with variables of the form $C^l(u)$, $R^l(u,v)$, for $u,v \in \Delta$. Sentences (9), however, require a different treatment. First, take

$$\square (O_1(\exists S)) u \rightarrow O_2(\exists S) (u),$$

for every $O_1 \subseteq O_2 \subseteq \mathcal{S}$ in $\mathcal{R}$, where each $O_2$ is $\mathcal{S}_P$, $\mathcal{S}_P$ or blank. Then we need CIs of the form $\exists \mathcal{R} \subseteq \mathcal{S}_P$, $\exists \mathcal{S}$ and $\exists \mathcal{R} \subseteq \mathcal{S}_P \exists \mathcal{S}$, for all $\mathcal{R}$ and $\mathcal{S}$ with defined $\max_{R,S}$ and $\min_{R,S}$, which are not entailed by (10). These integers can be represented in binary using $n$ bits, where $n$ is polynomial in $|\mathcal{R}|$. Assuming that $\max_{R,S} \geq 0$, we encode, for example, $\exists \mathcal{R} \subseteq \mathcal{S}_P \exists \mathcal{S}$ by

$$\square \bigl( \square \lnot (\exists \mathcal{R}) (u) \rightarrow \lnot (\exists \mathcal{S}) (u) \bigr),$$

(11)

$$\square \bigl( (\exists \mathcal{R}) (u) \land \lnot (\exists \mathcal{S}) (u) \rightarrow \exists \mathcal{S} \exists \mathcal{R} (u) \bigr),$$

(12)

$$\square \bigl( \lnot (\exists \mathcal{S}) (u) \rightarrow \exists \mathcal{S} \exists \mathcal{R} (u) \bigr),$$

(13)

where (12) is expressed by $O(n^2)$ formulas encoding the binary counter (similar to those in the proof of Theorem 8). To explain the meaning of (11)–(13), consider any $w \in \Delta^2$ in a model $\mathcal{I}$ of $\mathcal{K}$. If $w \in (\exists \mathcal{R})^{\mathcal{I}}(n)$ for infinitely many $n > 0$, then $w \in (\exists \mathcal{S})^{\mathcal{I}}(n)$ for all $n$, which is captured by (11). Otherwise, there is $n$ such that $w \in (\exists \mathcal{R})^{\mathcal{I}}(n)$ and $w \notin (\exists \mathcal{S})^{\mathcal{I}}(n)$ for all $m > n$, whence $w \in (\exists \mathcal{S})^{\mathcal{I}}(k)$, for any $k < n + \max_{R,S}$, which is captured by (12) and (13).

The LTL translation $\Psi_K$ of $\mathcal{K}$ is a conjunction of (7)–(8), (10) and (11)–(13) for all $\mathcal{R}$ and $\mathcal{S}$ with defined $\max_{R,S}$, and their counterparts for $\exists \mathcal{R} \subseteq \mathcal{S}_P \exists \mathcal{S}$. One can show that $\mathcal{K}$ is satisfiable iff $\Psi_K$ is satisfiable. The PSPACE lower bound follows from the fact that every LTL-formula is equisatisfiable with some LTL$^\mathsf{core}$ KB.

5 FO(RPR)-Rewritability of DL-Lite$^\mathsf{core}$/horn

We next investigate the data complexity of the satisfiability problem for temporal DL-Lite KBs. Again, out first result is negative; it is proved using Theorem 6 and a representation of the universal Turing machine by a set of tiles.

Theorem 12. There is a DL-Lite$^\mathsf{core}$/horn ontology $\mathcal{O}$ for which the satisfiability of $(\mathcal{O}, \mathcal{A})$, for a given $\mathcal{A}$, is undecidable.

We obtain our positive results using FO-rewritability. Let $\mathcal{L} \in \{ \mathsf{FO(<)}, \mathsf{FO(<,\exists n)}, \mathsf{FO(RPR)} \}$. Our first aim is to show that $\mathcal{L}$-rewritability of DL-Lite$^\mathsf{core}$/horn ontologies can be reduced to $\mathcal{L}$-rewritability of ontology-mediated atomic queries (or OMAQs) with LTL ontologies.

In general, by an OMAQ $q$ we mean a pair of the form $(\mathcal{O}, \mathcal{A})$ or $(\mathcal{P}, \mathcal{A})$, where $\mathcal{O}$ is an ontology, $\mathcal{A}$ a concept, and $\mathcal{P}$ a role name. A certain answer to $(\mathcal{O}, \mathcal{A})$ over an ABox $\mathcal{A}$ is any $(a,t) \in \text{ind}(\mathcal{A}) \times \text{tem}(\mathcal{A})$ such that $a^2 \in A^t(\mathcal{A})$ for every model $\mathcal{I}$ of $(\mathcal{O}, \mathcal{A})$; a certain answer to $(\mathcal{P}, \mathcal{A})$ over $\mathcal{A}$ is any $(a,b,t) \in \text{ind}(\mathcal{A}) \times \text{tem}(\mathcal{A})$ with $(a^2, b^2) \in P^t(\mathcal{A})$ for every $\mathcal{I} \models (\mathcal{O}, \mathcal{A})$. An $\mathcal{L}$-rewriting of $(\mathcal{O}, \mathcal{A})$ is an $\mathcal{L}$-formula $\Phi(x,t)$ such that $(a,t)$ is a certain answer to $(\mathcal{O}, \mathcal{A})$ over any ABox $\mathcal{A}$ if $\Phi_a \models \Phi(a,t)$; an $\mathcal{L}$-rewriting of $(\mathcal{P}, \mathcal{A})$ is defined similarly.

First, we show how to reduce the satisfiability problem for DL-Lite$^\mathsf{core}$/horn ontologies $\mathcal{O}$ to answering OMAQs $(\mathcal{O}', \mathcal{A}')$ with a $\bot$-free ontology $\mathcal{O}'$ and a concept name $\mathcal{A}'$. More precisely, for any ABox $\mathcal{A}$, the KB $(\mathcal{O}, \mathcal{A})$ is satisfiable iff $(\mathcal{O}', \mathcal{A}')$ has no certain answers over $\mathcal{A}$.

Let $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$. We define $\mathcal{O}' = \mathcal{T}' \cup \mathcal{R}'$ as follows. The RBox $\mathcal{R}'$ is obtained by replacing every occurrence of $\bot$ in $\mathcal{R}$ with a fresh role name $\mathcal{P}_2$, and adding the RI $\mathcal{R} \subseteq \mathcal{P}_1$, for any $\mathcal{P}$ inconsistent with $\mathcal{O}$ in the sense that $(\mathcal{O}, \{ \mathcal{P}(a,b) \})$ has no models. The TBox $\mathcal{T}'$ results from replacing every $\bot$ in $\mathcal{T}$ with a fresh concept $\mathcal{A}'$ and adding the CIs $\mathcal{P}_1 \subseteq \mathcal{A}'$, $\mathcal{P}_2 \subseteq \mathcal{A}'$, together with $\mathcal{A}' \subseteq \mathcal{P}_2 \mathcal{A}'$ and $\mathcal{A}' \subseteq \mathcal{P}_1 \mathcal{A}'$. We assume that $\mathcal{A}'$ is (or becomes) a notational variant of $\mathcal{A}'$. The consequence of the example of Theorem 10.

$$\Phi(x,t) = A(x,t) \vee \exists y(\{Q(x,y,t) \lor P(x,y,t-1)\})$$
as a rewriting of \((\mathcal{T}, A)\). It is readily checked that the following formula \(\Psi(x, y, t')\) is a rewriting of \((\mathcal{R}, Q)\):

\[
Q(x, y, t') \lor \left( \left( T_1(x, y, t') \lor P(x, y, t') \right) \land \left( T_2(x, y, t') \lor \exists y \left( (t'' < t') \land T_1(x, y, t'') \lor \exists y \left( (t''' < t'') \land T_2(x, y, t''') \right) \right) \right) \right).
\]

However, the result of replacing \(Q(x, y, t')\) with \(\Psi(x, y, t')\) is not an FO(\(-\))-rewriting of \((\mathcal{O}, A)\): when evaluated over \(A = \{ T(a, b, 0), P(a, b, 1) \}\), it does not return the certain answers \((a, 0)\) and \((a, 1)\); see the picture below:

(Note that these answers would be found had we evaluated the obtained ‘rewriting’ over \(Z\) rather than \(\{0, 1\}\). This time, we miss the CI \(\exists y \left( (t < t') \land \exists y \left( Q(x, y, t') \lor \exists y \left( G(x, y, t') \lor \right) \right) \right)\), and symmetrical ones for \(Z\).

We denote by \(\mathcal{T}_R\) the LTL\textsuperscript{\(□\)} TBox obtained from \(\mathcal{T}_R\) by replacing every basic concept \(B\) with its surrogate \(B^i\).

**Theorem 14.** A DL-Lite\textsuperscript{\(\text{hbox/horn}\)} OMAQ \((\mathcal{O}, A)\) with a \(\bot\)-free \(\mathcal{O} = \mathcal{T} \cup \mathcal{R}\) is L-rewritable whenever

- the LTL\textsuperscript{\(\text{hbox}\)} OMAQ \((\mathcal{T}_R; A)\) is L-rewritable and
- the LTL\textsuperscript{\(\text{horn}\)} OMAQ \((\mathcal{R}, R)\) is L-rewritable, for every temporalised role in \(\mathcal{R}\).

As a first consequence of Theorems 13 and 14, we obtain:

**Theorem 15.** Every DL-Lite\textsuperscript{\(\text{hbox/horn}\)} ontology is FO(RPR) rewriting.

Note that, as follows from (Artale et al. 2015, Theorem 9), satisfiability of LTL\textsuperscript{\(\text{horn}\)} KBs is NC\(_1\)-hard for data complexity, and so satisfiability of DL-Lite\textsuperscript{\(\text{hbox/horn}\)} ontologies is NC\(_1\)-complete.

**6 FO(\(-\), \(\exists\mathbf{\Xi}\))-Rewritability of DL-Lite\textsuperscript{\(\text{krom/core}\)}**

If \(\mathcal{O} = \mathcal{T} \cup \mathcal{R}\) is a DL-Lite\textsuperscript{\(\text{krom/core}\)} ontology, then the TBox \(\mathcal{T}_R\) constructed above in is DL-Lite\textsuperscript{\(\text{hbox/horn}\)} and so, by Theorem 14, we can show \(L\)-rewritability of \(\mathcal{O}\) by establishing \(L\)-rewritability of every LTL\textsuperscript{\(\text{horn}\)} OMAQ. It is known from (Artale et al. 2020) that LTL\textsuperscript{\(\text{krom}\)} OMAQs are FO(\(-\), \(\exists\mathbf{\Xi}\))-rewritable. Here we establish FO(\(-\), \(\exists\mathbf{\Xi}\))-rewritability of all LTL\textsuperscript{\(\text{horn}\)} OMAQs. The proof utilises the monotonicity of the \(\square\) operators, similarly to the proof of (Artale et al. 2020, Theorem 11). However, the latter relies on partially-ordered NFAs accepting the models of \((\mathcal{O}, A)\), which do not work in the presence of \(\mathcal{O}\). Our key observation here is that every model of \((\mathcal{O}, A)\) has at most \(O(|\mathcal{O}|)\) timestamps such that the same \(\Box\)-concepts hold between any two nearest of them. The placement of these timestamps and their concept types can be described by an FO(\(-\))-formula. However, to check whether these types are compatible (i.e., satisfiable in some model), we require FO(\(-\), \(\exists\mathbf{\Xi}\))-formulas similar to those in the proof of (Artale et al. 2020, Theorem 14).

**Theorem 16.** LTL\textsuperscript{\(\text{horn}\)} OMAQs are FO(\(-\), \(\exists\mathbf{\Xi}\))-rewritable.

**Proof.** Let \(q = (\mathcal{O}, A)\) be an LTL\textsuperscript{\(\text{horn}\)} OMAQ. We can assume that \(A\) occurs in \(\mathcal{O}\), which has no nested occurrences of temporal operators and contains CIs \(\Box B \equiv A \land B\), for every \(\Box B\) in \(\mathcal{O}\) with \(\Box \in \{ \Box P, \Box Q \}\). Define an NFA \(\mathfrak{N}_O\) that recognises ABoxes \(A\) consistent with \(\mathcal{O}\), represented as words \(X_{\text{min}_A}, \ldots, X_{\text{max}_A}\), where

\[
X_i = \{ B \mid B(i) \in A \land B \text{ occurs in } \mathcal{O} \}, \quad i \in \text{tem}(A).
\]

The set \(\mathfrak{T}\) of states in \(\mathfrak{N}_O\) comprises maximal sets \(\tau\) of concepts of \(\mathcal{O}\) consistent with \(\mathcal{O}\); we refer to such \(\tau\) as types for \(\mathcal{O}\). Now, for any \(\tau, \tau' \in \mathfrak{T}\) and an alphabet symbol \(\pi\), the NFA \(\mathfrak{N}_O\) has a transition \(\tau \rightarrow \tau'\) just in case the following conditions hold: (i) \(X \subseteq \tau\), (ii) \(\Box P C \in \tau\) iff \(C \in \tau\), (iii) \(C \notin \tau\), (iv) \(C \in \tau\) iff \(C \subseteq \tau\), and their past counterparts. As \(\tau \rightarrow \tau'\) implies \(\tau \rightarrow \Box \tau'\), for any \(X\), we omit \(\Box\) from \(\tau\). Since all \(\tau\) in \(\mathfrak{T}\) are consistent with \(\mathcal{O}\), every state in \(\mathfrak{N}_O\) has a \(\rightarrow\)-predecessor and a \(\rightarrow\)-successor. Thus, for any ABox \(A\) represented as \(X_0, X_1, \ldots, X_m\), a timestamp \(\ell (0 \leq \ell \leq m)\) is not a certain answer to \(q\) over \(A\) iff there is a path

\[
\pi = \tau_{-1} \rightarrow X_0 \tau_0 \rightarrow X_1 \tau_1 \rightarrow X_2 \ldots \rightarrow X_m \tau_m,
\]
Lemma 17: For any ABox \( A \), a timestamp \( \ell \) over \( A \) if there are \( d \leq 2|\mathcal{O}| + 2 \), a sequence \( \tau_0 \to \cdots \to \tau_m \) of types for \( \mathcal{O} \) and a sequence min \( A = s_0 < \cdots < s_d = \max A \) such that

- \( B \in \tau_i, \) for all \( B(s_i) \in A \);
- \( (B, n) \not\models (-B', n'), \) for \( s_i < n, n' < s_{i+1} \) with \( B(n), B'(n') \in A \);
- \( (L, s_i) \not\models (-L', n'), \) for \( L \in \tau_i \) and \( s_i < n' < s_{i+1} \) with \( B'(n') \in A \);
- \( (B, n) \not\models (-L', s_{i+1}), \) for \( s_i < n < s_{i+1} \) with \( B(n) \in A \) and \( L' \in \tau_{i+1} \);
- \( (L, s_i) \not\models (-L', s_{i+1}), \) for \( L \in \tau_i \) and \( s_i < s_{i+1} \) with \( L' \in \tau_{i+1} \);
- \( t = s_i \), for some \( i (0 \leq i \leq d) \) such that \( A \not\models \tau_i \).

We can now define an FO\((<, \equiv_{\mathbb{N}})\)-rewriting \( Q(t) \) of \( q \) by encoding the conditions of Lemma 17 as follows:

\[
Q(t) = -\left[ \bigvee_{d \leq 2|\mathcal{O}| + 2} \bigvee_{\tau_0 \to \cdots \to \tau_d} \left[ \exists t_0, \ldots, t_d \right.ight.
\left. \left( \text{path}_{\tau_0 \to \cdots \to \tau_d}(t_0, \ldots, t_d) \land \bigvee_{0 \leq i \leq d} \text{type}_{\tau_i}(t_i) \right) \bigvee \bigwedge_{0 \leq i \leq d} \text{type}_{\tau_i}(t_i) \right],
\]

where path\(_{\tau_0 \to \cdots \to \tau_d}(t_0, \ldots, t_d) \) is the formula

\[
(t_0 = \min) \land (t_d = \max) \land \bigwedge_{0 \leq i < d} (t_i < t_{i+1}) \land \bigwedge_{0 \leq i \leq d} \text{type}_{\tau_i}(t_i)
\]

and the disjunction is over all (possibly infinitely many) paths and type\(_i(t_i) \) is a conjunction of all \( \neg B(t) \) with \( B \not\models \tau_i \), for concept names \( B \) in \( \mathcal{O} \): the first conjunct ensures, by contraposition, that any \( B \) from \( \mathcal{X} \) also belongs to \( \tau_i \), while the second conjunct guarantees that \( A \not\models \tau_i \) in case \( \ell = t \).

We write \( \tau \to \tau' \) if \( \tau \) and \( \tau' \) satisfy \((ii)\), but necessarily \((ii)\). One can show that any path \( \tau_0 \to \cdots \to \tau_m \) in \( \mathcal{X} \) contains a subsequence

\[
\tau_{s_0} \rightarrow \cdots \tau_{s_d} \rightarrow \cdots \rightarrow \tau_{s_{d-1}} \rightarrow \tau_{s_d}
\]

such that \( 0 = s_0 < s_1 < \cdots < s_{d-1} < s_d = m \) for \( d \leq 2|\mathcal{O}| + 2 \) and, for all \( i < d \), either \( \square C, C \in \tau_i \) or \( C \not\in \tau_i \), for all \( \square C \in \mathcal{O} \), \( \square \in \{ \Box, \Diamond \} \), and all \( j \) such that \( (s_i, s_{i+1}) \).

To deal with the \( \square \)-operators, we consider the LTL\(_{\Box} \) ontology \( \mathcal{O} \) obtained from \( \mathcal{O} \) by first extending it with the CLs \( \Box C \subseteq \Box \Box C \) and \( \Box C \subseteq \Box S C \) for all \( \Box C \in \mathcal{O} \) and their past counterparts, which are obvious LTL\(_{\Box} \) tautologies, and then replacing every \( \Box C \) and \( \Box C \) with its surrogate, a fresh concept name. Let \( \mathcal{T}_{\Box} \) be the infinite directed graph whose vertices are pairs \( (L, n) \), for a simple literal \( L \) (a concept name or its negation) in \( \mathcal{O} \) and \( n \in \mathbb{Z} \). It contains an edge from \( (L, n) \) to \( (L', n+k) \), for \( k \in (-1, 0, 1) \). If \( \mathcal{O} \models L \subseteq \Box S C \), where \( \Box^k \) denotes \( \Box^k \) to \( C \) if \( k \geq 0 \) and \( \Box^{-k} \) if \( k < 0 \). We write \( (L_1, n_1) \Rightarrow (L_2, n_2) \) to \( \mathcal{GCO} \) has a path from \( (L_1, n_1) \) to \( (L_2, n_2) \), which means that \( \mathcal{O} \models L_1 \subseteq \Box^2 L_2 \). We slightly abuse notation and write, for example, \( (L, t) \in \tau \) for a type \( \tau \) in case \( L \) is the surrogate for \( \Box C \) and \( \tau \) contains \( \Box C \).

As a consequence of Theorems 13, 14 and 16, we obtain:

**Theorem 18.** DL-Lite\(_{\Box} \) ontologies are FO\((<, \equiv_{\mathbb{N}})\)-rewritable.

## 7 Conclusions

We extended the DL-Lite family of description logics by languages with Krom, Horn and arbitrary Boolean role inclusions and identified their computational complexity. We observed, in particular, that Boolean RIs make DL-Lite as expressive as FO\(^2\), while covering Krom RIs \( \top \subseteq R_1 \sqcup R_2 \) come for free as far as satisfiability is concerned.

We used those languages as a basis for defining a new type of temporal DLs. So far the main approach to designing well-behaved fragments of first-order temporal logic has been the monodicity principle, which disallows temporal operators before a formula with two or more free variables. The main contribution of this paper is to show that by restricting the use of classical connectives one can obtain natural and decidable fragments whose expressivity for binary relations is not captured by the monodicity principle.

Interesting directions of future work include establishing the tight combined complexity of DL-Lite\(_{\Box} \) and the data complexity of DL-Lite with Krom RIs. We also plan to investigate the problem of answering queries mediated by ontologies in our temporal languages. Answering unions of conjunctive queries (UCQs) is undecidable with DL-Lite\(_{\Box} \) ontologies (Rosati 2007) and 2ExpTime-complete for DL-Lite\(_{\Box} \). (Bárány, Gottlob, and Otto 2014; Bourhis et al. 2016; Hernich 2020). UCQs with DL-Lite\(_{\Box} \) ontologies are FO\((<, \equiv_{\mathbb{N}})\)-rewritable; with DL-Lite\(_{\Box} \) ontologies they are conNP-complete for data complexity. Temporal instance queries are FO\((<, \equiv_{\mathbb{N}})\)-rewritable for DL-Lite\(_{\Box} \) and FO\((<, \equiv_{\mathbb{N}})\)-rewritable for DL-Lite\(_{\Box} \) (Artale et al. 2015).

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References