Ontology Languages (COMP321)

Solution for Exercise 2

1. Let $T = \{ A \sqsubseteq \exists r.B, E \sqsubseteq A \}$. Show that $T \not\models A \sqsubseteq E$ by giving an interpretation $I$ such that $I \models T$ and $I \not\models A \sqsubseteq E$.

Solution: Let $I = (\Delta_I, I)$ with $\Delta_I = \{a, b\}$, $A^I = \{a\}$, $B^I = \{b\}$, $E^I = \emptyset$, $r^I = \{(a, b)\}$. Then $I \models T$ and $I \not\models A \sqsubseteq E$.

2. Let $T$ be an $\mathcal{EL}$-TBox consisting of the following concept inclusions:

   \[
   \text{Bird} \equiv \text{Vertebrate} \sqcap \exists \text{has.Wing} \\
   \text{Reptile} \sqsubseteq \text{Vertebrate} \sqcap \exists \text{lays.Egg}
   \]

   (a) Is $T$ in normal form? Explain.

   Solution: No, it is not.

   (b) Given $T$, compute an $\mathcal{EL}$-TBox $T'$ in normal form using the pre-processing algorithm from the lecture.

   Solution: We start by removing $\equiv$. Namely, $T$ is transformed to

   \[
   \text{Bird} \sqsubseteq \text{Vertebrate} \sqcap \exists \text{has.Wing} \\
   \text{Vertebrate} \sqcap \exists \text{has.Wing} \sqsubseteq \text{Bird} \\
   \text{Reptile} \sqsubseteq \text{Vertebrate} \sqcap \exists \text{lays.Egg}
   \]

   Now we remove conjunctions on the right-hand-side of inclusions and obtain

   \[
   \text{Bird} \sqsubseteq \text{Vertebrate} \\
   \text{Bird} \sqsubseteq \exists \text{has.Wing} \\
   \text{Vertebrate} \sqcap \exists \text{has.Wing} \sqsubseteq \text{Bird} \\
   \text{Reptile} \sqsubseteq \text{Vertebrate} \\
   \text{Reptile} \sqsubseteq \exists \text{lays.Egg}
   \]
It remains to simplify the third inclusion. Introduce a new concept name $X$ and rewrite the TBox to $\mathcal{T}'$:

\[
\begin{align*}
&\text{Bird } \sqsubseteq \text{Vertebrate} \\
&\text{Bird } \sqsubseteq \exists \text{has.Wing} \\
&\text{Vertebrate} \sqcap X \sqsubseteq \text{Bird} \\
&\exists \text{has.Wing} \sqsubseteq X \\
&\text{Reptile } \sqsubseteq \text{Vertebrate} \\
&\text{Reptile } \sqsubseteq \exists \text{lays.Egg}
\end{align*}
\]

$\mathcal{T}'$ is in normal form.

(c) Apply the algorithm deciding $A \sqsubseteq_{\mathcal{T}'} B$ (equivalently, $\mathcal{T}' \models A \sqsubseteq B$), where $A, B$ are concept names. Use the normalised TBox $\mathcal{T}'$ as input.

Solution: We initialise

\[
\begin{align*}
S(\text{Bird}) &= \{\text{Bird}\} \\
S(\text{Vertebrate}) &= \{\text{Vertebrate}\} \\
S(\text{Wing}) &= \{\text{Wing}\} \\
S(\text{Egg}) &= \{\text{Egg}\} \\
S(\text{Reptile}) &= \{\text{Reptile}\} \\
R(\text{has}) &= \emptyset \\
R(\text{lays}) &= \emptyset
\end{align*}
\]

- Apply (simpleR) to inclusion (1) and obtain
  \[S(\text{Bird}) = \{\text{Bird, Vertebrate}\}.
  \]
- Apply (simpleR) to inclusion (5) and obtain
  \[S(\text{Reptile}) = \{\text{Reptile, Vertebrate}\}.
  \]
- Apply (rightR) to inclusion (2) and obtain
  \[R(\text{has}) = \{(\text{Bird, Wing})\}.
  \]
• Apply (rightR) to inclusion (6) and obtain

\( R(lays) = \{(\text{Reptile, Egg})\} \).

• Apply (leftR) to inclusion (4) and obtain

\( S(\text{Bird}) = \{(\text{Bird, Vertebrate, X})\} \).

• No rules are applicable. Notice, for example, that applying (conjR) to inclusion (3) does not give us anything new.

Thus, we end up with

\[
\begin{align*}
S(\text{Bird}) &= \{\text{Bird, Vertebrate, X}\} \\
S(\text{Vertebrate}) &= \{\text{Vertebrate}\} \\
S(\text{Wing}) &= \{\text{Wing}\} \\
S(\text{Egg}) &= \{\text{Egg}\} \\
S(\text{Reptile}) &= \{\text{Reptile, Vertebrate}\} \\
R(\text{has}) &= \{\{(\text{Bird, Wing})\}\} \\
R(lays) &= \{\{(\text{Reptile, Egg})\}\}
\end{align*}
\]

(d) Using the output of the algorithm, decide whether

- \( \text{Reptile} \sqsubseteq_{\mathcal{T}} \text{Vertebrate} \)
- \( \text{Vertebrate} \sqsubseteq_{\mathcal{T}} \text{Bird} \)

Solution: In the first case, the answer is yes because

\( \text{Vertebrate} \in S(\text{Reptile}) \).

In the second case the answer is no because

\( \text{Bird} \not\in S(\text{Vertebrate}) \).
3. Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox containing the following concept inclusions:

$$\begin{align*}
A & \sqsubseteq X \\
A & \sqsubseteq Y \\
B & \sqsubseteq B' \\
X \cap Y & \sqsubseteq Z \\
Z & \sqsubseteq \exists r.B \\
\exists r.B' & \sqsubseteq A'
\end{align*}$$

(a) Is $\mathcal{T}$ in normal form?

Solution: Yes, it is in normal form.

Apply the algorithm for deciding $E \sqsubseteq_{\mathcal{T}} F$ (equivalently, $\mathcal{T} \models E \sqsubseteq F$), where $E, F$ are concept names. Use $\mathcal{T}$ as input.

Solution: Initialise $S(A) = \{A\}$, $S(B) = \{B\}$, $S(Y) = \{Y\}$, $S(X) = \{X\}$, $S(Z) = \{Z\}$, $S(A') = \{A'\}$, $S(B') = \{B'\}$, $R(r) = \emptyset$.

- Apply (simpleR) to inclusion (1) and obtain $S(A) = \{A, X\}$
- Apply (simpleR) to inclusion (2) and obtain $S(A) = \{A, X, Y\}$
- Apply (simpleR) to inclusion (3) and obtain $S(B) = \{B, B'\}$
- Apply (conjR) to inclusion (4) and obtain $S(A) = \{A, X, Y, Z\}$
- Apply (rightR) to inclusion (5) and obtain $R(r) = \{(A, B)\}$
- Apply (rightR) to inclusion (5) and obtain $R(r) = \{(Z, B), (A, B)\}$
- Apply (leftR) to inclusion (6) and obtain $S(A) = \{A, X, Y, Z, A'\}$ and $S(Z) = \{Z, A'\}$. 
No more rule is applicable. Thus $S(A) = \{A, X, Y, Z, A'\}$, $S(B) = \{B, B'\}$, $S(Y) = \{Y\}$, $S(X) = \{X\}$, $S(Z) = \{Z, A'\}$, $S(A') = \{A'\}$, $S(B') = \{B'\}$, $R(r) = \{(Z, B), (A, B)\}$.

(c) Using the output of the algorithm, decide whether

- $A \sqsubseteq_T Z$: Yes, because $Z \in S(A)$;
- $B \sqsubseteq_T Z$: No, because $Z \notin S(B)$;
- $X \sqsubseteq_T Y$: No, because $Y \notin S(X)$;
- $A \sqsubseteq_T A'$: Yes, because $A' \in S(A)$;
- $B \sqsubseteq_T B'$: Yes, because $B' \in S(B)$.